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Difficulties Experienced by Undergraduate Students in Proving Theorems of Real Analysis

Harsh Bahadur Chand

Department of Education, University Central Campus

Far West University, Mahendranagar, Nepal

Email: harsh_mpsteam21@kusoed.edu.np

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Abstract

Developing proofs is a very important task of undergraduate mathematics students. However, proving theorems in real analysis is a challenging task for many undergraduate students. In this context, this research was conducted to explore difficulties experienced by undergraduate students in proving theorems of real analysis. Narrative research design under the interpretive research paradigm was used for the study. Three students (higher achiever, average achiever, and below-average achiever) were selected purposively from the sixth semester of B.Ed. programme of the central campus of Far West University. Radical constructivism was selected as the theoretical basis of the study. Interview guideline was prepared as a research tool and interviews of participants were taken based on the guideline. For the analysis and interpretation, interview data were transcribed, coded and then themes were generated. Four themes were constructed that are inability to link statements logically, lack of skill of applying definitions/theorems, lack of skill of selecting an appropriate path of proving, and inability to grasp language/symbols and lengthy proof. The conclusion is that students can have the above types of difficulties in understanding/constructing proof of theorems but the type and extents of the difficulty experienced by learners depend upon the context and ability of learners. This study indicates that beliefs and practices guided by modernism is one of the reasons behind the difficulties in proof construction. The findings of this study would help teachers to select the appropriate pedagogical approach of teaching proofs and students to understand/construct proofs of Real Analysis.

Keywords: Narrative inquiry, proof construction, radical constructivism, mental ability, modernism

Introduction

I have been teaching real analysis at the undergraduate level since 2073. There is a 'Real Analysis I' at the fourth semester and 'Real Analysis II' at the fifth semester of the B.Ed.

programme. During facilitation of these courses, if I ask students to give examples of the concepts they usually restate examples given by me or written in the book. This may be the indication of rote memorization without understanding actual meaning. In addition, if I ask to explain the concepts, most of the time they respond with lack of confidence and they make mistakes in the use of quantifiers. In the answer sheets of mid-term examinations, the majority of students leave proof of the theorems but in the final examination, they write proofs in the answer sheets. One of the reasons behind this may be that students do not understand the content of analysis, proofs of theorems, examples, etc. but they memorize anyway for the final examination. Although some students secure good grades in the analysis course, the overall performance of students that I have been observing is not satisfactory in terms of understanding and constructing proofs.

Let me mention situation of some other Universities of Nepal regarding proof construction. In the year 2020, I got an opportunity to participate in a workshop on refresher courses of BSC. There were 37 teachers from Tribhuvan University (TU), Far west University (FWU), and Nepal Sanskrit University (NSU) as participants in that workshop. Real analysis was one of the main courses discussed in that workshop. We shared experiences about students' understanding of real analysis. Some common facts about proof construction that we shared were that students could not construct proofs independently; they face problems in understanding and memorizing/rewriting proofs, they could not know how to apply definitions to classify examples and non-examples of a concept, and they could not construct their examples of concepts. Thus, difficulty in proving theorems is a common problem of undergraduate level students of Nepal.

Real analysis is a very important subject for undergraduate students majoring in mathematics/mathematics education because it develops the ability of deductive reasoning and problem-solving (Bartle & Sherbert, 2005) and it forms a foundation for the other mathematical courses at the higher level. Whatever may be the reason, it can be observed that real analysis is included in the curriculum of undergraduate mathematics/mathematics education programmes of different Universities. Regarding Nepal, different universities Tribhuvan University (TU), Far West University (FWU), Kathmandu University (KU), and Mid West University (MWU) included real analysis in the curriculum of their Bachelor level Mathematics and Mathematics Education programmes. One of the common objectives that I found in those curricula is to develop an in-depth understanding and skill of developing proofs of theorems. The textbooks referred to in the curriculums of undergraduate real analysis course contains axioms, definitions, illustrative examples, theorem proving, and problem-solving (Apostal, 1997; Bartle & Sherbert, 2005; Gupta & Rani, 2003; Malik & Arora, 2010). The content included in the textbooks of real analysis indicates that understanding and constructing proofs of theorems are important activities of these courses. In particular, the major goal of the real analysis course is to develop the ability of proof construction.

