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# Equivalence of Möbius Transformation and Lorentz Transformation

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**Abstract**

*The equivalence between the Möbius group acting on $C_\infty$ and the proper orthochronous Lorentz group on world vectors is reviewed. This relationship will be used to characterize how celestial spheres change with Lorentz transformations.*

**Key-Words**: Möbius Transformation, Lorentz Transformation, World vector, stereographic projection

## Introduction

Many complex analysis courses cover Möbius transformations, named after German mathematician August Ferdinand Möbius (1790-1868). However, these transformations are worthwhile because they have a variety of intriguing and practical features. Furthermore, aside from being exactly the objective, conformal maps of the Riemann sphere, they yield unexpectedly in many fields of mathematics, such as projective geometry, group theory, and in general relatively [1].

### Möbius Transformation

A linear fractional transformation is a function of the form

S: $C_\infty \rightarrow C_\infty$ defined as

$$S(z) = \frac{az+b}{cz+b},$$

$\forall$ a, b, c, d $\in \tilde{\mathbb{C}}$. Such type of function is called a Möbius transformation if ad−b, c$\neq$ 0.

Möbius transformation, w = f(z) = $\frac{az+b}{cz+d}$

where |a| + |c| > 0, ad $\neq$ bc, and the coefficients a, b, c, d are complex or real numbers, z ∈ ∀ C (all complex numbers), so that w is not a constant function [2].

The complex numbers a, b, c, and d are called co-efficient of Möbius Transformation of S(z). The determinant $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ is called the determinant of Möbius Transformation S(z). The constants a, b, c, and do not uniquely determine. For any $\lambda \neq 0$, S(z) = $\frac{(\lambda a)z+(\lambda b)}{(\lambda c)z+(\lambda d)}$.

f' = $\frac{\partial f}{\partial z}$ = $\frac{dw-b}{-cw+a}$ does not cease to exist. The Möbius transformation f(z) is conformal at every point except its pole z = $-\frac{d}{c}$.

The inverse function z = $f^{-1}(w)$, (f ∘ $f^{-1}$ ≡ I, where I is the identity and can be computed as:

$f^{-1}(w)$ = $\frac{dw-b}{-cw+a}$

**Theorem 1**. A Möbius transformation is uniquely determined by three points

$z_{i,}$ where i= 1, 2, 3, $z_i \neq z_j$ ,i, j = 1, 2, 3.

Suppose $z_i$ and $w_i$ be given by $z_i \neq z_j$, $w_i \neq w_j$ , i, j = 1, 2, 3.

Now looking for a transformation w = f(z)

$$\forall \quad f(z_i) = w_i \qquad \text{(i)}$$

Let the cross-ratio (z,$z_1$ ,$z_2$ ,$z_3$) of the points z, $z_i$ where i = 1, 2, 3

$$T(z) = (z, z_1, z_2, z_3) = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)} \qquad \text{(ii)}$$

The function T(z); z ∈ $\mathbb{C}$; $z_i$−fixed one-to-one maps $\mathbb{C}$ . The transformation (i) is given by the composition (z,$z_1$,$z_2$,$z_3$) = (w, $w_1$, $w_2$, $w_3$).

w. w = f(z) is a transformation through the equation (i).

**Theorem2**. Möbius transformations form the group of transformations $\tilde{\mathbb{C}} \rightarrow \tilde{\mathbb{C}}$ generated (under composition) as:

- translations â maps of the form z↦ z + k where k ∈ $\tilde{\mathbb{C}}$ ;
- scaling or dilations â maps of the form z⟼ kz where non-zero k ∈ $\tilde{\mathbb{C}}$ and k= 0.

- inversion â the map $z \longmapsto \frac{1}{z}$.

**Decomposition and basic characteristics**

A Möbius transformation is the result of a series of simpler transformations. Assume f is any Möbius transformation [2]. Then

$f_1(z) = z + \frac{d}{c}$ (translation by d c)

- $f_2(z) = \frac{1}{z}$ (inversion and reflection with respect to the real axis)
- $f_3(z) = -\frac{(ad-bc)}{c^2} z$ (dilation and rotation)
- $f_4(z) = z + a c$ (translation by a c) then these functions can be composed, giving

$$f_4 \circ f_3 \circ f_2 \circ f_1(z) = z = f(w) = \frac{aw+b}{cw+d} \qquad \text{(iii)}$$

The inverse Möbius transformation is outstandingly defined by the equation, $g_1, g_2, g_3, g_4$, so each $g_i$ is the inverse of $f_i$ .Now the inverse formula is given as:

$$g_4 \circ g_3 \circ g_2 \circ g_1 (z) = w = f^{-1} (z) = \frac{dz-b}{-cz+a} \text{ [3].}$$

In an example of finding Möbius transformation of $s(z) = \frac{az+b}{cz+d}$,

Supposing $s_1(z) = z + 2$, $s_2 = 2z$, $s_3(z) = \frac{1}{z}$, $s_4(z) = iz$, $s_5(z) = z + i$. Now Set $S = s_5 \circ s_4 \circ s_3 \circ s_2 \circ s_1$.

