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Dynamics of Transcendental Entire Functions

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Abstract

The theory of complex dynamics, is specifically the iteration of entire mappings, has developed considerably since the work of Fatou and Julia. Although most of the traditional complex dynamics are centered on polynomials and rational maps, transcendental entire functions have rich, often counter intuitive, dynamics stemming from their singularities at infinity. This article provides an overview of fundamental breakthroughs, compiles new developments, and outlines outstanding issues in the dynamics of transcendental entire functions.

Keywords: normal family; Fatou set; Julia set; escaping set; fast escaping set.

AMS (MOS) Subject Classification: 37F10, 30D05.

Introduction

An entire function is defined as a complex-valued function that is holomorphic over the entire complex plane. If an entire function is not a polynomial, it is called a transcendental entire function. A fundamental feature of transcendental entire functions is that the point at infinity is always an essential singularity. This property was first illuminated by the theorems of Weierstrass, Casorati, and Picard in the late 19th century. The interplay between transcendentality and essential singularities lies at the heart of complex dynamical behavior. A domain for a transcendental entire function is simply or multiply connected. Note that, a domain $D \subset \mathbb{C}$ is said to be simply connected if it is connected and every closed curve γ in D can be continuously contracted to a single point within D. Otherwise, it is multiply connected. Note that, the unit disk $\mathbb{D}= \{z \in \mathbb{C}: |z| <1\}$ is simply connected, the entire complex plane $\mathbb{C}$ is also simply connected, and the complement of a closed disk $\mathbb{D}= \{z \in \mathbb{C}: |z| \leq 1\}$ is not simply connected.

Example
The functions: e^{z}, sin z, cos z are transcendental entire functions.

Understanding transcendental entire functions is crucial in fields such as value distribution theory, functional equations, and transcendental number theory.

Growth and classification of transcendental entire function

The growth of an entire function f is mainly characterized by:

Order (ρ)
The order of an entire function indicates how quickly the function grows as $|z| \to \infty$. It is characterized by:

$$\rho = \lim_{sup_{n\to\infty}} \frac{\log\log M(r)}{\log(r)};,$$

where $M(r) = max_{|z| = r}|f(z)|$

Entire functions are categorized based on the value of ρ as

- finite order if $\rho < \infty$
- infinite order if $\rho = \infty$.

Type (σ)
For functions whose order is finite, the type is characterized by:

$$\sigma = \lim_{sup_{r\to\infty}} \frac{\log M(r)}{r^{\rho}}.$$

Functions are divided into minimal, normal, and maximal types based on σ.

Minimal type when $\sigma = 0$. Growth is slower than e^{rp}.

Normal type when $0 < \sigma < \infty$. Growth is comparable to e^{rp}.

Maximal type when $\sigma = \infty$. Growth is faster than e^{rp} (Levin, 1964).

Note: Transcendental entire function is an exponential type if $\rho = 1$ and $0 < \sigma < \infty$ and is a finite lower order function, which takes into account the infimum growth rate.

Example

- The exponential function e^z has order 1 and type 1.
- The sine function sin z has order 1 and type 1.
- The Gamma function $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt$ is meromorphic, but its reciprocal $\frac{1}{\Gamma(z)}$ is entire of order 1.

An overview of transcendental dynamics

Definition 1

(Beardon, 2000) (n^th iterate): Let f: $\mathbb{C}\to\mathbb{C}$ be a holomorphic function. The n^{th} iterate of f, denoted f^n and is defined by $f^0(z) = z$, $f^1(z) = f(z)$, ... , $f^{n+1}(z) = f(f^n(z))$. This process generates a sequence $\{f^n(z)\}_{n=0}^{\infty}$ called the orbit of z under f, which is the central object of study in complex dynamics.

Example

For the function $f(z) = e^z$, $f^1(z) = e^z$, $f^2(z) = e^{e^z}$, ... , $f^n(z) = e^{e^{\cdots e^z}}$ (n times).