However, past studies (Doruk & Kaplan, 2015; Ekayanti, 2019) and my long experience of teaching real analysis indicates that students face many cognitive difficulties in understanding and constructing proofs of theorems in analysis. Regarding this, Widiati and Sthephani (2018) stated that students felt difficulty in initializing proof and selecting proof techniques, they didn't have a good understanding of prerequisite knowledge and they didn't understand proofs properly which are given in the book; in particular indirect proof is difficult to understand. Thus, the difficulty is not limited to construction of proofs only but there is difficulty in understanding proofs in textbook or developed by teachers. The result of Doruk and Kaplan (2015) is not distinct from these studies, they found that, of all the participants in the study only seven percent proved the theorem correctly, many others wrote invalid proof and some of them did not reply. Similarly, Nadlifah and Prabawanto (2017) mentioned that undergraduate students felt difficulty in proving because of problems in applying definitions, using logic, understanding symbols, notations, and concepts used in the theorem. From the findings of these studies, it can be inferred that students face problems in understanding and constructing proofs.

Moreover, in his study, Moore (1994) mentioned seven sources of difficulties in proving theorems: inability to state definitions, the lack of understanding of concepts, inadequate concept images, unable to construct examples, unable to apply definitions, poor understanding of language, and lack idea of initializing proof. Thus, how to initialize proof, how to make an appropriate understanding of concepts, and how to deal with mathematical language properly are difficult tasks for students that lead to problems in proving theorems. Similarly, Selden and Selden (2007) mentioned that undergraduate students face problems in using theorems and interpreting proofs, they might ignore hypotheses and apply different forms of theorems incorrectly, they might also feel difficulty in logical structures having no quantifiers and they might have difficulty in forming concept images. Thus, students' difficulty in proof may be because of factors related to understanding, applying, and the language factor. However, these studies do not.

In the context of Nepal, I did not find any study about difficulties faced by students in proving theorems of real analysis. But, many teachers, including me, accepted that students are facing problems in proving theorems. In addition, many studies that I have reviewed were conducted outside Nepal mentioned that undergraduate students face many difficulties in understanding and constructing the proofs. However, there is no study, particularly in Nepal, that discusses the students' experiences and their inner feelings regarding theorems proving of real analysis. Thus, there is a need for study that can reflect difficulties in understanding and developing proofs. In this context, this study was conducted with the purpose of exploring difficulties experienced by undergraduate students in proving theorems of Real Analysis. The research question of the study is 'how do students narrate their difficulties experienced in proving theorems of real analysis?'

Literature Review

To develop insight on different concepts used in my research, and to define such concepts operationally I did a thematic review. In addition, to orient my research in a specific direction from the problem selection to interpretation of data, I conducted the theoretical review. Moreover, I did empirical review to inform me and the audience of this study that what has already been known and what needs to be known (research gap). These reviews are presented below under different headings.

Empirical Review: Difficulties in Theorem Proving

Guler (2016) researched to explore academicians' view towards difficulties of prospective mathematics teachers' in proving theorems. He used a qualitative research design. Fifteen participants were selected consisting of an equal number of professors, associate professors, and assistant professors of different universities of Turkey. He used semi-structured interviews as research tool, and content analysis method for the analysis of data. He reported that according to the academicians, students feel difficulty in proof initialization, establishing relationships, and selecting proof techniques because of a lack of prerequisite knowledge, focusing on memorizing rather than understanding and methods of proving. However, this study did not consider the direct experience of students.

Widiati and Sthephani (2018) aimed to explore the difficulties experienced by undergraduate students in studying Real Analysis. They used a qualitative research design for the study. They selected 55 students of 4th-semester students of Mathematics Education by purposive sampling method. The tools used by them were questionnaires, interviews, and observation. The main findings they obtained were that that students felt difficulty in initializing proof and selecting proof techniques, they didn't have a good understanding of prerequisite knowledge and they didn't understand proofs properly which are given in the book in the particular indirect proof is difficult to understand, they memorized theorems without understanding, they felt problem to explain proofs given in the book, they felt difficulty in expressing their ideas in a nonverbal way.

Thus, there are many studies that discuss the type of difficulties that students face during understanding and constructing proofs. The above literatures mentioned many problems that are common to all these studies such as problems in initializing proof, establishing relationships, understanding proofs, and lack of prerequisite knowledge. However, these studies and no any other study that I reviewed did not explore students' experiences through narrative design. Thus, there is a gap in the knowledge, which can be filled up by exploring students' narrations of their experiences of difficulties in proving theorems.