Then the value of $S(z) = \frac{i}{2+4z} + i = \frac{2iz+5i}{2z+4}$

and evaluated value is: a = 2i, b = 5i, c = 2, and d = 4.

**Lorentz Transformation**

According to the theory of special relativity, the Lorentz transformations explain how two observers' fluctuating observations of space and time can be transformed into each other's frame of reference. It reflects the astounding reality that observers moving at various speeds account for distinct distances, time sequences, and, in some situations, event orderings. The Lorentz Transformations arose from Hendrik Antoon Lorentz's and others' attempts to explain observable properties of light propagating in what was assumed to be the luminiferous ether [2],[4],[5].

Later, Albert Einstein reinterpreted the transformation as a statement about the nature of space and time, deriving it from the postulates of relativity. Lorentz transformations explain the link between two inertial frames' coordinates [2].

### Method and Discussion

The Möbius transformation and Lorentz transformation are mathematical transformations that arise in different areas of mathematics and physics, and they are not equivalent in a direct sense. However, there is a connection between them that arises in the context of complex analysis and hyperbolic geometry.

For every P = (x, y, z) $\in S^2$ and w = u + iv $\in \check{\mathbb{C}}$, the stereographic projection SP has got the form

$$SP(P) = \frac{x}{1-z} + i\ \frac{y}{1-z} \qquad \text{(iv)}$$

and it's inverse transformation $Sp^{-1}$

$$Sp^{-1}\ (w) = \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1}\right),$$

Now expressing every complex number with homogeneous coordinates $\zeta = \frac{\delta}{\eta}$ , the equation (iv) can be written as:

$$x = \frac{\delta\bar{\eta}+\eta\bar{\delta}}{\eta\bar{\eta}+\delta\bar{\delta}},\ y = \frac{\delta\bar{\eta}-\eta\bar{\delta}}{i(\eta\bar{\eta}+\delta\bar{\delta})},\ z = \frac{\eta\bar{\eta}+\delta\bar{\delta}}{i(\eta\bar{\eta}+\delta\bar{\delta})} \qquad \text{(v)}$$

**World vector**

The transformation Y : $\mathbb{R}^4 \rightarrow$ H(v), being H(v), the set of 2×2 hermitian matrices, act like world-vectors as

$$\gamma(T,X,Y,Z) \rightarrow \frac{1}{\sqrt{2}} \begin{pmatrix} T+Z & X+iY \\ X-iY & T-Z \end{pmatrix}$$

(vi)

Since $\gamma$ is linear and bijective, so for every member of $\mathbb{R}^4$ consists one and only one member of H (v) and vice-versa. The following theorem proves the equivalency of Möbius Transformation and Lorentz Transformation [6], [7]:

**Theorem 3:** Every Möbius transformation corresponds to a unique Lorentz transformation.

For some A $\in$ Projective special linear group PSL (2, $\mathbb{C}$), we have to obtain a Lorentz transformation

$\Lambda A \in \mathcal{L}$. Suppose Q $\in$ PSL (2, $\mathbb{C}$) be the image of a world-vector $\gamma(T,X,Y,Z)$ then,

$$Q = \frac{1}{\sqrt{2}} \begin{pmatrix} T+Z & X+iY \\ X-iY & T-Z \end{pmatrix} = \begin{pmatrix} \delta\bar{\delta} & \delta\bar{\eta} \\ \eta\bar{\delta} & \eta\bar{\eta} \end{pmatrix} = \begin{pmatrix} \delta \\ \eta \end{pmatrix} \begin{pmatrix} \bar{\delta} & \bar{\eta} \end{pmatrix} \qquad \text{(vii)}$$

Now, $|Q| = \frac{X^2}{2}$ is obvious. The action of a Möbius transformation on a complex number ζ ∈ C∞ can be written as the action of a member of special linear group, SL (2, C) on that number's homogeneous coordinates (δ, η).

$$AQA^* = A\begin{pmatrix}\delta \\ \eta\end{pmatrix}\begin{pmatrix}\bar{\delta} & \bar{\eta}\end{pmatrix}A^* = \begin{pmatrix}\delta\bar{\delta} & \delta\bar{\eta} \\ \eta\bar{\delta} & \eta\bar{\eta}\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}T+Z & X+iY \\ X-iY & T-Z\end{pmatrix} = \check{Q}$$

In special theory of relativity, the Lorenz transformation and the Möbius transformation in complex space C are comparable. The Lorenz transformation explains how two observers move with constant velocity with regard to each other and encapsulate the space-time interval between two events. These two transformations are also important in the construction of various structures in j-space [8].

## Conclusion

The Möbius group and Lorentz transformations' equivalence is discussed. The foundations of special relativity are given briefly. This connection is mainly explored in the mathematical study of hyperbolic geometry and its relations to complex analysis. The direct link between Möbius transformations and Lorentz transformations is not as straightforward as, for example, the relationship between Lorentz transformations and Galilean transformations in the transition from classical to special relativity.

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