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Numerical Solution of Time-fractional BBMB Equation via LRPSM

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Abstract

The time-fractional Benjamin-Bona-Mahony-Burger (BBMB) differential equation plays an important role in explaining the unidirectional propagation of long waves in definite nonlinear differential systems. This work presents an analytical numerical solution of considered equation by Laplace transform with residual power series method (LRPSM) which is a generalized Taylor series together with Laplace transform and the residual error function. Using the proposed approach, the series solution of this equation is obtained. The analytical numerical solution shows that the LRPSM is a reliable and powerful method for solving the time fractional differential equations (FDEs) with less number of terms. The obtained results of numberical solutions are also compared with the exact solution and presented as absolute errors at different time levels.

Keywords: Laplace transforms, BBMB equation, residual power series method, Laplace residual function

Introduction

One of the generalised forms of the classical differential equations is the fractional differential equations (FDEs) which have the considerable applications in different branches of sciences from last decades. Researchers are using FDEs to discuss various scientific and social phenomenons happening in the real world, such as a mathematical model for optimal control of a pandemic has also been given as a fractional model [1]. It is noticeable that the idea of a fractional derivative traced back to the genesis of integral calculus [2]. Since, most FDEs do not have analytical solutions; there is a need to develop the approximate approach [3] to solve them. For instance, Odibat and Momani [4] have applied variational iteration method to solve non-linear FDEs in 2006. There are different types of numerical and analytical methods developed by the researchers that are applied to solve FDEs for approximate analytical solution such as VIM [5], HAT [6], G′/G expansion method [7], Cubic B-spline method [8], Cubic B-spline Collocation Scheme [9], homotopy analysis method [10], unified method [11], method of lines [12], Lie symmetry method [13], Adomian decomposition method [14] and many more.

The BBMB equation is a well-known equation that has been used in the analysis of the surface waves of long wavelength in liquids, acoustic-gravity waves in compressible fluids and hydro-magnetic waves in cold plasma [15]. Since, the wave of water form is extremely complicated for solving analytically. Hence quite a lot of investigations have been done in recent years to conclude the numerical solution of water wave form. Acoustic-gravity waves in fluids [16], thermodynamics, cracked rock, acoustic waves in anharmonic crystals are the applications of the BBMB equation with their special cases can be found in science and engineering.

To find the logical solutions to the BBMB differential equation homotopy analysis method (HAM) [17] is also used. Some researchers have also studied the solution of the BBMB equation using linearised difference scheme [18]. The linearised difference scheme by Cheng et.al. has been applied for handling the equation with preserving dissipation property for 2D BBMB equation. Majeed et al. [19] applied the cubic B-spline scheme for estimating the solution of non-homogeneous BBMB differential equation. Ostrovsky and Degasperis-Procesi have identified the analytical solution of BBMB equation using method based on the Laplace transform with ADM [20]. The researchers have also derived the BBMB and KdV equations on water signal model [21]. The space time fractional BBMB differential equation [22] has been solved analytically using ADM.

Recently, some analytical methods based on series expansion without linearization; discretization or perturbation have been recognised and successfully applied to a large number of FDEs emerging in non-linear and dynamical problems. One of the methods that are capable to solve these kinds of FDEs is Laplace transform with residual power series method [LRPSM]. However, there are so many analytical methods for numerical solutions of FDEs out of them LRPSM is used to solve BBMB equation in present paper. This is one of the most important methods which is first used in this equation.

Consider the time-fractional BBM-Burger differential equation defined as,

$$D_t^\alpha u - u_{xxt} + u_x + (\frac{u^2}{2})_x = 0, t > 0, 0 < \alpha \leq 1 \tag{1}$$

with initial situation $u(x,0) = f_0(x) = sech^2(\frac{x}{4})$ (2)

and the accurate solution when $\alpha = 1$ is $u(x,t) = sech^2(\frac{x}{4} - \frac{t}{4})$ (3)

This equation is a non-linear FDE and is solved by LRPSM in this paper. It is a latest method for solution of BBMB equation.

The manuscript arrangement is as follows: The first section deals with the introduction and highlight of the present approach. The second section presents the methodology in steps that need to implement the LRPSM to obtain the solution of BBMB equation. The third section presents the implementation of method to this equation for the numerical experiment. In the fourth section, the numerical simulations and graphs of solutions are presented and the conclusion of this work is given in the last section.

Methodology

The method of solution of one-dimensional time-fractional BBMB equation by LRPSM done as steps given below:

Step 1 Applying the Laplace transform on equation (1) as,

$$\mathcal{L}[D_t^\alpha u] - \mathcal{L}[u_{xxt}] + \mathcal{L}[u_x] + \mathcal{L}[(\frac{u^2}{2})_x] = 0 \tag{4}$$

From Laplace transform of fractional derivatives using the relation

$\mathcal{L}[D_t^\alpha u] = s^\alpha \mathcal{L}[u] - s^{\alpha-1} u(x,0)$ on equation (4), then it can be re-framed as,

$$U(x,s) = \frac{1}{s} f_0(x) + \frac{1}{s^\alpha}[\{U(x,s)\}_{xxt} - \{U(x,s)\}_x - \frac{1}{2}[\mathcal{L}\{(\mathcal{L}^{-1}[U(x,s)])^2\}]_x] \tag{5}$$

where $f_0(x) = u(x,0),\ U(x,s) = \mathcal{L}[u(x,t)]$

Step 2 The transformed function $U(x,s)$ can be written as

$$U(x,s) = \sum_{n=0}^{\infty} \frac{f_n(x)}{s^{n\alpha+1}} \tag{6}$$

Also the $k^{th}-$ truncated series of this relation (6) can be written as

$$U_k(x,s)=\sum_{n=0}^{k}\frac{f_n(x)}{s^{n\alpha+1}}$$

$$i.e.\, U_k(x,s)=\frac{f_0(x)}{s}+\sum_{n=1}^{k}\frac{f_n(x)}{s^{n\alpha+1}} \tag{7}$$

Again the $k^{th}-$Laplace residual function is,

$$\mathcal{L}Res_k(x,s)=U_k(x,s)-\frac{1}{s}f_0(x)-\frac{1}{s^\alpha}\Big[\{U_k(x,s)\}_{xxt}-\{U_k(x,s)\}_x-\frac{1}{2}[\mathcal{L}\{(\mathcal{L}^{-1}[U_k(x,s)])^2\}]_x\Big] \tag{8}$$

To find the values of $f_k(x), k=1,2,3,\ldots\ldots\ldots$ substitute the $k^{th}-$ truncated series (7) into the $k^{th}-$Laplace residual function (8) we get,

$$\mathcal{L}Res_k(x,s)=\frac{f_0(x)}{s}+\sum_{n=1}^{k}\frac{f_n(x)}{s^{n\alpha+1}}-\frac{f_0(x)}{s}-\frac{1}{s^\alpha}\left[\left\{\frac{f_0(x)}{s}+\sum_{n=1}^{k}\frac{f_n(x)}{s^{n\alpha+1}}\right\}_{xxt}-\left\{\frac{f_0(x)}{s}+\sum_{n=1}^{k}\frac{f_n(x)}{s^{n\alpha+1}}\right\}_x-\frac{1}{2}\left[\mathcal{L}\left\{\left(\mathcal{L}^{-1}\left[\frac{f_0(x)}{s}+\sum_{n=1}^{k}\frac{f_n(x)}{s^{n\alpha+1}}\right]\right)^2\right\}\right]_x\right]$$

$$=\sum_{n=1}^{k}\frac{f_n(x)}{s^{n\alpha+1}}-\frac{1}{s^\alpha}\left[\left\{\frac{sech^2\frac{x}{4}}{s}+\sum_{n=1}^{k}\frac{f_n(x)}{s^{n\alpha+1}}\right\}_{xxt}-\left\{\frac{sech^2\frac{x}{4}}{s}+\sum_{n=1}^{k}\frac{f_n(x)}{s^{n\alpha+1}}\right\}_x-\frac{1}{2}\left[\mathcal{L}\left\{\left(\mathcal{L}^{-1}\left[\frac{sech^2\frac{x}{4}}{s}+\sum_{n=1}^{k}\frac{f_n(x)}{s^{n\alpha+1}}\right]\right)^2\right\}\right]_x\right]$$

$$=\sum_{n=1}^{k}\frac{f_n(x)}{s^{n\alpha+1}}-\frac{1}{s^\alpha}\left[\left\{\frac{sech^2\frac{x}{4}}{s}+\sum_{n=1}^{k}\frac{f_n(x)}{s^{n\alpha+1}}\right\}_{xxt}-\left\{\frac{sech^2\frac{x}{4}}{s}+\sum_{n=1}^{k}\frac{f_n(x)}{s^{n\alpha+1}}\right\}_x-\frac{1}{2}\left[\mathcal{L}\left\{\left(\mathcal{L}^{-1}\left[\frac{sech^2\frac{x}{4}}{s}+\sum_{n=1}^{k}\frac{f_n(x)}{s^{n\alpha+1}}\right]\right)^2\right\}\right]_x\right]$$