Conceptual Review: Proof and Proof Techniques

Let me discuss some important concepts included in this research. One of the important concepts is proof. Rosen (2011) argued that "a proof is a valid argument that establishes the truth of a mathematical statement" (p.73). He mentioned that direct proof, proof by contraposition, trivial proofs, proof by contradiction, and proof by counterexamples are some techniques of proving theorems, where theorems are mathematical statements that can be shown true (p.74). Regarding mathematical proof, Bell (1976) mentioned that proof represents schematization of axioms, basic concepts, and theorems through the deductive method (as cited in Guler, 2016). For me, the proof is a valid argument that establishes the truth of the given statement; proof construction is developing proof of the theorems/ statements to be proved; understanding proof means relational understanding (Skemp, 1987) of proof that is presented in the book or presented by teachers or constructed by themselves so that students can make sense of it and understand it meaningfully.

Regarding techniques of proof, Bartle and Sherbert (2005) mentioned direct proof involves the construction of statements $R_1, R_2, \ldots, R_n$ such that $P \Rightarrow R_1, R_1 \Rightarrow R_2, \ldots, R_n \Rightarrow Q$ to prove the statement of the form $P \Rightarrow Q$. They also argued that proof by contrapositive is the method, in which we prove that not Q implies not P to prove that P implies Q, and in the proof by contradiction, we prove (P and (not Q) implies a contradiction to prove P implies Q. (pp. 340-342). Although there are several methods of proof, the reason behind mentioning these techniques is that in undergraduate real analysis these methods are used.

Theoretical Framework: Radical Constructivism

To orient my research in different steps such as problem selection, developing tool, collecting data, and interpreting data I selected radical constructivism (Mohrhoff, 2008). To construct proof and to understand proofs I assumed that learners must engage themselves actively in the knowledge construction/meaning-making and for these activities the role of cognition must be adaptive. These processes are similar to the learning described in radical constructivism (Mohrhoff, 2008). Therefore, I had selected this theory to orient my research.

Bodner et al. (2001) mentioned that 'knowledge is constructed in the mind of the learner' is the fundamental assumption of the constructivist theory of knowledge. This means that the learner himself/herself must construct knowledge/understanding and knowledge cannot be transmitted from one's head to the head of the others. Including some additional assumptions to this fundamental assumption, many forms of constructivism are developed. Among them, I have chosen radical constructivism. Its two fundamental assumptions mentioned by Mohrhoff (2008) are as follows: "knowledge is not passively received but is actively built up by the cognizing subject; the function of cognition is adaptive and serves the subject's organization of her experiential world, not the discovery of an ontological reality" (p.18). The first principle states that the cognizing subject (learner) must be involved actively in the construction of his/her knowledge. Referring to the second principle Belbase (2011) argued that a child learns something through the cognitive function of self-adaption of new ideas to the existing experiential knowledge. Thus, the second principle asserts that the experiences of learners are very important and the cognition of the learner organizes these experiences through adaptation. Therefore, opportunities for rich experiences should be given to the learner for better understanding. Furthermore, he argued that understanding requires a process of construction and for this disequilibration is a key. This is the same as mentioned by Hardy and Taylor (1997), who explained the role of assimilation, accommodation, equilibration, and disequilibration in the construction of knowledge. The meaning of these terms for von Glasersfeld (1995) is similar to their meaning as described by Piaget.

To decide what type of questions should be asked during interview with students, a framework for interviews based upon the principles of radical constructivism was prepared. In addition, in the process of the interview also, questions were asked to participants guided by radical constructivism. Furthermore, radical constructivism was used in the interpretation of difficulties that the students experienced, Thus, I used this theory in the development of data collection tools, the process of data collection and to describe their metacognitive aspect of proving theorems and to describe cognitive difficulties in constructing/understanding proofs of the theorems.

Methods and Procedures

I explored the difficulties experienced by students in proving theorems by standing in their shoes and realizing their feelings (Taylor & Madina, 2011) and I tried to speak, understand and interpret their thinking and meanings that they constructed (Kivunja & Kuyini, 2017) therefore I selected interpretative paradigm. My ontology was relativist and subjectivist (Guba & Lincoln, 1994) and I assumed multiple realities. My epistemology was subjectivist because I interpreted participants' experiences based on my cognitive process. Similarly, my axiology was value-laden because there was an influence of my values and beliefs in interpreting students' experiences (Taylor & Madina, 2011).