Definition 2

(Milnor, 2011) (Pre-periodic point): Let f: $\mathbb{C}\to\mathbb{C}$ be a holomorphic function. A point $Z_0 \in \mathbb{C}$ is called a pre-periodic point of f if $m \geq 0$ and $p \geq 1$ such that $f^{m+p}(Z_0) = f^m(Z_0)$.

Definition 3

(Milno, 2011) (Periodic Point) Let f: $\mathbb{C}\to\mathbb{C}$ be a holomorphic function. A point $Z_0 \in \mathbb{C}$ is called a periodic point of f if there exists a smallest positive integer $p \geq 1$ such that $f^p(Z_0) =$, where f^p denotes the p^{th} iterate of f.

Types of Periodic Points

(Milnor, 2011) Let Z_0 be a periodic point of f of period p. The multiplier of Z_0 is defined by $\lambda = (f^p)' Z_0$, where ($'$) represents the first derivative of f^p with respect to z. Then, Z_0 is classified according to the modulus of λ as follows:

- Attracting periodic point if $|\lambda| < 1$.
- Repelling periodic point if $|\lambda| > 1$.
- Parabolic periodic point if $|\lambda| = 1$.

In the study of complex dynamics, three main types of invariant sets are distinguished.

Definition 1

A set $E \subset \mathbb{C}$ is said to be forward invariant, backward invariant, and completely invariant if $f(E) \subseteq E$, $f^{-1}(E) \subseteq E$, and $f^{-1}(E) = E = f(E)$, respectively.

Example 1

1. For the map $f(z) = z + 1$, the set $S = \{z \in \mathbb{C}$: Re $(z) > 0\}$ is forward invariant but not backward invariant.
2. For the map $f(z) = z^2$ on $\mathbb{C}_\infty$, where $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$. The set $S = \cup_{n>1} f^{-n}(1)$ is backward invariant but not forward invariant.
3. For the map $f(z) = z^n$ $(n \geq 2)$, the set $S = \{z: |z| = 1\}$ is completely invariant.

Definition 2

(Carleson & Gamelin, 2013) (Normal family): A family $\mathscr{F}$ of holomorphic functions in a domain $S \subset \mathbb{C}_\infty$ is called a normal family in S if every sequence $(f_n)_{n\in N}$ in $\mathscr{F}$ contains a subsequence $(f_{n_k})_{n_k \in N}$ which converges uniformly on every compact subset of D either to a holomorphic function f or to ∞. The family $\mathscr{F}$ is called normal at a point $z \in \mathbb{C}$ if this point has a neighborhood where it is normal.

Example

For the function $f(z) = e^z$, $\mathscr{F} = \{f^n(z)\}$ is normal in $\mathbb{C}$.

Fatou and Julia sets of transcendental entire functions

Definition 1

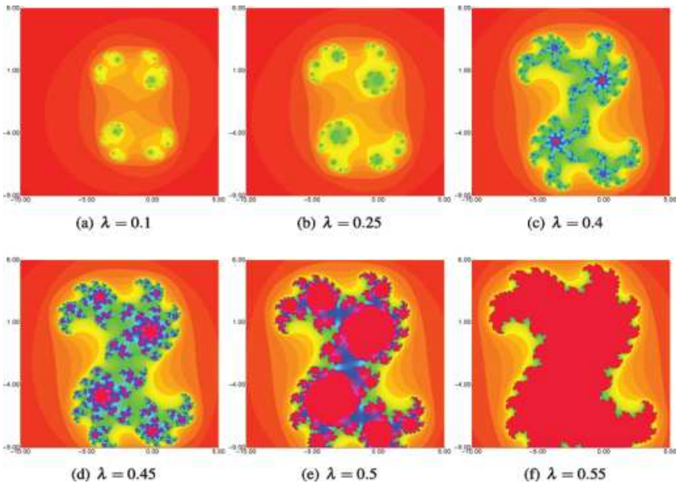
(Beardon, 2000) (Fatou and Julia sets): The Fatou set for the function f is defined by $F(f) = \{z \in \mathbb{C}$: f^n is normal in a neighborhood of $z\}$. The Julia set is the complement of the Fatou set. That is, $J(f) = \mathbb{C}_\infty \setminus F(f)$. It is the collection of points where the iterates are chaotic, sensitive to initial conditions.