$$=\sum_{n=1}^{k}\frac{f_n(x)}{s^{n\alpha+1}}-\frac{1}{s^\alpha}\left[\left\{\frac{sech^2\frac{x}{4}}{s}+\sum_{n=1}^{k}\frac{f_n(x)}{s^{n\alpha+1}}\right\}_{xxt}-\frac{1}{2s}sech^2\frac{x}{4}tanh\frac{x}{4}-\left\{\sum_{n=1}^{k}\frac{f_n(x)}{s^{n\alpha+1}}\right\}_x-\frac{1}{2}\left[\mathcal{L}\left\{\left(\mathcal{L}^{-1}\left[\frac{sech^2\frac{x}{4}}{s}+\sum_{n=1}^{k}\frac{f_n(x)}{s^{n\alpha+1}}\right]\right)^2\right\}\right]_x\right]$$

$$\text{i.e.}\,\mathcal{L}Res_k(x,s)=\sum_{n=1}^{k}\frac{f_n(x)}{s^{n\alpha+1}}-\frac{1}{s^\alpha}\left[\left\{\frac{sech^2\frac{x}{4}}{s}+\sum_{n=1}^{k}\frac{f_n(x)}{s^{n\alpha+1}}\right\}_{xxt}-\frac{1}{2s}sech^2\frac{x}{4}tanh\frac{x}{4}-\left\{\sum_{n=1}^{k}\frac{f_n(x)}{s^{n\alpha+1}}\right\}_x-\frac{1}{2}\left[\mathcal{L}\left\{\left(\mathcal{L}^{-1}\left[\frac{sech^2\frac{x}{4}}{s}+\sum_{n=1}^{k}\frac{f_n(x)}{s^{n\alpha+1}}\right]\right)^2\right\}\right]_x\right] \tag{9}$$

Step 3 By solving the following relation recursively the coefficients $f_n(x)$ can be obtained,

$$\lim_{s\to\infty} s^{k\alpha+1}\mathcal{L}Res_k(x,s)=0\; for\; 0<\alpha\leq 1, k=1,2,3,\ldots \tag{10}$$

Following are some useful relations which are used in LRPSM;

i) $\mathcal{L}Res(x,s) = 0$ *and* $\lim_{k\to\infty} \mathcal{L}Res_k(x,s) = \mathcal{L}Res(x,s)$, for $s > 0$.

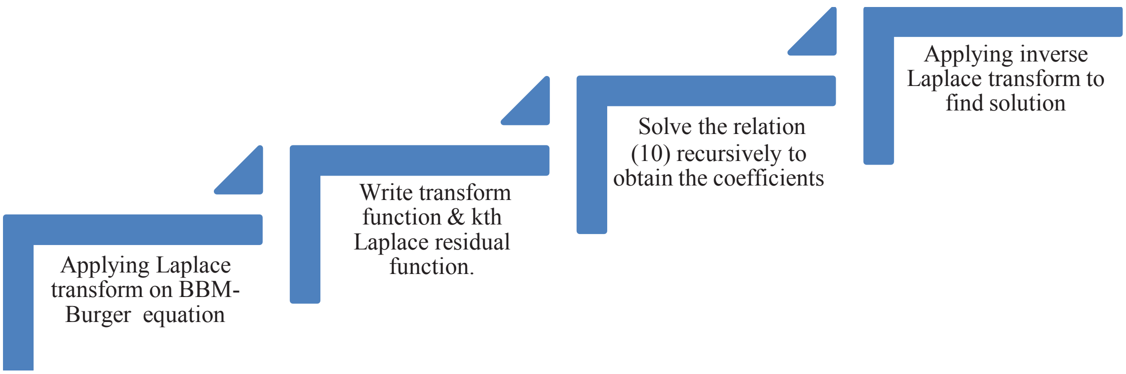
ii) $\lim_{s\to\infty} s\mathcal{L}Res(x,s) = 0$ *gives* $\lim_{s\to\infty} s\mathcal{L}Res_k(x,s) = 0$.

iii) $\lim_{s\to\infty} s^{k\alpha+1}\mathcal{L}Res(x,s) = \lim_{s\to\infty} s^{k\alpha+1}\mathcal{L}Res_k(x,s) = 0$ *for* $0 < \alpha \leq 1$.

Step 4 At last applying the inverse Laplace transform to $U_k(x,s)$ for obtaining the k^{th} approximate solution $u_k(x,t)$.

Figure 1

Pseudo code of the Methodology



Numerical Experiment

The numerical solution of BBMB equation with LRPSM can be done as follows:

Applying Laplace transform on equation (1) we get,

$$\mathcal{L}\{D_t^{\alpha}u - u_{xxt} + u_x + (\tfrac{u^2}{2})_x\} = 0 \quad (11)$$

$$or, \mathcal{L}(D_t^{\alpha}u) = \mathcal{L}(u_{xxt}) - \mathcal{L}(u_x) - \mathcal{L}(\tfrac{u^2}{2})_x$$

From Laplace transform of fractional derivatives using the relation

$\mathcal{L}[D_t^{\alpha}u(x,t)] = s^{\alpha}\mathcal{L}[u(x,t)] - s^{\alpha-1}u(x,0)$ on equation (11), then it can be re-framed as,

$$s^{\alpha}\mathcal{L}[u] - s^{\alpha-1}u(x,0) = \mathcal{L}(u_{xxt}) - \mathcal{L}(u_x) - \mathcal{L}(\tfrac{u^2}{2})_x$$

$$or, \mathcal{L}[u] = \tfrac{1}{s}u(x,0) + \tfrac{1}{s^{\alpha}}\{\mathcal{L}(u_{xxt}) - \mathcal{L}(u_x) - \mathcal{L}\left(\tfrac{u^2}{2}\right)_x\}$$

$$or, U(x,s) = \tfrac{1}{s}f_0(x) + \tfrac{1}{s^{\alpha}}[\{U(x,s)\}_{xxt} - \{U(x,s)\}_x - \tfrac{1}{2}[\mathcal{L}\{\mathcal{L}^{-1}[U(x,s)^2]\}]_x \quad (12)$$

where $\mathcal{L}[u(x,t)] = U(x,s)$ and $u(x,0) = f_0(x)$

The transformed function $U(x,s)$ can be written as

$$U(x,s)=\sum_{n=0}^{\infty}\frac{f_n(x)}{s^{n\alpha+1}} \tag{13}$$

Also the $k^{th}-$ truncated series of this relation (13) can be written as

$$U_k(x,s)=\sum_{n=0}^{k}\frac{f_n(x)}{s^{n\alpha+1}}$$

$$i.e. U_k(x,s)=\frac{f_0(x)}{s}+\sum_{n=1}^{k}\frac{f_n(x)}{s^{n\alpha+1}} \tag{14}$$

Again the $k^{th}-$Laplace residual function of (14) is

$$\mathcal{L}Res_k(x,s)=U_k(x,s)-\frac{1}{s}f_0(x)-\frac{1}{s^\alpha}\Big[\Big\{U_k(x,s)\}_{xxt}-\{U_k(x,s)\}_x-\frac{1}{2}[\mathcal{L}\{\mathcal{L}^{-1}[U_k(x,s)^2]\}]_x\Big] \tag{15}$$

To find the values of $f_k(x), k=1,2,3, \dots \dots \dots$ substitute the $k^{th}-$ truncated series (14) into $k^{th}-$Laplace residual function (15) we get,

$$\mathcal{L}Res_k(x,s)=\frac{f_0(x)}{s}+\sum_{n=1}^{k}\frac{f_n(x)}{s^{n\alpha+1}}-\frac{f_0(x)}{s}-\frac{1}{s^\alpha}\Big[\Big\{\frac{f_0(x)}{s}+\sum_{n=1}^{k}\frac{f_n(x)}{s^{n\alpha+1}}\Big\}_{xxt}-\Big\{\frac{f_0(x)}{s}+\sum_{n=1}^{k}\frac{f_n(x)}{s^{n\alpha+1}}\Big\} \frac{1}{2}\Big[\mathcal{L}\Big\{\mathcal{L}^{-1}\Big(\frac{f_0(x)}{s}+\sum_{n=1}^{k}\frac{f_n(x)}{s^{n\alpha+1}}\Big)^2\Big\}\Big]_x\Big]$$

$$=\sum_{n=1}^{k}\frac{f_n(x)}{s^{n\alpha+1}}-\frac{1}{s^\alpha}\Big[\Big\{\frac{sech^2\frac{x}{4}}{s}+\sum_{n=1}^{k}\frac{f_n(x)}{s^{n\alpha+1}}\Big\}_{xxt}-\Big\{\frac{sech^2\frac{x}{4}}{s}+\sum_{n=1}^{k}\frac{f_n(x)}{s^{n\alpha+1}}\Big\}_x-\frac{1}{2}\Big[\mathcal{L}\Big\{\mathcal{L}^{-1}\Big(\frac{sech^2\frac{x}{4}}{s}+\sum_{n=1}^{k}\frac{f_n(x)}{s^{n\alpha+1}}\Big)^2\Big\}\Big]_x\Big]$$

$$=\sum_{n=1}^{k}\frac{f_n(x)}{s^{n\alpha+1}}-\frac{1}{s^\alpha}\Big[\Big\{\frac{sech^2\frac{x}{4}}{s}+\sum_{n=1}^{k}\frac{f_n(x)}{s^{n\alpha+1}}\Big\}_{xxt}-\Big\{\frac{sech^2\frac{x}{4}}{s}+\sum_{n=1}^{k}\frac{f_n(x)}{s^{n\alpha+1}}\Big\}_x-\frac{1}{2}\Big[\mathcal{L}\Big\{\mathcal{L}^{-1}\Big(\frac{sech^2\frac{x}{4}}{s}+\sum_{n=1}^{k}\frac{f_n(x)}{s^{n\alpha+1}}\Big)^2\Big\}\Big]_x\Big]$$

$$=\sum_{n=1}^{k}\frac{f_n(x)}{s^{n\alpha+1}}-\frac{1}{s^\alpha}\Big[\{\sum_{n=1}^{k}\frac{f_n(x)}{s^{n\alpha+1}}\}_{xxt}+\frac{1}{2s}sech^2\frac{x}{4}tanh\frac{x}{4}-\{\sum_{n=1}^{k}\frac{f_n(x)}{s^{n\alpha+1}}\}_x-\frac{1}{2}\Big[\mathcal{L}\{\mathcal{L}^{-1}\Big(\frac{sech^2\frac{x}{4}}{s}+\sum_{n=1}^{k}\frac{f_n(x)}{s^{n\alpha+1}}\Big)^2\}\Big]_x\Big]$$