I used narrative research design for the study because I wanted to study what difficulties students' had been experiencing in theorem proving. In addition, I wanted to explore the experience at a fixed time, fixed context, and fixed issues. Thus, I wanted to explore personal experiences through their stories, therefore I selected a qualitative narrative research design, which is best in this situation (Creswell, 2012).

Since real analysis is included in the fourth and fifth semesters of the undergraduate Mathematics Education program, I decided to select sixth-semester students who had experiences of proving in both semesters. Therefore, I selected three students of the sixth semester of FWU purposively to include a diversity of mental abilities.

I used an open interview, as a data collection tool, including open-ended questions such as 'please tell me, what difficulties you experienced during understanding/constructing proof of theorems in Real Analysis' and 'Do you have any painful unforgettable events/moments regarding memorizing/understanding/rewriting/constructing proof of the theorems? If any, please tell me. I had also mentioned some guiding points to support the interview process such as difficulties during classroom teaching, self-study and during examinations, and so on.

To collect data, first of all, I contacted three students that I decided to select as participants and explained the purpose of my study. I gave them the freedom to decide whether or not to take part in the interview. Because of an internet problem, one of the students replied that she couldn't take part in the interview. Then I contacted another student, explained her details of my research, and then she became ready to participate in the interview. After that, I took their interviews one by one using the zoom app. I recorded the meeting by taking permission from the participants. During the interview, I tried to elicit experiences by asking sub-questions, and I was conscious of the time, context, and social situation of the students' stories.

For analysis and interpretation, I transcribed the interview data and prepared vignettes of each student. During preparing the vignette I considered temporality, context, and social aspects. Then, I coded and categorized the transcription and on that basis, I prepared four themes by the inductive method. After that, I analyzed the themes by using a content analysis method. Finally, I interpreted and discussed data based on the theoretical framework of the study.

I maintained quality standards by making the research credible, dependable, confirmable, transformable, and authentic. For credibility I checked my interpretations with the informants; for dependability, I used open-ended and emergent inquiry and keep a detailed record of the process; for transferability, I kept the detailed description of the social context of me and participants; for confirmability, I minimized my biases in data analysis; for fairness, I selected high achiever, average achiever, and low achiever and by mentioning my assumptions I ensured ontological authenticity. For the ethical purpose, I considered informed consent, confidentiality, privacy, and proper use of information.

Results and Discussion

For the purpose of analysis and interpretation of data, I transcribed the data and then I created the vignette of each student separately. The pseudonyms are used for the participants for the ethical purpose. The pseudonyms are Avinash, Bina, and Cristal respectively for the higher achiever, average achiever, and below achiever. From the vignette, I determined codes, and then I formed categories. From such categories, I developed the following four themes: inability to link statements logically, lack of skill of applying definitions/theorems, lack of skill of selecting an appropriate path of proving, and inability to grasp language/symbols and lengthy proof.

Inability to Link Statements Logically

In developing proofs of theorems it is necessary to establish logical links between different statements. Such statements might be axioms, definitions, and/or previously proved theorems. Let me mention here how participants' narrated their experiences of difficulties in connecting different mathematical statements while they tried to understand/construct proof of theorems. The parts of stories of three students are mentioned below.

Cristal was trying to write a proof of a long theorem in the mid-term examination but he was unable to prove it. He could recall isolated facts/definitions/theorems but became confused on which to write first and which next to that and so on.

Bina was attempting to solve a long question in the final examination of the fourth semester. She couldn't write complete proof because she forgot a single step, in the middle of the proof.

Avinash was constructing a proof of an unfamiliar theorem. He could remember all the definitions and theorems that are used to prove the theorem. But he was unable to prove the theorem because he was unable to make connections between such definitions and theorems.

Although the situation and context are different all three students experienced a similar problem of making logical connections between the statements. They memorized definitions and theorems as isolated facts but were unable to establish relationships between them. Although a higher achiever experienced such a problem during proof construction and other students experienced it during rewriting proof, the nature of the problem is common. Thus, students might have a plethora of definitions, theorems, and axioms in their minds but they might not be able to develop proof if they fail to make logical connections between them.

