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Wavy kinks kinematics and the impact on instantaneous mortality rate intensities

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Abstract

This study proposes an innovative approach to mortality modelling using Stirling interpolation, a numerical analysis technique that enables the estimation of mortality rates at intermediate ages or time points. The results show that the interpolation approach provides more accurate and robust estimates of mortality rates particularly at intermediate ages points. This study demonstrates the potential of Stirling interpolation in mortality modelling and highlights its advantages over traditional methods. The approach has significant implications for actuarial science including life insurance, annuities and pension funds. Mortality modelling is a crucial aspect of actuarial science as it enables the estimation of future mortality rates and the calculation of life insurance reserves, annuity values and pension fund liabilities. Traditional mortality models, such as the parametric model have been widely used in the insurance practice. Stirling interpolation is deployed as an alternative approach to mortality modelling. The objective of the study is to detect kinks and identify the impact on the force of mortality using Stirling's interpolation. From the results obtained, the interpolated value for the year shows a comparative significant increase between ages 9 and above or decrease from age 5 and 8, this could indicate trends in the population's health such as an improvement or worsening in health conditions, access to healthcare or public health interventions. Though the methodology has the potential to detect kinks due to random shocks in mortality rates, computational evidence confirm that the interpolative approach provides improved, robust and reasonably lower mortality estimates.

Keywords: Instantaneous, Intensities, Mortality estimate, Stirling Interpolation

1.0 Introduction

Wavy kinks known as mortality waves are a phenomenon observed in mortality rates particularly in the context of actuarial mortality tables. They refer to the characteristic waves or kinks in the mortality curves which exhibit non-smooth, non-monotonic behaviour especially at certain advanced ages. The importance of wavy kinks in actuarial mortality lies in their impact on mortality modelling, life insurance and pension plan valuation: (i) understanding wavy kinks helps actuaries develop more accurate mortality models which are essential for pricing life insurance products, calculating pensions and annuity liabilities and predicting population mortality trends, (ii) as populations age, the impact of wavy kinks on mortality rates becomes more significant. Accurately modelling these kinks is vital for predicting future mortality rates and managing longevity risks, (iii) wavy kinks can influence life insurance pricing as insurers need to account for the increased uncertainty and volatility in mortality rates. This may lead to higher premiums or reduced coverage to mitigate the potential losses, (iv) wavy kinks can provide valuable insights into demographic trends such as changes in population structure, advancements in medical technology and shifts in lifestyle factors. By analyzing these waves, actuaries can gain a deeper insight of the underlying factors driving mortality rates, (v) the presence of wavy kinks can increase the volatility of mortality rates making it more challenging to predict future mortality trends. This in turn can affect the stability of life insurers and pension plans as well as the overall solvency of life insurance firms.

Interpolation is a potent method of detecting kinks in mortality curves. Interpolation is a commonly used technique for modeling mortality rates and other demographic data due to its ability to provide smooth transitions between observed data points, especially when the data is sparse or unevenly spaced (Ndu et al., 2019). Interpolation creates a smooth curve that passes through all the data points. Mortality rates often change gradually across age groups or over time, and interpolation is particularly useful for generating smooth transitions between these points. This smoothness is essential in mortality modeling because abrupt changes would not accurately reflect the real-world trends in mortality, which generally evolve gradually.

Following Ndu et al.(2019), and Ogungbenle and Adeyele (2020), interpolation is a method for approximating a function at intermediate points based on a set of given points. It is particularly useful when the data is sparse or irregularly spaced. In the context of mortality modelling, Interpolation can be used to estimate mortality rates at intermediate ages or time points, based on a set of observed mortality rates. Elandt-Johnson and Johnson (1980) provides an introduction to numerical interpolation and its application in mortality computation. The authors showed that interpolation can provide more accurate estimates of mortality rates, particularly at intermediate ages compared to traditional methods.

Several analytic mortality models have been proposed that incorporate numerical interpolation. Hudec (2017) developed a life table model that uses polynomial interpolation to estimate mortality rates at intermediate ages. The authors demonstrated that their model can provide more accurate estimates of mortality rates, particularly for populations with limited data. Non-parametric mortality models have also been proposed that incorporate interpolation. Neil (1977) and McCutcheon (1983) developed non-parametric mortality models that involves interpolation to estimate mortality rates at intermediate ages. The authors demonstrated that their model can provide more accurate estimates of mortality rates, particularly for populations with sparse data.

According to Dickson et al. (2013); McCutcheon (1983); Pavlov and Mihova (2018), mortality rates tend to show gradual, continuous changes over time or across age intervals. Piecewise linear interpolation may not adequately capture these subtle variations. Interpolation, which fits a cubic polynomial between each pair of data points, is better suited for modeling the curvatures typically observed in mortality rates, such as age-related declines or fluctuations in mortality data over time.

Mortality data can sometimes contain outliers or small random variations that should not be reflected in the interpolation. Interpolation, by using higher-order polynomials, helps smooth out these minor fluctuations and prevents the model from overfitting to small data variations, which is crucial for maintaining the overall mortality trends across different cohorts. Interpolation can also be used to extrapolate mortality rates beyond the observed range, which may be necessary for projecting future cohorts. As noted by Hossain et al. (2023), mortality rates may need to be estimated for age groups or years where data is unavailable, such as for future populations. While interpolation allows for reasonably smooth extrapolation, this should be done cautiously, as it assumes that the underlying trends will continue similarly in the future.