Or, $\mathcal{L}Res_k(x,s) = \sum_{n=1}^{k}\frac{f_n(x)}{s^{n\alpha+1}} - \frac{1}{s^\alpha}\left[\{\sum_{n=1}^{k}\frac{f_n(x)}{s^{n\alpha+1}}\}_{xxt} + \frac{1}{2s}sech^2\frac{x}{4}tanh\frac{x}{4} - \{\sum_{n=1}^{k}\frac{f_n(x)}{s^{n\alpha+1}}\}_x - \frac{1}{2}\left[\mathcal{L}\left\{\mathcal{L}^{-1}\left(\frac{sech^2\frac{x}{4}}{s} + \sum_{n=1}^{k}\frac{f_n(x)}{s^{n\alpha+1}}\right)^2\right\}\right]_x\right]$ (16)

For $k=1$ from (16) the first Laplace residual function is,

$$\mathcal{L}Res_1(x,s) = \frac{f_1(x)}{s^{\alpha+1}} - \frac{1}{s^\alpha}\left[\{f_1(x)\}_{xxt}\frac{1}{s^{\alpha+1}} + \frac{1}{2s}sech^2\frac{x}{4}tanh\frac{x}{4} - \{f_1(x)\}_x\frac{1}{s^{\alpha+1}} - \frac{1}{2}\left[\mathcal{L}\left\{\mathcal{L}^{-1}\left(\frac{sech^2\frac{x}{4}}{s} + \frac{f_1(x)}{s^{\alpha+1}}\right)^2\right\}\right]_x\right] \quad (17)$$

$$= \frac{f_1(x)}{s^{\alpha+1}} - [\{f_1(x)\}_{xxt}\frac{1}{s^{2\alpha+1}} + \frac{1}{2s^{\alpha+1}}sech^2\frac{x}{4}tanh\frac{x}{4} - \{f_1(x)\}_x\frac{1}{s^{2\alpha+1}} - \frac{1}{2}\left[\mathcal{L}\left\{\mathcal{L}^{-1}\left(\frac{sech^4\frac{x}{4}}{s^2} + \frac{2sech^2\frac{x}{4}f_1(x)}{s^{\alpha+2}} + \frac{\{f_1(x)\}^2}{s^{2\alpha+2}}\right\}\right]_x]$$

$$= \frac{f_1(x)}{s^{\alpha+1}} - \{f_1(x)\}_{xxt}\frac{1}{s^{2\alpha+1}} - \frac{1}{2s^{\alpha+1}}sech^2\frac{x}{4}tanh\frac{x}{4} + \{f_1(x)\}_x\frac{1}{s^{2\alpha+1}} + \frac{1}{2s^\alpha}\mathcal{L}[sech^4\frac{x}{4}\frac{t}{1!} + 2sech^2\frac{x}{4}f_1(x)\frac{t^{\alpha+1}}{(\alpha+1)!} + \frac{\{f_1(x)\}^2}{(2\alpha+1)!}\,t^{2\alpha+1}]$$

$$= \frac{f_1(x)}{s^{\alpha+1}} - \{f_1(x)\}_{xxt}\frac{1}{s^{2\alpha+1}} - \frac{1}{2s^{\alpha+1}}sech^2\frac{x}{4}tanh\frac{x}{4} + \{f_1(x)\}_x\frac{1}{s^{2\alpha+1}} + \frac{1}{2s^\alpha}[sech^4\frac{x}{4}\frac{1}{s^2} + 2sech^2\frac{x}{4}f_1(x)\frac{1}{(\alpha+1)!}\frac{(\alpha+1)!}{s^{\alpha+2}} + \frac{\{f_1(x)\}^2}{(2\alpha+1)!}\frac{(2\alpha+1)!}{s^{2\alpha+2}}]$$

$$= \frac{f_1(x)}{s^{\alpha+1}} - \{f_1(x)\}_{xxt}\frac{1}{s^{2\alpha+1}} - \frac{1}{2s^{\alpha+1}}sech^2\frac{x}{4}tanh\frac{x}{4} + \{f_1(x)\}_x\frac{1}{s^{2\alpha+1}} + \frac{1}{2}sech^4\frac{x}{4}\frac{1}{s^{\alpha+2}} + sech^2\frac{x}{4}f_1(x)\frac{1}{s^{2\alpha+2}} + \frac{\{f_1(x)\}^2}{s^{3\alpha+2}}$$

Now, the relation $\lim_{s\to\infty}(s^{\alpha+1}\,\mathcal{L}Res_1(x,s)) = 0\ for\ k=1$ gives that,

$$f_1(x) - \frac{1}{2}sech^2\frac{x}{4}tanh\frac{x}{4} = 0$$

i.e. $f_1(x) = \frac{1}{2}sech^2\frac{x}{4}tanh\frac{x}{4}$

For $k=2$ from (16) the second Laplace residual function is,

$$\mathcal{L}Res_2(x,s) = \sum_{n=1}^{2}\frac{f_n(x)}{s^{n\alpha+1}} - \frac{1}{s^\alpha}\left[\left\{\sum_{n=1}^{2}\frac{f_n(x)}{s^{n\alpha+1}}\right\}_{xxt} + \frac{1}{2s}sech^2\frac{x}{4}tanh\frac{x}{4} - \left\{\sum_{n=1}^{2}\frac{f_n(x)}{s^{n\alpha+1}}\right\}_x - \frac{1}{2}\left[\mathcal{L}\left\{\mathcal{L}^{-1}\left(\frac{sech^2\frac{x}{4}}{s} + \sum_{n=1}^{2}\frac{f_n(x)}{s^{n\alpha+1}}\right)^2\right\}\right]_x\right] \quad (18)$$

$$= \frac{f_1(x)}{s^{\alpha+1}} + \frac{f_2(x)}{s^{2\alpha+1}} - \frac{1}{s^\alpha}\left[\left\{\frac{f_1(x)}{s^{\alpha+1}} + \frac{f_2(x)}{s^{2\alpha+1}}\right\}_{xxt} + \frac{1}{2s}sech^2\frac{x}{4}tanh\frac{x}{4} - \left\{\frac{f_1(x)}{s^{\alpha+1}} + \frac{f_2(x)}{s^{2\alpha+1}}\right\}_x - \frac{1}{2}\left[\mathcal{L}\left\{\mathcal{L}^{-1}\left(\frac{sech^2(\frac{x}{4})}{s} + \frac{f_1(x)}{s^{\alpha+1}} + \frac{f_2(x)}{s^{2\alpha+1}}\right)\right\}^2\right]_x\right]$$

$$= \frac{f_1(x)}{s^{\alpha+1}} + \frac{f_2(x)}{s^{2\alpha+1}} - \frac{1}{s^\alpha}\left[\left\{\frac{f_1(x)}{s^{\alpha+1}}\right\}_{xxt} + \left\{\frac{f_2(x)}{s^{2\alpha+1}}\right\}_{xxt} + \frac{1}{2s}sech^2\frac{x}{4}tanh\frac{x}{4} - \left\{\frac{f_1(x)}{s^{\alpha+1}}\right\}_x - \left\{\frac{f_2(x)}{s^{2\alpha+1}}\right\}_x - \frac{1}{2}\left[\mathcal{L}\left\{\mathcal{L}^{-1}\left(\frac{sech^2(\frac{x}{4})}{s} + \frac{f_1(x)}{s^{\alpha+1}} + \frac{f_2(x)}{s^{2\alpha+1}}\right)\right\}^2\right]_x\right]$$