The shared experiences also indicate that students might try to memorize the whole proof without understanding it. It seems that students have focused only on instrumental understanding and they neglected relational understanding (Skemp, 1987). Forgetting a single statement prevented them to complete the proof means that they memorize proof as a collection of sequential steps without knowing the logical relations between such sequential steps. One of the reasons behind this might be that the students were following a banking concept of education (Freire, 2013). In other words, there might be transmissionists' disempowering teaching approach (Rai & Shyangtan, 2020).The other reason might be that students might felt that mathematics is a pure body of objective knowledge that should be reproduced in the same way as they found in the textbook or/and as they taught by their teachers (Luitel, 2009). Moreover, another cause behind the above condition might be teacher's beliefs guided by modernism concepts such as the role of a teacher as a transformer of objective knowledge (Kestel & Korkmaz, 2019). Since the nature of the difficulty they faced seemed different according to their mental ability, I realized that my teaching approach might not have addressed the diversity of students because my teaching approach might be guided by a one-size-fits-all approach of modernism (Luitel, 2009). In addition, it seems that the students did emphasize on reproduction of proofs rather than meaning-making and they might not be using their cognition to construct meaning (Mohrhoff, 2008) in the process of understanding /developing proofs.

The result that I obtained here is similar to the finding of Isnani et al. (2020) who stated that while proving theorems on the limit of a sequence, students faced problems in logical abilities and mathematical connection skills. Similarly, Guler (2016) mentioned that prospective teachers directly memorize proofs without questioning and that they fail to establish a link between hypothesis and conclusion. In addition, Weber (2001) mentioned that students fail to use synthetic knowledge in proving. Thus, the results obtained here are similar to the findings of different studies in some aspects. However, the nature of difficulty might be different according to the ability of the learner, and the context of learning is not described by other studies and hence these findings are new.

Lack of Skills of Applying Definitions and Theorems

Being able to define concepts and statements of theorems is one thing and applying them in unfamiliar conditions is another thing. It means that students might state them by rote memorization without any level of understanding but to apply them appropriately in new situations. Let me mention the experiences of students that they shared during the interview.

Cristal tried to understand proof from the note copy of his friend but he was unable to comprehend the reasons behind different steps of the proof. Although, he noticed that particular definitions and theorems are used in the proof but he did not understand how they were used in the proof and why they were used in different ways.

Bina usually got confused in her understanding. She did not know whether she understood or not the definitions/proofs but she usually became able to rewrite them just after their study, however, she could not give examples of concepts using definitions and could not explain how and why any particular definitions/theorems are applied in the proof of a theorem. She could not connect definitions and theorems with other definitions and theorems.

Avinash was not satisfied in the classroom while proving theorems because he did not know the reason behind the use of definitions and theorems in different forms in different situations. When he was developing proof of an unfamiliar theorem, he was unable to use a definition in proving even though he stated the definition clearly.

The experience of participants indicates that students felt problems in applying definitions in the forms different from stated in the definitions. The reason behind this might be that they memorize definitions or they do not know different forms (conditionals that follow from the definitions and their negations). Similarly, they might have problems in forming converse, inverse, contrapositive, and negation of a theorem. For example, in the above narrative, Cristal did not understand the form of the definition used in the proof even though he identified the definition used. Similarly, Bina could not construct an example of a concept, which means that she might not understand the actual meaning of the concept and might have believed that being able to recall definition is the understanding of the concept. Moreover, poor understanding of definitions/theorems and inability to make different forms of them might be the reason behind her failure to connect them in new situations. Even higher achievers failed to apply a definition where the negation form of definition was applicable. However, he could understand how definitions are used in the readymade proofs. Thus, higher achievers experienced problems in constructing proofs, and other students experienced such problems in understanding proofs. In this way nature of difficulty depends upon the mental ability and students might fail to use definitions and theorems in their different application forms.

From the above analysis, I concluded that students might not have sufficient concept images (Tall & Vinner, 1981) so that they could not translate given definitions in different application forms. The reason behind this might be that teachers and students might have considered definitions as unchangeable truths, definitions and theorems are abstract, and they have no relation with real-life context. Thus, teaching and learning definitions and theorems of real analysis might be decontextualized (Luitel, 2009) so that students were unable to form appropriate concept images of their context. Moreover, my past images (as a teacher) such as mathematics teaching are making students able to reproduce definitions and theorems, and mathematics as a collection of unchangeable abstract definitions and theorems might be disempowering factors for the meaningful learning of definitions and theorems. In addition, great emphasis on the transmission of knowledge of curriculum and its practice guided by modernism perspective (Luitel, 2009; Kahraman, 2015) might also be a disempowering factor. Furthermore, according to radical constructivism, learners construct the meaning of new knowledge by adapting it to the existing cognitive structure (Glasersfeld, 1995). It means that for learning to be meaningful, a cognizing subject must adept new knowledge by using his/her cognition. The connection of new knowledge with existing knowledge is necessary for it. However, students might not have a proper connection in learning definitions because of decontextualized teaching/learning practice so that they were unable to understand meaningfully and in turn, they became unable to apply in unfamiliar situations.