Mortality data is often unevenly spaced, particularly when examining specific age groups or time intervals. Interpolation is flexible in this regard, as it does not require equally spaced data points, making it ideal for working with demographic data that may have gaps or irregular reporting frequencies. Hudec (2017) and Ogungbenle (2023) observed that mortality rates tend to decrease in early childhood, stabilize in middle age and rise significantly in older age. Interpolation captures these variations more accurately than simpler methods. Actuaries and demographers often use interpolation to model life tables, survival functions, and mortality rates. This method is known for its balance between accuracy and computational efficiency, making it a practical choice for mortality modeling in actuarial contexts and general demographic analysis.

Interpolation strikes a balance between smoothness, accuracy and flexibility, making it a suitable method for modeling mortality rates. Its ability to handle irregular data, capture smooth transitions and trends in mortality, and adapt for both interpolation and extrapolation makes it a highly valuable tool. As noted by Kostaki and Panousis (2001) and Rabbi and Karmaker (2013), the study of the force of mortality, often referred to as mortality analysis, is crucial for the insurance industry. According to Putra et al. (2019), mortality analysis involves examining and predicting the probability of death or survival for individuals or groups over a set period. By studying mortality data, insurance companies can identify demographic trends, changes in life expectancy, and emerging risk factors, enabling them to develop new products or adjust existing ones to meet the needs of the insured. This insight allows insurers to create innovative products that address specific mortality-related risks, such as critical illness or longevity.

Understanding the force of mortality is essential for managing longevity risk, which refers to the possibility that policyholders may live longer than expected. Longevity risk can affect the profitability and sustainability of insurance companies, particularly in life annuities and pension products. Actuaries use mortality analysis to estimate the potential liability associated with longevity risk and develop risk management strategies to mitigate its impact. Reinsurance companies, which provide coverage to primary insurers, also rely on mortality analysis. By understanding the force of mortality, reinsurers can assess the mortality risks taken on by primary insurers and price their reinsurance policies accordingly, thereby helping to distribute and manage risk more effectively across the insurance industry.

1.1 Mathematical Theory

Let T_x be the time of death of (x) and define

$$p_x = \mathbf{P}\left(T_x > x+1 \middle| T_x > x\right) \tag{1}$$

$$\Rightarrow_t p_x = \mathbf{P}\left(T_x > x+t \middle| T_x > x\right) \tag{2}$$

$$\begin{aligned} f_T\left(s+n\right) &= {}_n p_x \times {}_s p_{x+n} \mu_{x+n+s} = {}_n p_x \times {}_s p_{x+n} \mu_{x+n}\left(s\right) \\ &= {}_n p_x \times f_{T_{x+n}}\left(s\right) \end{aligned} \tag{2a}$$

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$$ {}_{t+\Delta t}p_x = \mathbf{P}\left(T_x > x+t+\Delta t \middle| T_x > x\right) \quad (3) $$

and

$$ {}_{t}q_x = \mathbf{P}\left(T_x < x+t \middle| T_x > x\right) \quad (4) $$

The force of mortality μ_{x+t} is defined as

$$ \mu_{x+t} = \lim_{\Delta t \to 0}\left(\frac{\mathbf{P}\left(T_x \le x+t+\Delta t \middle| T_x > x+t\right)}{\Delta t}\right) \quad (5) $$

$$ \mu_{x+t} = \lim_{\Delta t \to 0}\left(\frac{\mathbf{P}\left(x+t < T_x \le x+t+\Delta t \middle| T_x > t\right)}{\Delta t}\right) \times \left(\frac{1}{\mathbf{P}\left(T_x > x+t \middle| T_x > t\right)}\right) \quad (6) $$

$$ P\left(x+t < T_x \le x+t+\Delta t \middle| T_x > t\right) = \left({}_{t}p_x - {}_{t+\Delta t}p_x\right) \quad (7) $$

$$ \mu_{x+t} = \lim_{\Delta t \to 0}\left(\frac{{}_{t}p_x - {}_{t+\Delta t}p_x}{\Delta t}\right) \times \left(\frac{1}{{}_{t}p_x}\right) \quad (8) $$

$$ \Rightarrow \mu_{x+t} = -\left(\frac{1}{{}_{t}p_x}\right) \times \lim_{\Delta t \to 0}\left(\frac{{}_{t+\Delta t}p_x - {}_{t}p_x}{\Delta t}\right) \quad (9) $$

$$ \Rightarrow \mu_{x+t} = -\left(\frac{1}{{}_{t}p_x}\right) \times \frac{d\left({}_{t}p_x\right)}{dt} \quad (10) $$

$$ \Rightarrow \mu_{x+t} = -\frac{d}{dt}\log_e\left({}_{t}p_x\right) \quad (11) $$

The function μ_x is difficult to obtain and this can be done through estimation using the survival function l_x. Expanding the survivor's function l_x in Taylor's series

$$ l_{x+\theta} = l_x + \theta l_x' + \frac{\theta^2 l_x''}{2} + \frac{\theta^3 l_x'''}{6} + \frac{\theta^4 l_x^{iv}}{24} + \frac{\theta^5 l_x^{v}}{120} + \ldots \quad (12) $$

Using the first five terms, we get

when $\theta = -2$; $l_{x-2} = l_x - 2l_x' + 2l_x'' - \frac{4}{3}l_x''' + \frac{2}{3}l_x^{iv}$ (13)

when $\theta = -1$; $l_{x-1} = l_x - l_x' + \frac{1}{2}l_x'' - \frac{1}{6}l_x''' + \frac{1}{24}l_x^{iv}$ (14)

when $\theta = 1$; $l_{x+1} = l_x + l_x' + \frac{1}{2}l_x'' + \frac{1}{6}l_x''' + \frac{1}{24}l_x^{iv}$ (15)

when $\theta = 2$; $l_{x+2} = l_x + 2l_x' + 2l_x'' + \frac{4}{3!}l_x''' + \frac{2}{3!}l_x^{iv}$ (16)