Example

(Devaney, 2018; Eremenko, 1989). For the function $f(z) = \lambda e^z$, where $\lambda \in \mathbb{C} \setminus \{0\}$, the Fatou set contains baker domains where iterates tend to infinity and the

Julia set consists of infinitely many curves called "hairs" or "Cantor bouquets".

(a)



(b)

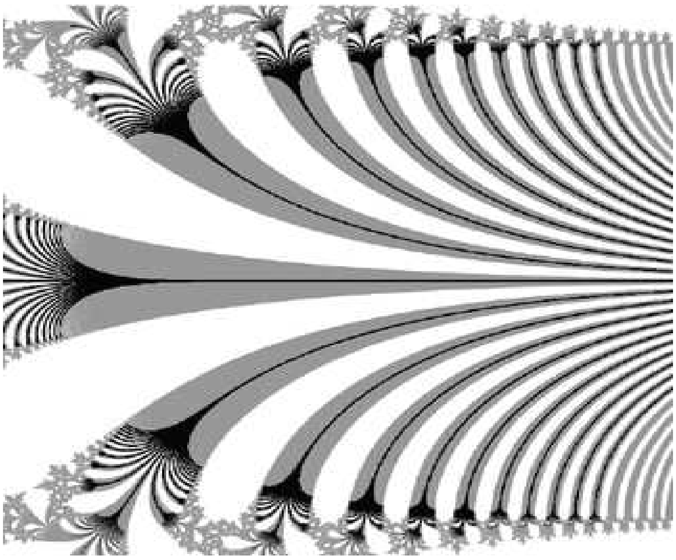


Figure (a) and (b): Fatou and Julia sets of transcendental entire function of $f(z) = \lambda e^z$. (Devaney, 2018; Eremenko, 1989)

Definition 2

(Milnor, 2011) (Fatou component): The connected components of the Fatou set are called Fatou components. Fatou components are further classified as follows:

Pre-periodic domain

A pre-periodic domain is the Fatou component U such that $f^{n+p}(U) = f^n(U)$ for some $n, p \in \mathbb{N}$. For $n = 0$, $f^p(U) = U$, then U is a Fatou component of period p.

(Bergweiler, 1993) (Wandering domain)

A wandering domain is a Fatou component U such that $\{f^n(U)\}_{n\geq 0}$ is a sequence of pair wise disjoint components of the Fatou set.

For a transcendental entire function f, the following type of periodic Fatou components may exist.

Attracting Domain

A Fatou component U such that all points in U converges to an attracting periodic point z_0, i.e. $f^p(z_0) = z_0$ and $|(f^p)'(z_0)| < 1$ (Carleson &Gamelin, 2013).

Parabolic Domain

A Fatou component where the iterates converge to a parabolic periodic point z_0, satisfying $f^p(z_0) = z_0$ and $|(f^p)'(z_0)| = 1$, with $(f^p)'(z_0)$ a root of unity, but the convergence is not linearizable (Carleson & Gamelin, 2013).

Siegel Disk

A Fatou component where the dynamics are conjugate to an irrational rotation of a disk, i.e. the function behaves like an irrational rotation near a fixed point z_0 with $|(f^p)'(z_0)| = 1$, but the multiplier is not a root of unity (Carleson & Gamelin, 2013).

Baker Domain

A periodic Fatou component U such that $f^n(z) \rightarrow \infty$ for all $z \in U$, but the function is not conjugate to a translation. These appear only in the dynamics of transcendental entire functions (Carleson & Gamelin, 2013).

Theorem 1

(Beardon, 2000) Let f be a transcendental entire function. Then, every bounded component of the Fatou set $F(f)$ is simply connected.