Moore (1994) stated that students' concept images were not sufficient and they did not know the way of using definitions in proving theorems. Thus, my result is similar to the result of his findings regarding definitions. Doruk and Kaplan (2015) also found that one of the main causes of the difficulty in proving is a lack of understanding of definitions and less skill of using them in proving. My research added that a lack of meaningful understanding of concepts and theorems creates problems for higher achievers in applying them in proof construction and for others in constructing as well as understanding proofs.

Lack of Skill of Selecting Appropriate Path of Proving

To construct proof one needs to know the appropriate technique of proof and sketch of a proof. There might be different difficulties related to the way of proving while students try to develop proof. Let me explore some difficulties experienced by the participants in proving theorems.

When Cristal tried to understand proof from the notebook he did not get the reasons behind the selection of the first statement of the proof. Moreover, He did not know why that particular technique was selected for proving. Why a teacher starts a proof from a particular method and why the teacher selects a particular method of proof was always a curiosity for him.

Bina has a history of forgetting any proof after 5/6 days. She felt difficulty in memorizing long theorems, having several steps, in the fourth semester. Moreover, she could not construct even a very short proof in real analysis independently.

In the classroom, Avinash became always curious to know the reason behind the first step of proof and the particular technique selected by his teacher in proving theorems. When he confronted the unfamiliar theorem he was unable to sketch proof. In his self-study also, he believed that he understood the ready-made proofs but was unable to know the reason behind the path of proving.

From the narrative presented above, I reflected that how to select the appropriate path/technique of proof is problematic for the students regardless of their mental ability. However, it was found that higher achievers are more conscious about techniques of proof in comparison to average achievers and low achievers. Whatever may be the case, how to initialize proof and how to select a way of proving (direct method, indirect method, contrapositive, contradiction, proof by cases, counterexample) are great problems for students in real analysis courses. Although the participants understand the proof presented by the teacher or given in the textbook they hardly know the reason behind initiation and sketch of a proof. In particular, when they encounter new theorems, these two difficulties are the most common. There may be different reasons behind this. They might not have sufficient storage of mathematical contents, sufficient knowledge of techniques of proof, and less skill of rule of inferences. Moreover, the method of teaching might not be motivational, contextual, and meaningful.

The experiences of students indicate that they tried to memorize theorems rather than understanding them. The reason behind this might be that they considered proofs as structured, unchangeable, and incorrigible (Luitel, 2013). Therefore, they might be tired to memorize the content of proof in the same way as presented by their teacher keeping fear of mistakes if they use their language. Their feeling might be that teachers and textbooks are a source of knowledge and university/school is the center of learning that is why they might have been following them. As a teacher, I realized that my teaching approach was teacher-centered in the sense that while teaching proof I presented the content in the classroom considering me as a source of knowledge and considering students as a receiver of a body of knowledge (Luitel & Taylor, 2005). In other words, I didn't engage students in knowledge construction assuming them as a co-constructor of subjective knowledge (Kestel, & Korkmaz, 2019). Thus, my teaching as depositing money in a bank (Freire, 1996) might be one of the causes behind their inability of understanding or constructing proof of the theorems. Pritchard and Woollard (2013) mentioned that according to constructivism, learners construct knowledge by assimilating new information into his/her schema. Therefore, without proper and sufficient schema understanding of new materials might be almost impossible. Constructing proof also requires that students must synthesize the idea they have already learned based on the schema. From this point of view, students didn't have sufficient schema to construct a proof.

Moore (1994) mentioned that initialization of proof is problematic while proving theorems. Moreover, Weber (2001) stated that lack of strategic knowledge is the main cause of the failure of undergraduates in proving theorems. In addition, Guler (2016) mentioned that proof initialization and selecting appropriate techniques are sources of difficulties in proving.

Thus, my findings are similar to many research findings. However, the new thing for this study is that while studying proof of the theorem higher some students might be conscious of proof techniques and others might not. Moreover, these difficulties are not limited to only constructing proofs but also the matter of remembering and understanding proofs presented by teachers and presented in the books.