$$l_{x-2} - l_{x+2} = -\left(4l_x' + \frac{8}{3}l_x'''\right) \quad (17)$$

$$l_{x-1} - l_{x+1} = -\left(2l_x' + \frac{1}{3}l_x'''\right) \quad (18)$$

Multiply (17) by 8 and subtracting from (16) to get

$$\left(l_{x-2} - l_{x+2}\right) - 8\left(l_{x-1} - l_{x+1}\right) = -4l_x' + 16l_x' = 12l_x' \quad (19)$$

$$\Rightarrow \frac{dl_x}{dx} = \frac{\left[\left(l_{x-2} - l_{x+2}\right) - 8\left(l_{x-1} - l_{x+1}\right)\right]}{12} \quad (20)$$

$$\Rightarrow \mu_x = -\frac{1}{l_x}\frac{dl_x}{dx} = \frac{\left[8l_{x-1} - 8l_{x+1} + l_{x+2} - l_{x-2}\right]}{12l_x} \quad (21)$$

$$\Rightarrow \mu_{x+t} = \frac{\left[8l_{x+t-1} - 8l_{x+t+1} + l_{x+t+2} - l_{x+t-2}\right]}{12l_{x+t}} \quad (22)$$

Consequently,

$$12l_x \times \mathbf{P}\left(T_0 \le x + h \middle| T_0 \ge x\right) = \left[8hl_{x-1} - 8hl_{x+1} + hl_{x+2} - hl_{x-2}\right] \quad (23)$$

1.2 Application of Mortality Intensity to Abstract Underwriting Under Impulsive Mortality Severity Conditions

Extreme events represent the consequences of impulsive forces of natural disasters occurring suddenly and last for a short time with the attendant abnormal mortality risk resulting in sudden and massive deaths. The pervasive consequences from these extreme events to the financial health of life insurance offices has the potential of being enormous and can cause decline in insurer's income. Mortality risk consequences can fall under two major categories: (i) the effect of extra risk arising from the conditions where proposal for life insurance has not been accepted at standard rating. The extra risk further describes the underwriter's phenomenal aversion of the degree with which the scheme holder is in mortality risk than a standard risk. This type of extra risk is measured by either a monotone addition or constant addition to the mortality rate intensity because impaired lives are observed to experience mortality rate of a normal life who is m years older where m is a function of the severity of physiological impairment. (ii) the impulsive force of extra severity risk can be measured by the Heaviside function $H(x,\xi)$.

$$\frac{d}{dx}H(x,\xi)=\delta(x,\xi) \tag{24}$$

where δ is the Dirac-delta function. The intensity of mortality risk due to severity under disastrous condition is then actuarially measured by these generalized functions.

Let $H(.)$ be the heavy side and let $\delta(.)$ be the Dirac-delta generalized functions and let $I(t \geq s)$ be the indicator function. The presence of $\delta(.)$ is to kill the insured more instantly and rapidly. In the study by Gerber et al, (2003), the following relationships was defined

$$H(t-s)=I(t \geq s) \tag{25}$$

Differentiating (25), we have

$$H'(t-s)\Delta t = H(t-s+\Delta t)-H(t-s) \tag{26}$$

But the indicator function is

$$I(t+\Delta t \geq s)-I(t \geq s)=I(t \leq s \leq t+\Delta t) \tag{27}$$

By definition,

$$\delta(t-s)\psi=\begin{cases}\infty & if \quad t=s \\ 0 & if \quad t \neq s\end{cases} \tag{28}$$

By the sifting property of Dirac-delta function,

$$\int_{-\infty}^{\infty}\delta(t-s)f(t)dt=f(s) \tag{29}$$

According to Meghwal (2020),

$$\delta(t-s)f(t)=\delta(s-t)f(s)\Rightarrow\int_{-\infty}^{\infty}\delta(t-s)f(t)dt=\int_{-\infty}^{\infty}\delta(s-t)f(s)ds \tag{30}$$

Therefore, by reason of (30), we have

$$\int_{-\infty}^{\infty}\delta(t-s)\psi dt=\psi\int_{-\infty}^{\infty}\delta(t-s)dt=\psi\times(1)=\psi \tag{31}$$

$$\int_{\alpha}^{\beta}H(t-s)f(t)dt=\int_{s}^{\beta}f(t)dt,\ \text{if}\quad \alpha<s<\beta \tag{32}$$

Recall the force of mortality under no impact of mortality intensity is

$$\lambda_x^{(1)}(t,s,\psi)=\mu_{x+t}=-\frac{1}{l_{x+t}}\frac{dl_{x+t}}{d(x+t)}=-\frac{1}{l_{x+t}}\frac{dl_{x+t}}{dt} \tag{33}$$

To measure the impact of extra severity intensity risk especially when there is wanton destruction of life at an instant, we define the following mortality intensity components.

$$\lambda_x^{(2)}(t,s,\psi)=\delta(t-s)\psi \tag{34}$$

$$\lambda_x^{(3)}(t,s,\psi)=\eta(t) \tag{35}$$

The function $\eta(t)$ represents an addition to the force of mortality and describes underwriting extra risks of the form $\mu_{x+t}+\eta(t)$. Consequently, the extra mortality can be modelled by increasing the force of interest correspondingly. μ_{x+t} describes the force of mortality under no impact of extra severity mortality. Consequently, $\delta(t-s)\psi$ is the extra forcing mortality risk factor under severe conditions. Consequently, the aggregate mortality intensity $\lambda_x(t,s,\psi)$ can be decomposed as force of mortality, extra mortality risk and force of severity under disastrous contingencies such as plane crash, ship wreck or war

$$\lambda_x(t,s,\psi)=\lambda_x^{(1)}(t,s,\psi)+\lambda_x^{(2)}(t,s,\psi)+\lambda_x^{(3)}(t,s,\psi) \tag{36}$$

$$\lambda_x(t,s,\psi)=\mu_{x+t}+\delta(t-s)\psi+\eta(t) \tag{37}$$

Note the superscripts are not exponents or derivatives. The insured can be underwritten under the influence of both (i) extra underwriting mortality risk either because of the sum assured in excess of the policy retention level or not accepted on normal rating and (ii) severe mortality risk.