Theorem 2

(Bergweiler, 1993) (Fundamental properties of Fatou and Julia sets) Let f be a transcendental entire function. Then:

1. $F(f)$ is open, and $J(f)$ is closed and perfect (contains no isolated points).
2. $F(f)$ and $J(f)$ are completely invariant under f. That is, $f(F(f)) \subset F(f)$, $f^{-1}(F(f)) \subset F(f)$ and similarly for $J(f)$.
3. The Julia set $J(f)$ is nonempty and infinite.
4. Every neighborhood of a point in $J(f)$ contains infinitely many periodic points of f.

Theorem 3

(Bergweiler, 1993) (Existence of wandering domains) There exist transcendental entire functions that possess wandering domains; that is, Fatou components U such that

$f^m(U) \cap f^n(U) = \emptyset$ for all $m \neq n$.

An example is given by Baker's function $f(z) = z + 1 + e^{-z}$, which has an invariant Baker domain in which $f^n(z) \to \infty$.

Escaping set of transcendental entire function

Definition 1

(Eremenko, 1989) (Escaping Set): The escaping set of an entire function f is defined as:

$I(f) = \{z \in C: f^n(z) \to \infty \text{ as } n \to \infty\}$.

Example 1

$f(z) = \lambda e^z$, $0 < \lambda < \frac{1}{e}$. For λ in this range, f has an attracting fixed point z_0, but there also exists points whose iterates escape to infinity.

Theorem 1

(Eremenk, 1989) (Eremenko's Theorem on the escaping set) Let f be a transcendental entire function. Then,

1. $I(f) \neq \emptyset$.
2. The boundary of $I(f)$ equals the Julia set, i.e. $\partial I(f) = J(f)$.
3. Every component of $I(f)$ is unbounded.

Theorem 2

(Martí-Pete et al., 2025) There exists a transcendental entire function f, such that some connected component of the escaping set $I(f)$ is bounded.

Definition 2

(Rippon & Stallard, 2005) (Fast escaping set): Let $f:\mathbb{C} \to \mathbb{C}$ be a transcendental entire function. The fast-escaping set $A(f)$ is defined as; $A(f)=\{z\in\mathbb{C}: \exists L\in N \text{ such that } |f^{n+L}(z)| \geq M^n(R, f) \text{ for all } n \geq 0\}$,

where $M(r, f) = \max_{|z|=r}|f(z)|$ denotes the maximum modulus of f, and $R > 0$ is chosen such that $M(r, f) > r$ for all $r \geq R$.

Example 2

(Rippon & Stallard, 2005) Consider the transcendental entire function $f(z) = e^Z$. Then, the maximum modulus of f is

$$M(r, f) = \max_{|z|=r} |e^Z| = e^r.$$

Thus, $M^n(R, f) = e^{e^{\cdots e^Z}}$ (n times). For any $z \in C$ with sufficiently large Re (z), we have

$$f^n(z) = e^{e^{\cdots e^Z}} \to \infty \text{ as } n \to \infty,$$

and such points lie in the fast-escaping set $A(f)$. Hence, $A(f)$ for $f(z) = e^Z$ includes all points with a large real part, and their orbits escape to infinity at a rate comparable to the iterated exponential growth.

Let f: $\mathbb{C} \longrightarrow \mathbb{C}$ be a transcendental entire function and let $U \subset \mathbb{C}$ be a simply connected domain such that $U \cap S(f) = \emptyset$, where $S(f)$ denotes the set of singular values of f. A pullback of U under f *is* any connected component V of the pre-image $f^{-1}(U) = \{z \in \mathbb{C}: f(z) \in U\}$.That is, V is a pullback of U $\Leftrightarrow$V is a connected component of $f^{-1}(U)$.

Define inductively, $U_0 = U$ and for $n \geq 1$, U_n is any connected component of $f^{-n}(U)$. Then, the sequence $\{U_n\}_{n=0}^{\infty}$ is called an iterated pullback chain of U if $f(U_{n+1}) \subset U_n$.

Theorem 3

(Bergweiler & Rempe, 2025) Let f be a transcendental entire function and let U be a multiply connected Fatou component. Then $U \subset A(f)$ where $A(f)$ is the fast-escaping set.