$$= \frac{sech^2\frac{x}{4}tanh\frac{x}{4}}{2s^{\alpha+1}} + \frac{f_2(x)}{s^{2\alpha+1}} - \frac{(f_1(x))_{xxt}}{s^{2\alpha+1}} - \frac{\{f_2(x)\}_{xxt}}{s^{3\alpha+1}} - \frac{sech^2\frac{x}{4}tanh\frac{x}{4}}{2s^{\alpha+1}} + \frac{\{f_1(x)\}_x}{s^{2\alpha+1}} + \frac{\{f_2(x)\}_x}{s^{3\alpha+1}} + \frac{1}{2s^\alpha}\left[\mathcal{L}\left\{\mathcal{L}^{-1}\left(\frac{sech^4\frac{x}{4}}{s^2} + \frac{\{f_1(x)\}^2}{s^{2\alpha+2}} + \frac{\{f_2(x)\}^2}{s^{4\alpha+2}} + \frac{2sech^2\frac{x}{4}f_1(x)}{s^{\alpha+2}} + \frac{2f_1(x)f_2(x)}{s^{3\alpha+2}} + \frac{2sech^2\frac{x}{4}f_2(x)}{s^{2\alpha+2}}\right)\right]_x$$

$$= \frac{f_2(x)}{s^{2\alpha+1}} - \frac{(f_1(x))_{xxt}}{s^{2\alpha+1}} - \frac{\{f_2(x)\}_{xxt}}{s^{3\alpha+1}} + \frac{\{f_1(x)\}_x}{s^{2\alpha+1}} + \frac{\{f_2(x)\}_x}{s^{3\alpha+1}} + \frac{1}{2s^\alpha}\left[\mathcal{L}\left\{sech^4\frac{x}{4}\frac{t}{1!} + \{f_1(x)\}^2\frac{t^{2\alpha+1}}{(2\alpha+1)!} + \{f_2(x)\}^2\frac{t^{4\alpha+1}}{(4\alpha+1)!} + 2sech^2\frac{x}{4}f_1(x)\frac{t^{\alpha+1}}{(\alpha+1)!} + 2f_1(x)f_2(x)\frac{t^{3\alpha+1}}{(3\alpha+1)!} + 2sech^2\frac{x}{4}f_2(x)\frac{t^{2\alpha+1}}{(2\alpha+1)!}\right]_x$$

$$= \frac{f_2(x)}{s^{2\alpha+1}} - \frac{(f_1(x))_{xxt}}{s^{2\alpha+1}} - \frac{\{f_2(x)\}_{xxt}}{s^{3\alpha+1}} + \frac{\{f_1(x)\}_x}{s^{2\alpha+1}} + \frac{\{f_2(x)\}_x}{s^{3\alpha+1}} + \frac{1}{2s^\alpha}\left[sech^4\frac{x}{4}\frac{1}{s^2} + \{f_1(x)\}^2\frac{1}{(2\alpha+1)!}\frac{(2\alpha+1)!}{s^{2\alpha+2}} + \{f_2(x)\}^2\frac{1}{(4\alpha+1)!}\frac{(4\alpha+1)!}{s^{4\alpha+2}} + 2sech^2\frac{x}{4}f_1(x)\frac{1}{(\alpha+1)!}\frac{(\alpha+1)!}{s^{\alpha+2}} + 2f_1(x)f_2(x)\frac{1}{(3\alpha+1)!}\frac{(3\alpha+1)!}{s^{3\alpha+2}} + 2sech^2\frac{x}{4}f_2(x)\frac{1}{(2\alpha+1)!}\frac{(2\alpha+1)!}{s^{2\alpha+2}}\right]_x$$

$$= \frac{f_2(x)}{s^{2\alpha+1}} - \frac{(f_1(x))_{xxt}}{s^{2\alpha+1}} - \frac{\{f_2(x)\}_{xxt}}{s^{3\alpha+1}} + \frac{\{f_1(x)\}_x}{s^{2\alpha+1}} + \frac{\{f_2(x)\}_x}{s^{3\alpha+1}} + \left[sech^4\frac{x}{4}\frac{1}{s^{\alpha+2}} + \frac{1}{2}\{f_1(x)\}^2\frac{1}{s^{3\alpha+2}} + \frac{1}{2}\{f_2(x)\}^2\frac{1}{s^{5\alpha+2}} + sech^2\frac{x}{4}f_1(x)\frac{1}{s^{2\alpha+2}} + f_1(x)f_2(x)\frac{1}{s^{4\alpha+2}} + sech^2\frac{x}{4}f_2(x)\frac{1}{s^{3\alpha+2}}\right]_x$$

Now, the relation $\lim_{s\to\infty}(s^{2\alpha+1}\,\mathcal{L}Res_2(x,s)) = 0\ for\ k = 2$, gives us that

$$f_2(x) + \{f_1(x)\}_x = 0$$

$$f_2(x) = -\{f_1(x)\}_x$$

$$f_2(x) = -\frac{1}{8}sech^4\frac{x}{4} + \frac{1}{4}sech^2\frac{x}{4}tanh^2\frac{x}{4}$$

For $k = 3$ from (16) the third Laplace residual function is,

$$\mathcal{L}Res_3(x,s) = \sum_{n=1}^{3}\frac{f_n(x)}{s^{n\alpha+1}} - \frac{1}{s^\alpha}\left[\left\{\sum_{n=1}^{3}\frac{f_n(x)}{s^{n\alpha+1}}\right\}_{xxt} + \frac{1}{2s}sech^2\frac{x}{4}tanh\frac{x}{4} - \left\{\sum_{n=1}^{3}\frac{f_n(x)}{s^{n\alpha+1}}\right\}_x - \frac{1}{2}\left[\mathcal{L}\left\{\mathcal{L}^{-1}\left(\frac{sech^2\frac{x}{4}}{s} + \sum_{n=1}^{3}\frac{f_n(x)}{s^{n\alpha+1}}\right)^2\right\}\right]_x\right] \quad (19)$$

$$= \frac{f_1(x)}{s^{\alpha+1}} + \frac{f_2(x)}{s^{2\alpha+1}} + \frac{f_3(x)}{s^{3\alpha+1}} - \frac{1}{s^\alpha}\left[\left\{\frac{f_1(x)}{s^{\alpha+1}} + \frac{f_2(x)}{s^{2\alpha+1}} + \frac{f_3(x)}{s^{3\alpha+1}}\right\}_{xxt} + \frac{1}{2s} sech^2\frac{x}{4} tanh\frac{x}{4} - \left\{\frac{f_1(x)}{s^{\alpha+1}} + \frac{f_2(x)}{s^{2\alpha+1}} + \frac{f_3(x)}{s^{3\alpha+1}}\right\}_x - \frac{1}{2}\left[\mathcal{L}\left\{\mathcal{L}^{-1}\left(\frac{sech^2(\frac{x}{4})}{s} + \frac{f_1(x)}{s^{\alpha+1}} + \frac{f_2(x)}{s^{2\alpha+1}} + \frac{f_3(x)}{s^{3\alpha+1}}\right)\right\}^2\right]_x\right]$$

$$= \frac{f_1(x)}{s^{\alpha+1}} + \frac{f_2(x)}{s^{2\alpha+1}} + \frac{f_3(x)}{s^{3\alpha+1}} - \frac{1}{s^\alpha}\left[\left\{\frac{f_1(x)}{s^{\alpha+1}}\right\}_{xxt} + \left\{\frac{f_2(x)}{s^{2\alpha+1}}\right\}_{xxt} + \; + \left\{\frac{f_3(x)}{s^{3\alpha+1}}\right\}_{xxt} + \frac{1}{2s} sech^2\frac{x}{4} tanh\frac{x}{4} - \left\{\frac{f_1(x)}{s^{\alpha+1}}\right\}_x - \left\{\frac{f_2(x)}{s^{2\alpha+1}}\right\}_x - \left\{\frac{f_3(x)}{s^{3\alpha+1}}\right\}_x - \frac{1}{2}\left[\mathcal{L}\left\{\mathcal{L}^{-1}\left(\frac{sech^2(\frac{x}{4})}{s} + \frac{f_1(x)}{s^{\alpha+1}} + \frac{f_2(x)}{s^{2\alpha+1}} + \frac{f_3(x)}{s^{3\alpha+1}}\right)\right\}^2\right]_x\right]$$

$$= \frac{sech^2\frac{x}{4}tanh\frac{x}{4}}{2s^{\alpha+1}} + \frac{f_2(x)}{s^{2\alpha+1}} + \frac{f_3(x)}{s^{3\alpha+1}} - \frac{(f_1(x))_{xxt}}{s^{2\alpha+1}} - \frac{\{f_2(x)\}_{xxt}}{s^{3\alpha+1}} - \frac{\{f_3(x)\}_{xxt}}{s^{4\alpha+1}} - \frac{sech^2\frac{x}{4}tanh\frac{x}{4}}{2s^{\alpha+1}} + \frac{\{f_1(x)\}_x}{s^{2\alpha+1}} + \frac{\{f_2(x)\}_x}{s^{3\alpha+1}} + \frac{\{f_3(x)\}_x}{s^{4\alpha+1}} + \frac{1}{2s^\alpha}\left[\mathcal{L}\{\mathcal{L}^{-1}\left(\frac{sech^4\frac{x}{4}}{s^2} + \frac{\{f_1(x)\}^2}{s^{2\alpha+2}} + \frac{\{f_2(x)\}^2}{s^{4\alpha+2}} + \frac{\{f_3(x)\}^2}{s^{6\alpha+2}} + \frac{2sech^2\frac{x}{4}f_1(x)}{s^{\alpha+2}} + \frac{2f_1(x)f_2(x)}{s^{3\alpha+2}} + \frac{2sech^2\frac{x}{4}f_2(x)}{s^{2\alpha+2}} + \frac{2sech^2\frac{x}{4}f_3(x)}{s^{3\alpha+2}} + \frac{2f_1(x)f_3(x)}{s^{4\alpha+2}} + \frac{2f_2(x)f_3(x)}{s^{5\alpha+2}}\right)\}\right]_x$$