According to Legua et al. (2006); Gerber et al. (2003); Venetis (2014); Elnour (2018) and Meghwal (2020), the function $\delta(.)$ is defined as

$$\delta(t-s)=\begin{cases}\infty & if \quad t=s\\ 0 & if \quad t\neq s\end{cases} \tag{38}$$

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Now, the Heavy side is therefore

$$H(t-s)=\begin{cases}1 & if \quad t\geq s\\ 0 & if \quad t<s\end{cases} \tag{39}$$

Furthermore, following Amaku et al. (2021); Kovacevic, Jovanovic, Ferrando (2021), we have

$$H(t-s)f(t)=\begin{cases}f(t) & if \quad t\geq s\\ 0 & if \quad t<s\end{cases} \tag{40}$$

$$\Rightarrow H(t-s)=\int_{-\infty}^{t}\delta(\zeta-s)d\zeta \tag{41}$$

However, the impact at the initial time $s=0$ is as follows

$$\Rightarrow \delta(t-0)=\begin{cases}\infty & if \quad t=0\\ 0 & if \quad t\neq 0\end{cases} \tag{42}$$

$$\Rightarrow H(t-0)=\begin{cases}1 & if \quad t\geq 0\\ 0 & if \quad t<0\end{cases} \tag{43}$$

$$H(t-0)=\int_{-\infty}^{t}\delta(\zeta-0)d\zeta \tag{44}$$

$$H'(\zeta-s)=\delta(\zeta-s) \tag{45}$$

$$H'(\zeta-s)\Delta\zeta=\delta(\zeta-s)\Delta\zeta \tag{46}$$

$$H'(\zeta-s)\Delta\zeta=H(\zeta-s+\Delta\zeta)-H(\zeta-s) \tag{47}$$

Therefore,

$$\delta(\zeta-s)\Delta\zeta=H(\zeta-s+\Delta\zeta)-H(\zeta-s) \tag{48}$$

If $I(.)$ is an indicator function, then following Gerber et al. (2003), we have

$$I(\zeta+\Delta\zeta\geq s)-I(\zeta\geq s)=I(\zeta\leq s\leq\zeta+\Delta\zeta)=H'(\zeta-s)\Delta\zeta \tag{49}$$

Consequently,

$$\delta(\zeta-s)\Delta\zeta=I(\zeta+\Delta\zeta\geq s)-I(\zeta\geq s)=I(\zeta\leq s\leq\zeta+\Delta\zeta) \tag{50}$$

$$\int_{-\infty}^{\infty}I(\zeta\leq s\leq\zeta+\Delta\zeta)f(\zeta)d\zeta=f(s) \tag{51}$$

We can now begin to evaluate the intensity of massive death rate at an instant

Therefore, given that

$$\lambda_x(t,s,\psi)=\mu_{x+t}+\delta(t-s)\psi+\eta(t) \tag{52}$$

$$\lambda_x(\theta,s,\psi)=\mu_{x+\theta}+\delta(\theta-s)\psi+\eta(\theta) \tag{53}$$

The probability of survival of a live aged x surviving to age $x+t$ under no impact is

$${}_t p_x=\frac{l_{x+t}}{l_x}=e^{-\int_0^t \mu_{x+\zeta}d\zeta} \tag{54}$$

while the probability of survival under impact of severe condition is given by

$${}_t \overline{p}_x(s,\psi)=e^{-\int_0^t(\lambda_x(\theta,s,\psi))d\theta} \tag{55}$$

The bar indicates the effect of impact of severity on the survival probability functions

$${}_t \overline{p}_x(s,\psi)=\exp\left(-\left(\int_0^t \mu_{x+\theta}+\delta(\theta-s)\psi+\eta(\theta)d\theta\right)\right) \tag{56}$$

$${}_t \overline{p}_x(s,\psi)$$
$$=\exp\left(-\left(\int_0^t \mu_{x+\theta}d\theta\right)\right)\times\exp\left(-\left(\int_0^t \delta(\theta-s)\psi d\theta\right)\right)\times\exp\left(-\left(\int_0^t \eta(\theta)d\theta\right)\right) \tag{57}$$

Recalling that , $H(t-s)=\int_{-\infty}^t \delta(\zeta-s)d\zeta$ (58)

$${}_t \overline{p}_x(s,\psi)=\left[\exp\left(-\int_0^t \mu_{x+\theta}d\theta\right)\right]\exp\left(\left(-H(t-s)\psi\right)\right)\exp\left(-\left(\int_0^t \eta(\theta)d\theta\right)\right) \tag{59}$$

$${}_t \overline{p}_x(s,\psi)=\exp\left(-\left(\int_0^t \eta(\theta)d\theta\right)\right)\left[\exp\left(-\int_0^t \mu_{x+\theta}d\theta\right)\right]\exp\left(\left(-H(t-s)\psi\right)\right) \tag{60}$$

Recalling again that, ${}_t p_x=e^{-\int_0^t \mu_{x+\theta}d\theta}$ (61)

Therefore,

$${}_t \overline{p}_x(s,\psi)=({}_t p_x)\exp\left(-\left(\int_0^t \eta(\theta)d\theta\right)\right)\exp\left(\left(-H(t-s)\psi\right)\right) \tag{62}$$

$$ {}_t\bar{p}_x(s,\psi) = e^{-\left(\int_0^t \eta(\theta)d\theta\right)}\left({}_t p_x\right)e^{-H(t-s)\psi} \tag{63}$$