Inability to Grasp Language/Symbols and Lengthy Proof

Sometimes students might face difficulties in grasping the meaning of language and symbols used in the theorem, definitions, and proofs. In addition, the length of proof might matter in developing and understanding proof of the theorems. Let me explore some experiences of participants regarding the language, symbols, and length of a proof.

When Cristal tried to prove the statement 'Dedekind's property is equivalent to the completeness axiom in real number' in the mid-term examination was unable to develop proof. The reasons behind this were language used in the theorem and lengthy proof written in paragraph format. He believed that he could rewrite the theorem if the proof presented by the teacher was step-wise rather than in paragraph format.

Bina was unable to understand the proof when she tried to understand it from the book but she believed that she understood when she read from the note copy of her friend in which the proof presented by the teacher was written. Moreover, she couldn't write a proof of the theorem in the final examination because of lengthy proof. She said that "I feel difficult to understand and remember if proof contains symbols and language both and if it is lengthy'.

Avinash was unable to complete the theorem in the final exam because it contained several steps. In particular, he got confused in making connections because of lengthy proof.

The experience shared by participants indicates that the length of proof matters in understanding, remembering, and constructing proof of theorems. However, language and symbols were sources of difficulty for average achievers and below achiever students but they didn't matter for higher achievers significantly. Moreover, the paragraph format of proof might be difficult in comparison to step-wise proof written in symbolic form for some students who try to memorize proof rather than understanding. But, such difficulty was not experienced by higher achievers who focused on understanding and constructing proof rather than the only memorization. It indicates that paragraph format of writing might be difficult if students focus on rote learning rather than understanding but it might not be problematic for all. Similarly, language and symbols might create problems for those who try to remember proof without concerning understanding. However, creating lengthy proof might be problematic for all students.

The reason behind the difficulty in language might be that students highly believed in the universality of language and symbolic systems borrowed from western culture. Moreover, we (teacher and students) emphasized common language and common culture (Kestel, & Korkmaz, 2019) rather than local language and local culture.

From the experiences shared by participants, it seems that students and teachers were following modernism in the sense that spoon-feeding was emphasized by the teacher and vomiting was emphasized by students. Moreover, students might have been following the foundationalist view of mathematics, which states that mathematics is based on unchangeable truth and cannot be corrigible, altered, and replaced (Luitel, 2009). Such thought might have been forced them to memorize proofs. Students might have been focusing on symbols only rather than their meaning. Thus, participants seemed to emphasize receiving readymade knowledge from teachers or textbooks rather than engaging themselves in meaning-making. Now, let me discuss it from the lenses of constructivism. Bodner et al. (2001) mentioned that, according to constructivism, knowledge is constructed in the mind of the learner and hence it cannot be transmitted from the teacher's head to students' head. However, the participants seem to depend upon the lecture of the teacher rather than constructing proof/making meaning themselves. Students were not engaged in constructing knowledge but they tried to memorize proofs. Therefore, the role of students as the passive receiver might be one of the causes behind difficulties in proving concerning language, symbols, and length of proof.

Moore (2001) mentioned that one of the sources of difficulty in proving theorems is that students become unable to comprehend and apply mathematical language and symbols. Similarly, Mujib (2015) stated that language and mathematical notation are the main difficulties faced by students in proving. The finding of Widati and Stephani (2018) was not different from the above results concerning language. However, I got the new result that difficulty due to language used and style of writing proof might not be the same for all students. Moreover, some students might face such difficulties in constructing and others might face such difficulties also in memorizing and understanding.

Conclusion

There might be difficulties in understanding/constructing proof of theorems for every student but the nature of difficulty may vary according to their mental ability. While developing/understanding proof of theorems students might experience difficulty to link different mathematical statements (axioms, definitions, and theorems) logically, to apply definitions/theorems, to select appropriate paths of proving (sketching proof), and to comprehend language/symbols used in the proofs. However, the type and extent of the difficulty experienced by learners depend upon the context and ability of learners. Principles of modernism followed by teachers during instruction might be one of the reasons behind such difficulties. By studying this study teachers can be aware of possible difficulties in developing and understanding proofs of theorems so that by addressing such difficulties they can make their teaching of so-called abstract courses meaningful. In addition, students can make their understanding meaningful and they can develop their skills of constructing proofs by studying this research report.

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