$$= \frac{f_2(x)}{s^{2\alpha+1}} + \frac{f_3(x)}{s^{3\alpha+1}} - \frac{(f_1(x))_{xxt}}{s^{2\alpha+1}} - \frac{\{f_2(x)\}_{xxt}}{s^{3\alpha+1}} - \frac{\{f_3(x)\}_{xxt}}{s^{4\alpha+1}} + \frac{\{f_1(x)\}_x}{s^{2\alpha+1}} + \frac{\{f_2(x)\}_x}{s^{3\alpha+1}} + \frac{\{f_3(x)\}_x}{s^{4\alpha+1}} + \frac{1}{2s^\alpha}\Big[\mathcal{L}\{sech^4\frac{x}{4}\frac{t}{1!} + \{f_1(x)\}^2\frac{t^{2\alpha+1}}{(2\alpha+1)!} + \{f_2(x)\}^2\frac{t^{4\alpha+1}}{(4\alpha+1)!} + \{f_3(x)\}^2\frac{t^{6\alpha+1}}{(6\alpha+1)!} + 2sech^2\frac{x}{4}f_1(x)\frac{t^{\alpha+1}}{(\alpha+1)!} + 2f_1(x)f_2(x)\frac{t^{3\alpha+1}}{(3\alpha+1)!} + 2sech^2\frac{x}{4}f_2(x)\frac{t^{2\alpha+1}}{(2\alpha+1)!} + 2sech^2\frac{x}{4}f_3(x)\frac{t^{3\alpha+1}}{(3\alpha+1)!} + 2f_1(x)f_3(x)\frac{t^{4\alpha+1}}{(4\alpha+1)!} + 2f_2(x)f_3(x)\frac{t^{5\alpha+1}}{(5\alpha+1)!}\Big]_x$$

$$= \frac{f_2(x)}{s^{2\alpha+1}} + \frac{f_3(x)}{s^{3\alpha+1}} - \frac{(f_1(x))_{xxt}}{s^{2\alpha+1}} - \frac{\{f_2(x)\}_{xxt}}{s^{3\alpha+1}} - \frac{\{f_3(x)\}_{xxt}}{s^{4\alpha+1}} + \frac{\{f_1(x)\}_x}{s^{2\alpha+1}} + \frac{\{f_2(x)\}_x}{s^{3\alpha+1}} + \frac{\{f_3(x)\}_x}{s^{4\alpha+1}} + \frac{1}{2s^\alpha}\Big[sech^4\frac{x}{4}\frac{1}{s^2} + \{f_1(x)\}^2\frac{1}{(2\alpha+1)!}\frac{(2\alpha+1)!}{s^{2\alpha+2}} + \{f_2(x)\}^2\frac{1}{(4\alpha+1)!}\frac{(4\alpha+1)!}{s^{4\alpha+2}} + \{f_3(x)\}^2\frac{1}{(6\alpha+1)!}\frac{(6\alpha+1)!}{s^{6\alpha+2}} + 2sech^2\frac{x}{4}f_1(x)\frac{1}{(\alpha+1)!}\frac{(\alpha+1)!}{s^{\alpha+2}} + 2f_1(x)f_2(x)\frac{1}{(3\alpha+1)!}\frac{(3\alpha+1)!}{s^{3\alpha+2}} + 2sech^2\frac{x}{4}f_2(x)\frac{1}{(2\alpha+1)!}\frac{(2\alpha+1)!}{s^{2\alpha+2}} + 2sech^2\frac{x}{4}f_3(x)\frac{1}{(3\alpha+1)!}\frac{(3\alpha+1)!}{s^{3\alpha+2}} + 2f_1(x)f_3(x)\frac{1}{(4\alpha+1)!}\frac{(4\alpha+1)!}{s^{4\alpha+2}} + 2f_2(x)f_3(x)\frac{1}{(5\alpha+1)!}\frac{(5\alpha+1)!}{s^{5\alpha+2}}\Big]_x$$

$$= \frac{-\frac{1}{8}sech^4\frac{x}{4}+\frac{1}{4}sech^2\frac{x}{4}tanh^2\frac{x}{4}}{s^{2\alpha+1}} + \frac{f_3(x)}{s^{3\alpha+1}} - \frac{(f_1(x))_{xxt}}{s^{2\alpha+1}} - \frac{\{f_2(x)\}_{xxt}}{s^{3\alpha+1}} - \frac{\{f_3(x)\}_{xxt}}{s^{4\alpha+1}} + \frac{\frac{1}{8}sech^4\frac{x}{4}-\frac{1}{4}sech^2\frac{x}{4}tanh^2\frac{x}{4}}{s^{2\alpha+1}} + \frac{\{f_2(x)\}_x}{s^{3\alpha+1}} + \frac{\{f_3(x)\}_x}{s^{4\alpha+1}} + \Big[sech^4\frac{x}{4}\frac{1}{s^{\alpha+2}} + \frac{1}{2}\{f_1(x)\}^2\frac{1}{s^{3\alpha+2}} + \frac{1}{2}\{f_2(x)\}^2\frac{1}{s^{5\alpha+2}} + \frac{1}{2}\{f_3(x)\}^2\frac{1}{s^{7\alpha+2}} + sech^2\frac{x}{4}f_1(x)\frac{1}{s^{2\alpha+2}} + f_1(x)f_2(x)\frac{1}{s^{4\alpha+2}} + sech^2\frac{x}{4}f_2(x)\frac{1}{s^{3\alpha+2}} + sech^2\frac{x}{4}f_3(x)\frac{1}{s^{4\alpha+2}} + f_1(x)f_3(x)\frac{1}{s^{5\alpha+2}} + f_2(x)f_3(x)\frac{1}{s^{6\alpha+2}}\Big]_x$$

$$= \frac{f_3(x)}{s^{3\alpha+1}} - \frac{(f_1(x))_{xxt}}{s^{2\alpha+1}} - \frac{\{f_2(x)\}_{xxt}}{s^{3\alpha+1}} - \frac{\{f_3(x)\}_{xxt}}{s^{4\alpha+1}} + \frac{\{f_2(x)\}_x}{s^{3\alpha+1}} + \frac{\{f_3(x)\}_x}{s^{4\alpha+1}} + \Big[sech^4\frac{x}{4}\frac{1}{s^{\alpha+2}} + \frac{1}{2}\{f_1(x)\}^2\frac{1}{s^{3\alpha+2}} + \frac{1}{2}\{f_2(x)\}^2\frac{1}{s^{5\alpha+2}} + \frac{1}{2}\{f_3(x)\}^2\frac{1}{s^{7\alpha+2}} + sech^2\frac{x}{4}f_1(x)\frac{1}{s^{2\alpha+2}} + f_1(x)f_2(x)\frac{1}{s^{4\alpha+2}} + sech^2\frac{x}{4}f_2(x)\frac{1}{s^{3\alpha+2}} + sech^2\frac{x}{4}f_3(x)\frac{1}{s^{4\alpha+2}} + f_1(x)f_3(x)\frac{1}{s^{5\alpha+2}} + f_2(x)f_3(x)\frac{1}{s^{6\alpha+2}}\Big]_x$$

Now, the relation $\lim_{s\to\infty}(s^{3\alpha+1}\,\mathcal{L}Res_3(x,s)) = 0 \; for \; k = 3$, gives us that

$$f_3(x) + \{f_2(x)\}_x = 0$$

i.e. $f_3(x) = -\{f_2(x)\}_x$

i.e. $f_3(x) = -\frac{1}{12}sech^4\frac{x}{4}tanh\frac{x}{4} + \frac{1}{4}sech^2\frac{x}{4}tanh^3\frac{x}{4}$