Differentiating (63) with respect to t

$$\frac{\partial}{\partial t}\left[{}_t\bar{p}_x(s,\psi)\right] = \frac{\partial}{\partial t}\left[\left({}_t p_x\right)e^{-\left(\int_0^t \eta(\theta)d\theta\right)}e^{-H(t-s)\psi}\right] = e^{-H(t-s)\psi}\left[{}_t p_x\right]\frac{\partial}{\partial t}e^{-\left(\int_0^t \eta(\theta)d\theta\right)}$$
$$+\left[{}_t p_x\right]e^{-\left(\int_0^t \eta(\theta)d\theta\right)}\frac{\partial}{\partial t}e^{-H(t-s)\psi} + e^{-\left(\int_0^t \eta(\theta)d\theta\right)}e^{-H(t-s)\psi}\frac{\partial}{\partial t}\left[{}_t p_x\right] \tag{64}$$

$$\Rightarrow \frac{\partial}{\partial t}\left[{}_t\bar{p}_x(s,\psi)\right] = e^{-H(t-s)\psi}\left[{}_t p_x\right]\left(-\eta(t)e^{-\left(\int_0^t \eta(\theta)d\theta\right)}\right)$$
$$-\left[{}_t p_x\right]e^{-\left(\int_0^t \eta(\theta)d\theta\right)}\left[\psi \times H'(t-s)\right]e^{-H(t-s)\psi} + e^{-\left(\int_0^t \eta(\theta)d\theta\right)}e^{-H(t-s)\psi}\frac{\partial}{\partial t}\left[{}_t p_x\right] \tag{65}$$

Differentiating ${}_t p_x$ with respect to t

$$\frac{\partial}{\partial t}\left[\left({}_t p_x\right)\right] = \frac{\partial}{\partial t}\left(\frac{l_{x+t}}{l_x}\right) = \frac{l'_{x+t}}{l_x} = \frac{-l_{x+t}\mu_{x+t}}{l_x} = -\left({}_t p_x\right)\mu_{x+t} \tag{66}$$

Then (65) becomes

$$\frac{\partial}{\partial t}\left[{}_t\bar{p}_x(s,\psi)\right] = -e^{-H(t-s)\psi}\left[{}_t p_x\right]\left(\eta(t)e^{-\left(\int_0^t \eta(\theta)d\theta\right)}\right)$$
$$-\left[{}_t p_x\right]e^{-\left(\int_0^t \eta(\theta)d\theta\right)}\left[\psi \times \delta(t-s)\right]e^{-H(t-s)\psi} - e^{-\left(\int_0^t \eta(\theta)d\theta\right)}e^{-H(t-s)\psi}\left({}_t p_x\right)\mu_{x+t} \tag{67}$$

The probability density function for a live aged x under impact of extra mortality and severity risk is therefore derived as

$$-\frac{\partial}{\partial t}\left[{}_t\bar{p}_x(s,\psi)\right] = e^{-H(t-s)\psi}\left[{}_t p_x\right]\left(\eta(t)e^{-\left(\int_0^t \eta(\theta)d\theta\right)}\right)$$
$$+\left[{}_t p_x\right]e^{-\left(\int_0^t \eta(\theta)d\theta\right)}\left[\psi \times \delta(t-s)\right]e^{-H(t-s)\psi} + e^{-\left(\int_0^t \eta(\theta)d\theta\right)}e^{-H(t-s)\psi}\left({}_t p_x\right)\mu_{x+t} \tag{68}$$

$$-\frac{\partial}{\partial t}\left[{}_t\bar{p}_x(s,\psi)\right] = \left\{\left(\eta(t)\right) + \left[\psi \times \delta(t-s)\right] + \mu_{x+t}\right\}\left({}_t p_x\right)e^{-H(t-s)\psi}e^{-\left(\int_0^t \eta(\theta)d\theta\right)} \tag{69}$$

Hence the death density is finally obtained as

$$\overline{f}_{T(x)}(s,\psi)=\left[\left(\eta(t)\right)+\left[\psi\times\delta(t-s)\right]+\mu_{x+t}\right]\left[{}_tp_x\right]e^{-H(t-s)\psi}e^{-\left(\int_0^t\eta(\theta)d\theta\right)} \quad (70)$$

$${}_t\overline{q}_x(s,\psi)=1-e^{-\left(\int_0^t\eta(\theta)d\theta\right)}\left({}_tp_x\right)e^{-H(t-s)\psi} \quad (71)$$

Differentiating (63) with respect to x,

$$\frac{\partial}{\partial x}\left[{}_t\overline{p}_x(s,\psi)\right]=\frac{\partial}{\partial x}\left[\left({}_tp_x\right)e^{-\left(\int_0^t\eta(\theta)d\theta\right)}e^{-H(t-s)\psi}\right]=e^{-H(t-s)\psi}\left[{}_tp_x\right]\frac{\partial}{\partial x}e^{-\left(\int_0^t\eta(\theta)d\theta\right)}$$
$$+\left[{}_tp_x\right]e^{-\left(\int_0^t\eta(\theta)d\theta\right)}\frac{\partial}{\partial x}e^{-H(t-s)\psi}+e^{-\left(\int_0^t\eta(\theta)d\theta\right)}e^{-H(t-s)\psi}\frac{\partial}{\partial x}\left[{}_tp_x\right] \quad (72)$$