Proof

Let γ_n be the sequence of curves constructed via the standard pullback construction for multiply connected Fatou components. From results on escaping curves intersecting fast escaping sets, one obtains that for sufficiently large n, $\gamma_n \cap A(f) \neq \emptyset$. Since the fast-escaping set is completely invariant, $f^{-1}(A(f)) = A(f)$, it follows that if a forward image of a curve intersects $A(f)$, then the original curve must also intersect $A(f)$. Hence, $\gamma \cap A(f) \neq \emptyset$, and therefore $U \cap A(f) \neq \emptyset$. Using the closure properties of fast-escaping sets inside the multiply connected Fatou components then yields $U \subset A(f)$.

Theorem 1

(Rippon & Stallard, 2005) Let $A(f)$ denotes the fast-escaping set of f. Then,

- $A(f) \neq \emptyset$.

- Every component of A (*f*) is unbounded.
- $J(f) = \partial A(f)$.
- $J(f) \cap A(f) \neq \emptyset$.

Definition 1

(Rippon & Stallard, 2012) (Spider's Web): A set $E \subset \mathbb{C}$ is called a spider's web if it is connected and there exists a sequence of bounded, simply connected domains $G_1 \subset G_2 \subset G_3 \subset \cdots$ such that, $\partial G_n \subset E$ for all $n \in \mathbb{N}$, and $\bigcup_{n=1} G_n = \mathbb{C}$.

Example 1

Consider the transcendental entire function $f(z) = e^z$. For this function, the fast-escaping set $A(f)$ forms a spider's web. This means that $A(f)$ is connected and there exists an increasing sequence of bounded, simply connected domains whose boundaries lie in $A(f)$, and whose union covers the entire complex plane. Hence, every compact subset of $\mathbb{C}$ is contained within some bounded region surrounded by $A(f)$.

Theorem 2

(Rempe, 2024) For certain transcendental entire functions (e.g. $f(z) = \cosh z$), the fast-escaping set $A(f)$ is a spider's web.

Theorem 3

(Evdoridou et al., 2023) For any order $\rho \in (1/2,1)$, there exists a transcendental entire function of order ρ with an unbounded wandering domain contained in the fast-escaping set.

Theorem 4

(Rippon & Stallard, 2012) Let $R > 0$ and define $A_R(f) := \{z: |f^n(z)| > M^n(R, f), \forall\, n \in \mathbb{N}\}$. If $\mathbb{C} \setminus A_R(f)$ has a bounded component, then $A_R(f)$, $A(f)$, and $I(f)$ each form a spider's web.

Theorem 5

(Bergweiler & Rempe, 2025) Let *f* be a transcendental entire function. Then, every multiply connected Fatou component of *f* is wandering.

Proof

Suppose that, if possible, there exists a multiply connected Fatou component U that is pre-periodic. Passing to an iterate, assume U is invariant. For $z \in U$, growth estimates for multiply connected invariant Fatou components imply $\log|f^n(z)| = O(n)$, however we know that $z \in A(f)$ is the fast-escaping set. But points in the fast-escaping set satisfy asymptotically faster growth, contradicting the previous estimate. Hence, U cannot be pre-periodic, so it must be wandering.

Conclusion

The dynamics of transcendental entire function has been studied intensively for the last 100 years. Within this period, structure and properties of Fatou, Julia and escaping sets are studied. In this article, we collected some important results of the transcendental dynamics.

This review highlights how the essential singularity at infinity drives complex phenomena such as intricate Julia set geometries, diverse Fatou components, and the fundamental role of escaping and fast-escaping sets. Over the past decades, a synthesis of analytic, topological, and geometric methods has deepened our understanding of these systems, leading to major advances including the structure theory of $I(f)$ and $A(f)$, the emergence of spider's web configurations, and comprehensive classifications of periodic, pre-periodic, Baker, and wandering domains.

As we know, the dynamics of transcendental entire functions form a vibrant and rapidly developing research area, where classical complex analysis intersects with modern dynamical theory. The interplay between deep theoretical results, powerful analytic techniques, and open foundational problems ensures that this field will continue to inspire significant mathematical exploration and shape future developments in complex dynamics.

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