For $k = 4$ from (16) the fourth Laplace residual function is,

$$\mathcal{L}Res_4(x,s) = \sum_{n=1}^{4}\frac{f_n(x)}{s^{n\alpha+1}} - \frac{1}{s^\alpha}[\left\{\sum_{n=1}^{4}\frac{f_n(x)}{s^{n\alpha+1}}\right\}_{xxt} + \frac{1}{2s}sech^2\frac{x}{4}tanh\frac{x}{4} - \left\{\sum_{n=1}^{4}\frac{f_n(x)}{s^{n\alpha+1}}\right\}_x - \frac{1}{2}\left[\mathcal{L}\left\{\mathcal{L}^{-1}\left(\frac{sech^2\frac{x}{4}}{s} + \sum_{n=1}^{4}\frac{f_n(x)}{s^{n\alpha+1}}\right)^2\right\}\right]_x] \tag{20}$$

$$= \frac{f_1(x)}{s^{\alpha+1}} + \frac{f_2(x)}{s^{2\alpha+1}} + \frac{f_3(x)}{s^{3\alpha+1}} + \frac{f_4(x)}{s^{4\alpha+1}} - \frac{1}{s^\alpha}[\left\{\frac{f_1(x)}{s^{\alpha+1}} + \frac{f_2(x)}{s^{2\alpha+1}} + \frac{f_3(x)}{s^{3\alpha+1}} + \frac{f_4(x)}{s^{4\alpha+1}}\right\}_{xxt} + \frac{1}{2s}sech^2\frac{x}{4}tanh\frac{x}{4} - \left\{\frac{f_1(x)}{s^{\alpha+1}} + \frac{f_2(x)}{s^{2\alpha+1}} + \frac{f_3(x)}{s^{3\alpha+1}} + \frac{f_4(x)}{s^{4\alpha+1}}\right\}_x - \frac{1}{2}[\mathcal{L}\left\{\mathcal{L}^{-1}(\frac{sech^2(\frac{x}{4})}{s} + \frac{f_1(x)}{s^{\alpha+1}} + \frac{f_2(x)}{s^{2\alpha+1}} + \frac{f_3(x)}{s^{3\alpha+1}} + \frac{f_4(x)}{s^{4\alpha+1}})\right\}^2]_x]$$

$$= \frac{f_1(x)}{s^{\alpha+1}} + \frac{f_2(x)}{s^{2\alpha+1}} + \frac{f_3(x)}{s^{3\alpha+1}} + \frac{f_4(x)}{s^{4\alpha+1}} - \frac{1}{s^\alpha}[\left\{\frac{f_1(x)}{s^{\alpha+1}}\right\}_{xxt} + \left\{\frac{f_2(x)}{s^{2\alpha+1}}\right\}_{xxt} + \left\{\frac{f_3(x)}{s^{3\alpha+1}}\right\}_{xxt} + \left\{\frac{f_4(x)}{s^{4\alpha+1}}\right\}_{xxt} + \frac{1}{2s}sech^2\frac{x}{4}tanh\frac{x}{4} - \left\{\frac{f_1(x)}{s^{\alpha+1}}\right\}_x - \left\{\frac{f_2(x)}{s^{2\alpha+1}}\right\}_x - \left\{\frac{f_3(x)}{s^{3\alpha+1}}\right\}_x - \left\{\frac{f_4(x)}{s^{4\alpha+1}}\right\}_x - \frac{1}{2}[\mathcal{L}\left\{\mathcal{L}^{-1}(\frac{sech^2(\frac{x}{4})}{s} + \frac{f_1(x)}{s^{\alpha+1}} + \frac{f_2(x)}{s^{2\alpha+1}} + \frac{f_3(x)}{s^{3\alpha+1}} + \frac{f_4(x)}{s^{4\alpha+1}})\right\}^2]_x]$$

$$= \frac{sech^2\frac{x}{4}tanh\frac{x}{4}}{2s^{\alpha+1}} + \frac{f_2(x)}{s^{2\alpha+1}} + \frac{f_3(x)}{s^{3\alpha+1}} + \frac{f_4(x)}{s^{4\alpha+1}} - \frac{(f_1(x))_{xxt}}{s^{2\alpha+1}} - \frac{\{f_2(x)\}_{xxt}}{s^{3\alpha+1}} - \frac{\{f_3(x)\}_{xxt}}{s^{4\alpha+1}} - \frac{\{f_4(x)\}_{xxt}}{s^{5\alpha+1}} - \frac{sech^2\frac{x}{4}tanh\frac{x}{4}}{2s^{\alpha+1}} + \frac{\{f_1(x)\}_x}{s^{2\alpha+1}} + \frac{\{f_2(x)\}_x}{s^{3\alpha+1}} + \frac{\{f_3(x)\}_x}{s^{4\alpha+1}} + \frac{\{f_4(x)\}_x}{s^{5\alpha+1}} + \frac{1}{2s^\alpha}[\mathcal{L}\{\mathcal{L}^{-1}\left(\frac{sech^4\frac{x}{4}}{s^2} + \frac{\{f_1(x)\}^2}{s^{2\alpha+2}} + \frac{\{f_2(x)\}^2}{s^{4\alpha+2}} + \frac{\{f_3(x)\}^2}{s^{6\alpha+2}} + \frac{\{f_4(x)\}^2}{s^{8\alpha+2}} + \frac{2sech^2\frac{x}{4}f_1(x)}{s^{\alpha+2}} + \frac{2f_1(x)f_2(x)}{s^{3\alpha+2}} + \frac{2sech^2\frac{x}{4}f_2(x)}{s^{2\alpha+2}} + \frac{2sech^2\frac{x}{4}f_3(x)}{s^{3\alpha+2}} + \frac{2f_1(x)f_3(x)}{s^{4\alpha+2}} + \frac{2f_2(x)f_3(x)}{s^{5\alpha+2}} + \frac{2sech^2\frac{x}{4}f_4(x)}{s^{4\alpha+2}} + \frac{2f_1(x)f_4(x)}{s^{5\alpha+2}} + \frac{2f_2(x)f_4(x)}{s^{6\alpha+2}} + \frac{2f_3(x)f_4(x)}{s^{7\alpha+2}}\right)\}]_x$$

$$= \frac{f_2(x)}{s^{2\alpha+1}} + \frac{f_3(x)}{s^{3\alpha+1}} + \frac{f_4(x)}{s^{4\alpha+1}} - \frac{(f_1(x))_{xxt}}{s^{2\alpha+1}} - \frac{\{f_2(x)\}_{xxt}}{s^{3\alpha+1}} - \frac{\{f_3(x)\}_{xxt}}{s^{4\alpha+1}} - \frac{\{f_4(x)\}_{xxt}}{s^{5\alpha+1}} + \frac{\{f_1(x)\}_x}{s^{2\alpha+1}} + \frac{\{f_2(x)\}_x}{s^{3\alpha+1}} + \frac{\{f_3(x)\}_x}{s^{4\alpha+1}} + \frac{\{f_4(x)\}_x}{s^{5\alpha+1}} + \frac{1}{2s^\alpha}[\mathcal{L}\{sech^4\frac{x}{4}\frac{t}{1!} + \{f_1(x)\}^2\frac{t^{2\alpha+1}}{(2\alpha+1)!} + \{f_2(x)\}^2\frac{t^{4\alpha+1}}{(4\alpha+1)!} + \{f_3(x)\}^2\frac{t^{6\alpha+1}}{(6\alpha+1)!} + \{f_4(x)\}^2\frac{t^{8\alpha+1}}{(8\alpha+1)!} + 2sech^2\frac{x}{4}f_1(x)\frac{t^{\alpha+1}}{(\alpha+1)!} + 2f_1(x)f_2(x)\frac{t^{3\alpha+1}}{(3\alpha+1)!} + 2sech^2\frac{x}{4}f_2(x)\frac{t^{2\alpha+1}}{(2\alpha+1)!} + 2sech^2\frac{x}{4}f_3(x)\frac{t^{3\alpha+1}}{(3\alpha+1)!} + 2f_1(x)f_3(x)\frac{t^{4\alpha+1}}{(4\alpha+1)!} + 2f_2(x)f_3(x)\frac{t^{5\alpha+1}}{(5\alpha+1)!} + 2sech^2\frac{x}{4}f_4(x)\frac{t^{4\alpha+1}}{(4\alpha+1)!} + 2f_1(x)f_4(x)\frac{t^{5\alpha+1}}{(5\alpha+1)!} + 2f_2(x)f_4(x)\frac{t^{6\alpha+1}}{(6\alpha+1)!} + 2f_3(x)f_4(x)\frac{t^{7\alpha+1}}{(7\alpha+1)!}]_x$$

$$= \frac{f_2(x)}{s^{2\alpha+1}} + \frac{f_3(x)}{s^{3\alpha+1}} + \frac{f_4(x)}{s^{4\alpha+1}} - \frac{(f_1(x))_{xxt}}{s^{2\alpha+1}} - \frac{\{f_2(x)\}_{xxt}}{s^{3\alpha+1}} - \frac{\{f_3(x)\}_{xxt}}{s^{4\alpha+1}} - \frac{\{f_4(x)\}_{xxt}}{s^{5\alpha+1}} + \frac{\{f_1(x)\}_x}{s^{2\alpha+1}} + \frac{\{f_2(x)\}_x}{s^{3\alpha+1}} + \frac{\{f_3(x)\}_x}{s^{4\alpha+1}} + \frac{\{f_4(x)\}_x}{s^{5\alpha+1}} + \frac{1}{2s^\alpha}[sech^4\frac{x}{4}\frac{1}{s^2} + \{f_1(x)\}^2\frac{1}{(2\alpha+1)!}\frac{(2\alpha+1)!}{s^{2\alpha+2}} + \{f_2(x)\}^2\frac{1}{(4\alpha+1)!}\frac{(4\alpha+1)!}{s^{4\alpha+2}} + \{f_3(x)\}^2\frac{1}{(6\alpha+1)!}\frac{(6\alpha+1)!}{s^{6\alpha+2}} + \{f_4(x)\}^2\frac{1}{(8\alpha+1)!}\frac{(8\alpha+1)!}{s^{8\alpha+2}} + 2sech^2\frac{x}{4}f_1(x)\frac{1}{(\alpha+1)!}\frac{(\alpha+1)!}{s^{\alpha+2}} + 2f_1(x)f_2(x)\frac{1}{(3\alpha+1)!}\frac{(3\alpha+1)!}{s^{3\alpha+2}} + 2sech^2\frac{x}{4}f_2(x)\frac{1}{(2\alpha+1)!}\frac{(2\alpha+1)!}{s^{2\alpha+2}} + 2sech^2\frac{x}{4}f_3(x)\frac{1}{(3\alpha+1)!}\frac{(3\alpha+1)!}{s^{3\alpha+2}} +$$