Now differentiating ${}_tp_x$ with respect to x, we obtain

$$\frac{\partial}{\partial x}\left[{}_tp_x\right]=\frac{\partial}{\partial x}\left[\frac{l_{x+t}}{l_x}\right]=\left[\frac{l_xl'_{x+t}-l'_xl_{x+t}}{l_x^2}\right]=\left[\frac{-l_xl_{x+t}\mu_{x+t}-l_{x+t}\left(-l_x\mu_x\right)}{l_x^2}\right]$$
$$=\left[\frac{-l_xl_{x+t}\mu_{x+t}+l_{x+t}\left(l_x\mu_x\right)}{l_x^2}\right] \quad (73)$$

$$\frac{\partial}{\partial x}\left[{}_tp_x\right]=\left[\frac{l_{x+t}\mu_x-l_{x+t}\mu_{x+t}}{l_x}\right]=\left[\frac{l_{x+t}\mu_x}{l_x}\right]+\left[\frac{-l_{x+t}\mu_{x+t}}{l_x}\right]=\left({}_tp_x\right)\mu_x-\left({}_tp_x\right)\mu_{x+t} \quad (74)$$

$$\frac{\partial}{\partial x}\left[{}_tp_x\right]=\left[\frac{-\mu_{x+t}+l_{x+t}\mu_x}{l_x}\right]=-\mu_{x+t}+{}_tp_x\mu_x \quad (75)$$

$$\frac{\partial}{\partial x}\left[{}_t\overline{p}_x(s,\psi)\right]=e^{-\left(\int_0^t\eta(\theta)d\theta\right)}e^{-H(t-s)\psi}\left(-\mu_{x+t}+{}_tp_x\mu_x\right) \quad (76)$$

Recalling from (63)

$${}_t\overline{p}_x(s,\psi)={}_tp_xe^{-H(t-s)\psi}e^{-\left(\int_0^t\eta(\theta)d\theta\right)} \quad (77)$$

$$\frac{\bar{l}_{x+t}(s,\psi)}{l_{x+t}}=e^{-\left(\int_0^t\mu(x+\zeta)d\theta\right)}e^{-\left(\int_0^t\eta(\theta)d\theta\right)}e^{-H(t-s)\psi} \quad (78)$$

$$\bar{l}_{x+t}(s,\psi)=l_{x+t}\times\left(e^{-\left(\int_0^t\mu(x+\zeta)d\theta\right)}e^{-\left(\int_0^t\eta(\theta)d\theta\right)}e^{-H(t-s)\psi}\right) \quad (79)$$

$$\bar{l}_{x+t}\left(s,\psi\right)=\left(\int_{0}^{\infty} l_{x+t+\beta}\mu_{x+t+\beta}d\beta\right)\times e^{-\left(\int_{0}^{t}\mu(x+\zeta)d\theta\right)}e^{-\left(\int_{0}^{t}\eta(\theta)d\theta\right)}e^{-H(t-s)\psi} \tag{80}$$

Setting $x=0$ in (80) yields the survival function at time t

$$\bar{l}_{t}\left(s,\psi\right)=\left(\int_{0}^{\infty} l_{t+\beta}\mu_{t+\beta}d\beta\right)\times e^{-\left(\int_{0}^{t}\mu_{\zeta}d\theta\right)}e^{-\left(\int_{0}^{t}\eta(\theta)d\theta\right)}e^{-H(t-s)\psi} \tag{81}$$

Since t is arbitrary, we can set $t=x$ in (81) to obtain

$$\bar{l}_{x}\left(s,\psi\right)=\left(\int_{0}^{\infty} l_{x+\beta}\mu_{x+\beta}d\beta\right)\times e^{-\left(\int_{0}^{x}\mu(\zeta)d\theta+\int_{0}^{x}\eta(\theta)d\theta\right)}e^{-H(x-s)\psi} \tag{82}$$

2.0 Material and Methods

2.1 Stirling Interpolation

$$\begin{aligned} y_v &= \Delta^0 y_0 + v\left(\frac{\Delta y_{-1}+\Delta y_0}{2}\right)+\frac{v^2}{2!}\Delta^2 y_{-1}+\frac{v\left(v^2-1\right)}{3!}\left(\frac{\Delta^3 y_{-1}+\Delta^3 y_{-2}}{2}\right) \\ &+\frac{v^2\left(v^2-1\right)}{4!}\Delta^4 y_{-2}+\frac{\left(v+2\right)\left(v+1\right)v\left(v-1\right)\left(v-2\right)}{5!}\left(\frac{\Delta^5 l_{-3}+\Delta^5 l_{2}}{2}\right) \end{aligned} \tag{83}$$

Set $v=t$ and $y=l$ in (83) to get

$$\begin{aligned} \Rightarrow l_{0+t} &= \frac{\Delta^0 l_0}{0!}+\frac{t}{1!}\left(\frac{\Delta l_{-1}+\Delta l_0}{2}\right)+\frac{t^2}{2!}\Delta^2 l_{-1}+\frac{t\left(t^2-1\right)}{3!}\left(\frac{\Delta^3 l_{-1}+\Delta^3 l_{-2}}{2}\right)+\frac{t^2\left(t^2-1\right)}{4!}\Delta^4 l_{-2} \\ &+\frac{\left(t+2\right)\left(t+1\right)t\left(t-1\right)\left(t-2\right)}{5!}\left(\frac{\Delta^5 l_{-3}+\Delta^5 l_{-2}}{2}\right) \end{aligned} \tag{84}$$

Now $x+0+t\rightarrow x+t$

$$\begin{aligned} l_{x+t} &= \frac{\Delta^0 l_x}{0!}+\frac{t}{1!}\left(\frac{\Delta l_{x-1}+\Delta l_x}{2}\right)+\frac{t^2}{2!}\Delta^2 l_{x-1}+\frac{t\left(t^2-1\right)}{3!}\left(\frac{\Delta^3 l_{x-1}+\Delta^3 l_{x-2}}{2}\right) \\ &+\frac{t^2\left(t^2-1\right)}{3!}\Delta^4 l_{x-2}+\frac{\left(t+2\right)\left(t+1\right)t\left(t-1\right)\left(t-2\right)}{5!}\left(\frac{\Delta^5 l_{x-3}+\Delta^5 l_{x-2}}{2}\right) \end{aligned} \tag{85}$$

$$\begin{aligned} l_{x+t} &= l_x+\frac{t}{1!}\left(\frac{\Delta l_{x-1}+\Delta l_x}{2}\right)+\frac{t^2}{2!}\Delta^2 l_{x-1}+\frac{\left(t^3-t\right)}{3!}\left(\frac{\Delta^3 l_{x-1}+\Delta^3 l_{x-2}}{2}\right)+\frac{\left(t^4-t^2\right)}{4!}\Delta^4 l_{x-2} \\ &+\frac{\left(t^5-5t^3+4t\right)}{5!}\left(\frac{\Delta^5 l_{x-3}+\Delta^5 l_{x-2}}{2}\right) \end{aligned} \tag{86}$$

Differentiate the survival function l_{x+t} with respect to t obtain

$$\frac{dl_x}{dx}=+\frac{\Delta l_{x-1}+\Delta l_x}{2}+\frac{2t}{2!}\Delta^2 l_{x-1}+\frac{(3t^2-1)}{3!}\left(\frac{\Delta^3 l_{x-1}+\Delta^3 l_{x-2}}{2}\right)+\frac{(4t^3-2t)}{4!}\Delta^4 l_{x-2}+\frac{(5t^4-15t^2+4)}{5!}\left(\frac{\Delta^5 l_{x-3}+\Delta^5 l_{x-2}}{2}\right) \tag{87}$$

By definition, $\mu_x=-\frac{1}{l_x}\frac{dl_x}{dx}$ (88)

$$-\frac{1}{l_x}\frac{dl_x}{dx}=\frac{1}{l_x}\left[\begin{array}{l}-\frac{\Delta l_{x-1}+\Delta l_x}{2}-\frac{2t}{2!}\Delta^2 l_{x-1}-\frac{(3t^2-1)}{3!}\left(\frac{\Delta^3 l_{x-1}+\Delta^3 l_{x-2}}{2}\right)\\-\frac{(4t^3-2t)}{4!}\Delta^4 l_{x-2}-\frac{(5t^4-15t^2+4)}{5!}\left(\frac{\Delta^5 l_{x-3}+\Delta^5 l_{x-2}}{2}\right)\end{array}\right] \tag{89}$$

Set $t=0$ in $-\frac{1}{l_x}\frac{dl_{x+t}}{dt}\Rightarrow-\frac{1}{l_x}\frac{dl_x}{dx}$

$$\Rightarrow\mu_x=-\frac{1}{l_x}\frac{dl_x}{dx}=-\left(\frac{\Delta l_{x-1}+\Delta l_x}{2l_x}\right)-\frac{2(0)}{2!l_x}\Delta^2 l_{x-1}-\frac{\left(3(0)^2-1\right)}{3!l_x}\left(\frac{\Delta^3 l_{x-1}+\Delta^3 l_{x-2}}{2l_x}\right)-\frac{\left(4(0)^3-2(0)\right)}{4!l_x}\Delta^4 l_{x-2}-\frac{\left(5(0)^4-15(0)^2+4\right)}{5!l_x}\left(\frac{\Delta^5 l_{x-3}+\Delta^5 l_{x-2}}{2l_x}\right) \tag{90}$$

Now taking the successive forward differencing yields

$$\Delta^1 l_x=l_{x+1}-l_x \tag{91}$$

$$\Delta l_{x-1}=l_x-l_{x-1} \tag{92}$$

$$\Delta^2 l_{x-1}=\Delta(l_x-l_{x-1}) \tag{93}$$

$$\Delta^2 l_{x-1}=\Delta l_x-\Delta l_{x-1} \tag{94}$$

$$\Delta^2 l_{x-1}=l_{x+1}-l_x-(l_x-l_{x-1}) \tag{95}$$

$$\Delta^2 l_{x-1}=l_{x+1}-l_x-l_x+l_{x-1} \tag{96}$$

$$\Delta^2 l_{x-1}=l_{x+1}-2l_x+l_{x-1} \tag{97}$$

$$\Delta^3 l_{x-1}=\Delta l_{x+1}-2\Delta l_x+\Delta l_{x-1} \tag{98}$$

$$\Delta^3 l_{x-1}=l_{x+2}-l_{x+1}-2(l_{x+1}-l_x)+l_x-l_{x-1} \tag{99}$$

$$\Delta^3 l_{x-1}=l_{x+2}-l_{x+1}-2l_{x+1}+2l_x+l_x-l_{x-1} \tag{100}$$

$$\Delta^3 l_{x-1} = l_{x+2} - 3l_{x+1} + 3l_x - l_{x-1} \quad (101)$$

$$\Delta l_{x-2} = l_{x-1} - l_{x-2} \quad (102)$$

$$\Delta^2 l_{x-2} = \Delta l_{x-1} - \Delta l_{x-2} \quad (103)$$

$$\Delta^2 l_{x-2} = l_x - l_{x-1} - (l_{x-1} - l_{x-2}) \quad (104)$$

$$\Delta^2 l_{x-2} = l_x - l_{x-1} - l_{x-1} + l_{x-2} \quad (105)$$

$$\Delta^2 l_{x-2} = l_x - 2l_{x-1} + l_{x-2} \quad (106)$$

$$\Delta^3 l_{x-2} = \Delta l_x - 2\Delta l_{x-1} + \Delta l_{x-2} \quad (107)$$

$$\Delta^3 l_{x-2} = l_{x+1} - l_x - 2(l_x - l_{x-1}) + l_{x-1} - l_{x-2} \quad (108)$$