$2sech^2\frac{x}{4}f_4(x)\frac{1}{(4\alpha+1)!}\frac{(4\alpha+1)!}{s^{4\alpha+2}}+2f_1(x)f_3(x)\frac{1}{(4\alpha+1)!}\frac{(4\alpha+1)!}{s^{4\alpha+2}}+2f_2(x)f_3(x)\frac{1}{(5\alpha+1)!}\frac{(5\alpha+1)!}{s^{5\alpha+2}}+$
$2f_1(x)f_4(x)\frac{1}{(5\alpha+1)!}\frac{(5\alpha+1)!}{s^{5\alpha+2}}+2f_2(x)f_4(x)\frac{1}{(6\alpha+1)!}\frac{(6\alpha+1)!}{s^{6\alpha+2}}+2f_3(x)f_4(x)\frac{1}{(7\alpha+1)!}\frac{(7\alpha+1)!}{s^{7\alpha+2}}]_x$

$=\frac{-\frac{1}{8}sech^4\frac{x}{4}+\frac{1}{4}sech^2\frac{x}{4}tanh^2\frac{x}{4}}{s^{2\alpha+1}}+\frac{f_3(x)}{s^{3\alpha+1}}+\frac{f_4(x)}{s^{4\alpha+1}}-\frac{(f_1(x))_{xxt}}{s^{2\alpha+1}}-\frac{\{f_2(x)\}_{xxt}}{s^{3\alpha+1}}-\frac{\{f_3(x)\}_{xxt}}{s^{4\alpha+1}}-\frac{\{f_4(x)\}_{xxt}}{s^{5\alpha+1}}+$
$\frac{\frac{1}{8}sech^4\frac{x}{4}-\frac{1}{4}sech^2\frac{x}{4}tanh^2\frac{x}{4}}{s^{2\alpha+1}}+\frac{\{f_2(x)\}_x}{s^{3\alpha+1}}+\frac{\{f_3(x)\}_x}{s^{4\alpha+1}}+\frac{\{f_4(x)\}_x}{s^{5\alpha+1}}+[sech^4\frac{x}{4}\frac{1}{s^{\alpha+2}}+\frac{1}{2}\{f_1(x)\}^2\frac{1}{s^{3\alpha+2}}+$
$\frac{1}{2}\{f_2(x)\}^2\frac{1}{s^{5\alpha+2}}+\frac{1}{2}\{f_3(x)\}^2\frac{1}{s^{7\alpha+2}}+\frac{1}{2}\{f_4(x)\}^2\frac{1}{s^{9\alpha+2}}+sech^2\frac{x}{4}f_1(x)\frac{1}{s^{2\alpha+2}}+f_1(x)f_2(x)\frac{1}{s^{4\alpha+2}}+$
$sech^2\frac{x}{4}f_2(x)\frac{1}{s^{3\alpha+2}}+sech^2\frac{x}{4}f_3(x)\frac{1}{s^{4\alpha+2}}+sech^2\frac{x}{4}f_4(x)\frac{1}{s^{5\alpha+2}}+f_1(x)f_3(x)\frac{1}{s^{5\alpha+2}}+$
$f_2(x)f_3(x)\frac{1}{s^{6\alpha+2}}+f_1(x)f_4(x)\frac{1}{s^{6\alpha+2}}+f_2(x)f_4(x)\frac{1}{s^{7\alpha+2}}+f_3(x)f_4(x)\frac{1}{s^{8\alpha+2}}]_x$

$=-\frac{\frac{1}{4}sech^4\frac{x}{4}tanh\frac{x}{4}-\frac{1}{8}sech^2\frac{x}{4}tanh^3\frac{x}{4}}{s^{3\alpha+1}}+\frac{f_4(x)}{s^{4\alpha+1}}-\frac{(f_1(x))_{xxt}}{s^{2\alpha+1}}-\frac{\{f_2(x)\}_{xxt}}{s^{3\alpha+1}}-\frac{\{f_3(x)\}_{xxt}}{s^{4\alpha+1}}-\frac{\{f_4(x)\}_{xxt}}{s^{5\alpha+1}}+$
$\frac{\frac{1}{4}sech^4\frac{x}{4}tanh\frac{x}{4}-\frac{1}{8}sech^2\frac{x}{4}tanh^3\frac{x}{4}}{s^{3\alpha+1}}+\frac{\{f_3(x)\}_x}{s^{4\alpha+1}}+\frac{\{f_4(x)\}_x}{s^{5\alpha+1}}+[sech^4\frac{x}{4}\frac{1}{s^{\alpha+2}}+\frac{1}{2}\{f_1(x)\}^2\frac{1}{s^{3\alpha+2}}+$
$\frac{1}{2}\{f_2(x)\}^2\frac{1}{s^{5\alpha+2}}+\frac{1}{2}\{f_3(x)\}^2\frac{1}{s^{7\alpha+2}}+\frac{1}{2}\{f_4(x)\}^2\frac{1}{s^{9\alpha+2}}+sech^2\frac{x}{4}f_1(x)\frac{1}{s^{2\alpha+2}}+f_1(x)f_2(x)\frac{1}{s^{4\alpha+2}}+$
$sech^2\frac{x}{4}f_2(x)\frac{1}{s^{3\alpha+2}}+sech^2\frac{x}{4}f_3(x)\frac{1}{s^{4\alpha+2}}+sech^2\frac{x}{4}f_4(x)\frac{1}{s^{5\alpha+2}}+f_1(x)f_3(x)\frac{1}{s^{5\alpha+2}}+$
$f_2(x)f_3(x)\frac{1}{s^{6\alpha+2}}+f_1(x)f_4(x)\frac{1}{s^{6\alpha+2}}+f_2(x)f_4(x)\frac{1}{s^{7\alpha+2}}+f_3(x)f_4(x)\frac{1}{s^{8\alpha+2}}]_x$

$=\frac{f_4(x)}{s^{4\alpha+1}}-\frac{(f_1(x))_{xxt}}{s^{2\alpha+1}}-\frac{\{f_2(x)\}_{xxt}}{s^{3\alpha+1}}-\frac{\{f_3(x)\}_{xxt}}{s^{4\alpha+1}}-\frac{\{f_4(x)\}_{xxt}}{s^{5\alpha+1}}+\frac{\{f_3(x)\}_x}{s^{4\alpha+1}}+\frac{\{f_4(x)\}_x}{s^{5\alpha+1}}+[sech^4\frac{x}{4}\frac{1}{s^{\alpha+2}}+$
$\frac{1}{2}\{f_1(x)\}^2\frac{1}{s^{3\alpha+2}}+\frac{1}{2}\{f_2(x)\}^2\frac{1}{s^{5\alpha+2}}+\frac{1}{2}\{f_3(x)\}^2\frac{1}{s^{7\alpha+2}}+sech^2\frac{x}{4}f_1(x)\frac{1}{s^{2\alpha+2}}+f_1(x)f_2(x)\frac{1}{s^{4\alpha+2}}+$
$sech^2\frac{x}{4}f_2(x)\frac{1}{s^{3\alpha+2}}+sech^2\frac{x}{4}f_3(x)\frac{1}{s^{4\alpha+2}}+sech^2\frac{x}{4}f_4(x)\frac{1}{s^{5\alpha+2}}+f_1(x)f_3(x)\frac{1}{s^{5\alpha+2}}+$
$f_2(x)f_3(x)\frac{1}{s^{6\alpha+2}}+f_1(x)f_4(x)\frac{1}{s^{6\alpha+2}}+f_2(x)f_4(x)\frac{1}{s^{7\alpha+2}}+f_3(x)f_4(x)\frac{1}{s^{8\alpha+2}}]_x$

Now, the relation $\lim_{s\to\infty}(s^{4\alpha+1}\,\mathcal{L}Res_4(x,s))=0\ for\ k=4$, gives us that

$f_4(x)+\{f_3(x)\}_x=0$

i.e. $f_4(x)=-\{f_3(x)\}_x$

i.e. $f_4(x)=\frac{1}{48}sech^6\frac{x}{4}-\frac{13}{48}sech^4\frac{x}{4}tanh^2\frac{x}{4}+\frac{1}{8}sech^2\frac{x}{4}tanh^4\frac{x}{4}$

Hence by Laplace residual power series solution of given equation in infinite form is,

$$U(x,s)=sech^2\frac{x}{4}\frac{1}{s}+\frac{1}{2}sech^2\frac{x}{4}tanh\frac{x}{4}\frac{1}{s^{\alpha+1}}+\left(\frac{1}{4}sech^2\frac{x}{4}tanh^2\frac{x}{4}-\frac{1}{8}sech^4\frac{x}{4}\right)\frac{1}{s^{2\alpha+1}}+\left(-\frac{1}{12}sech^4\frac{x}{4}tanh\frac{x}{4}+\frac{1}{4}sech^2\frac{x}{4}tanh^3\frac{x}{4}\right)\frac{1}{s^{3\alpha+1}}+(\frac{1}{48}sech^6\frac{x}{4}-\frac{13}{48}sech^4\frac{x}{4}tanh^2\frac{x}{4}+\frac{1}{8}sech^2\frac{x}{4}tanh^4\frac{x}{4})\frac{1}{s^{4\alpha+1}}\ldots \quad (21)$$

At last by taking inverse Laplace transform in (21) then the required solution of given equation by using LRPSM as,

$$u(x,t) = sech^2\frac{x}{4} + \frac{1}{2}sech^2\frac{x}{4}tanh\frac{x}{4}\frac{t^{\alpha}}{\alpha!} + \left\{\frac{1}{4}sech^2\frac{x}{4}tanh^2\frac{x}{4} - \frac{1}{8}sech^4\frac{x}{4}\right\}\frac{t^{2\alpha}}{(2\alpha)!} + \left(-\frac{1}{12}sech^4\frac{x}{4}tanh\frac{x}{4} + \frac{1}{4}sech^2\frac{x}{4}tanh^3\frac{x}{4}\right)\frac{t^{3\alpha}}{(3\alpha)!} + \left(\frac{1}{48}sech^6\frac{x}{4} - \frac{13}{48}sech^4\frac{x}{4}tanh^2\frac{x}{4} + \frac{1}{8}sech^2\frac{x}{4}tanh^4\frac{x}{4}\right)\frac{t^{4\alpha}}{(4\alpha)!} \ldots (22)$$

Results and discussion

The approximate analytical solution of BBMB differential equation for different values of t is calculated and compared with exact solution for different values of t. The graph of both the solutions with different values of t=0.02, 0.04, 0.06, 0.08 and 0.10 and alpha=0.5 are shown in Figure 2, Figure 3, Figure 4, Figure 5 and Figure 6 when x varies from -20 to 20.

Figure 2

When t=0.02 and alpha=0.5

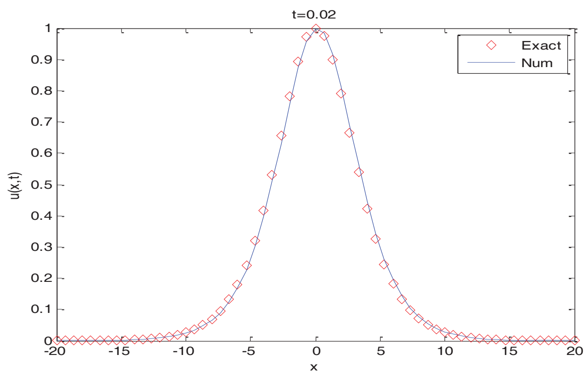


Figure 3

When t=0.04 and alpha=0.5

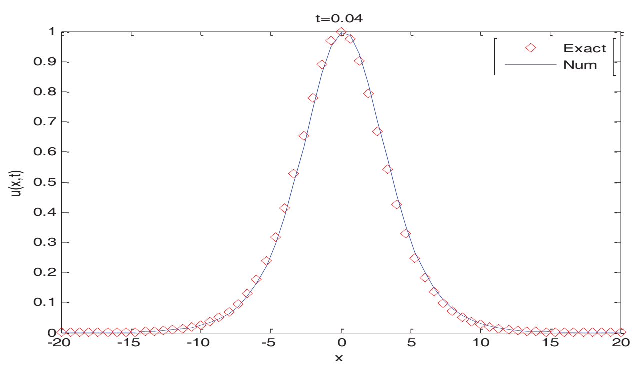


Figure 4

When t=0.06 and alpha=0.5

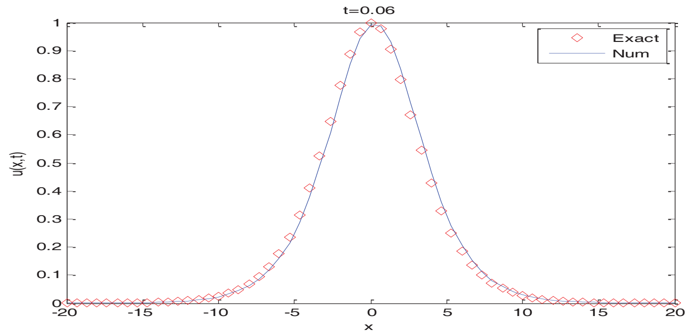


Figure 5

When t=0.08 and alpha=0.5

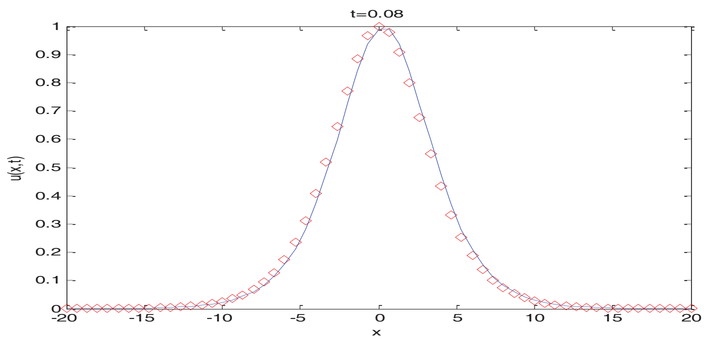


Figure 6

When t=0.10 and alpha=0.5

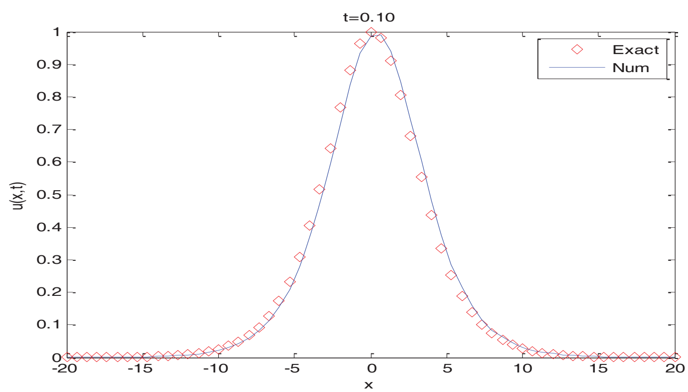


Table 1

Absolute errors of solutions of BBMB equation as prescribed points when alpha=0.5.

X	t=0.02	t=0.04	t=0.06	t=0.08	t=0.10
-20	1.1865e-05	1.5332e-05	1.7464e-05	1.8924e-05	1.9966e-05
-16	8.7574e-05	1.1317e-04	1.2891e-04	1.3969e-04	1.4739e-04
-12	6.4193e-04	8.2977e-04	9.4538e-04	1.0246e-03	1.0813e-03
-8	4.4721e-03	5.7918e-03	6.6090e-03	7.1731e-03	7.5798e-03
-4	2.1629e-02	2.8390e-02	3.2751e-02	3.5890e-02	3.8263e-02
4	2.3164e-02	3.1422e-02	3.7242e-02	4.1803e-02	4.5560e-02
8	5.1000e-03	7.0400e-03	8.4701e-03	9.6394e-03	1.0644e-02
12	7.3863e-04	1.0222e-03	1.2325e-03	1.4055e-03	1.5548e-03
16	1.0089e-04	1.3967e-04	1.6845e-04	1.9214e-04	2.1262e-04
20	1.3671e-05	1.8926e-05	2.2828e-05	2.6040e-05	2.8816e-05

Table 2

Absolute errors of solutions of BBMB equation at point t=0.02 for prescribed values of alpha.

x/alpha	0.25	0.50	0.75	1.0
-20	3.0789e-05	1.1865e-05	3.3528e-06	3.0187e-11
-16	2.2728e-04	8.7574e-05	2.4746e-05	2.2288e-10
-12	1.6676e-03	6.4193e-04	1.8133e-04	1.6375e-09
-8	1.1696e-02	4.4721e-03	1.2600e-03	1.1591e-08
-4	5.9167e-02	2.1629e-02	5.9830e-03	6.0683e-08
4	7.1211e-02	2.3164e-02	6.1328e-03	6.0859e-08
8	1.6985e-02	5.1000e-03	1.3209e-03	1.1646e-08
12	2.4895e-03	7.3863e-04	1.9070e-04	1.6457e-09
16	3.4060e-04	1.0089e-04	2.6036e-05	2.2400e-10
20	4.6162e-05	1.3671e-05	3.5278e-06	3.0338e-11

Conclusion

In this paper, a new and reliable scheme is constructed for the solution of one dimensional time-fractional BBMB differential equation by LRPSM. The advantage of this method is to decrease the computational work required for finding the solution in residual power series form after applying Laplace transform. The coefficients of power series solution form are determined after applying Laplace transform with RPSM in the above successive steps. The one dimensional time-fractional BBMB differential equation is solved by using the LRPSM which proved its ability to solve non-linear fractional differential equations with sufficient accuracy and reliable calculation steps. When number of terms increased in numerical solution then the solution becomes closer towards the exact solution of this equation. This verified that the solution of this equation by LRPSM is reliable as well as accurate.

In short, LRPSM is an effective, trustworthy and suitable method for finding analytical solution of BBMB equation with sufficient accuracy.

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