$$\Delta^3 l_{x-2} = l_{x+1} - l_x - 2l_x + 2l_{x-1} + l_{x-1} - l_{x-2} \quad (109)$$

$$\Delta^3 l_{x-2} = l_{x+1} - 3l_x + 3l_{x-1} - l_{x-2} \quad (110)$$

$$\Delta^4 l_{x-2} = \Delta l_{x+1} - 3\Delta l_x + 3\Delta l_{x-1} - \Delta l_{x-2} \quad (111)$$

$$\Delta^4 l_{x-2} = l_{x+2} - l_{x+1} - 3(l_{x+1} - l_x) + 3(l_x - l_{x-1}) - 1(l_{x-1} - l_{x-2}) \quad (112)$$

$$\Delta^4 l_{x-2} = l_{x+2} - l_{x+1} - 3l_{x+1} + 3l_x + 3l_x - 3l_{x-1} - l_{x-1} + l_{x-2} \quad (113)$$

$$\Delta^4 l_{x-2} = l_{x+2} - 4l_{x+1} + 6l_x - 4l_{x-1} + l_{x-2} \quad (114)$$

$$\Delta^5 l_{x-2} = \Delta l_{x+2} - 4\Delta l_{x+1} + \Delta 6 l_x - 4\Delta l_{x-1} + \Delta l_{x-2} \quad (115)$$

$$\Delta^5 l_{x-2} = l_{x+3} - l_{x+2} - 4(l_{x+2} - l_{x+1}) + 6(l_{x+1} - l_x) - 4(l_x - l_{x-1}) + l_{x-1} - l_{x-2} \quad (116)$$

$$\Delta^5 l_{x-2} = l_{x+3} - l_{x+2} - 4l_{x+2} + 4l_{x+1} + 6l_{x+1} - 6l_x - 4l_x + 4l_{x-1} + l_{x-1} - l_{x-2} \quad (117)$$

$$\Delta^5 l_{x-2} = l_{x+3} - 5l_{x+2} + 10l_{x+1} - 10l_x + 5l_{x-1} - l_{x-2} \quad (118)$$

$$\Delta l_{x-3} = l_{x-2} - l_{x-3} \quad (119)$$

$$\Delta^2 l_{x-3} = \Delta l_{x-2} - \Delta l_{x-3} \quad (120)$$

$$\Delta^2 l_{x-3} = l_{x-1} - l_{x-2} - (l_{x-2} - l_{x-3}) \quad (121)$$

$$\Delta^2 l_{x-3} = l_{x-1} - l_{x-2} - l_{x-2} + l_{x-3} \quad (122)$$

$$\Delta^2 l_{x-3} = l_{x-1} - 2l_{x-2} + l_{x-3} \quad (123)$$

$$\Delta^3 l_{x-3} = \Delta l_{x-1} - 2\Delta l_{x-2} + \Delta l_{x-3} \quad (124)$$

$$\Delta^3 l_{x-3} = l_x - l_{x-1} - 2(l_{x-1} - l_{x-2}) + l_{x-2} - l_{x-3} \quad (125)$$

$$\Delta^3 l_{x-3} = l_x - l_{x-1} - 2l_{x-1} + 2l_{x-2} + l_{x-2} - l_{x-3} \quad (126)$$

$$\Delta^3 l_{x-3} = l_x - 3l_{x-1} + 3l_{x-2} - l_{x-3} \tag{127}$$

$$\Delta^4 l_{x-3} = \Delta l_x - 3\Delta l_{x-1} + 3\Delta l_{x-2} - \Delta l_{x-3} \tag{128}$$

$$\Delta^4 l_{x-3} = l_{x+1} - l_x - 3(l_x - l_{x-1}) + 3(l_{x-1} - l_{x-2}) - (l_{x-2} - l_{x-3}) \tag{129}$$

$$\Delta^4 l_{x-3} = l_{x+1} - l_x - 3l_x + 3l_{x-1} + 3l_{x-1} - 3l_{x-2} - l_{x-2} + l_{x-3} \tag{130}$$

$$\Delta^4 l_{x-3} = l_{x+1} - 4l_x + 6l_{x-1} - 4l_{x-2} + l_{x-3} \tag{131}$$

$$\Delta^5 l_{x-3} = \Delta l_{x+1} - 4\Delta l_x + 6\Delta l_{x-1} - 4\Delta l_{x-2} + \Delta l_{x-3} \tag{132}$$

$$\Delta^5 l_{x-3} = l_{x+2} - l_{x+1} - 4(l_{x+1} - l_x) + 6(l_x - l_{x-1}) - 4(l_{x-1} - l_{x-2}) + l_{x-2} - l_{x-3} \tag{133}$$

$$\Delta^5 l_{x-3} = l_{x+2} - l_{x+1} - 4l_{x+1} + 4l_x + 6l_x - 6l_{x-1} - 4l_{x-1} + 4l_{x-2} + l_{x-2} - l_{x-3} \tag{134}$$

$$\Delta^5 l_{x-3} = l_{x+2} - 5l_{x+1} + 10l_x - 10l_{x-1} + 5l_{x-2} - l_{x-3} \tag{135}$$

$$\mu_x = -\frac{1}{l_x}\frac{dl_x}{dx} = -\left(\frac{\Delta l_{x-1} + \Delta l_x}{2l_x}\right) + \frac{1}{3!l_x}\left(\frac{\Delta^3 l_{x-1} + \Delta^3 l_{x-2}}{2l_x}\right) - \frac{4}{5!}\left(\frac{\Delta^5 l_{x-3} + \Delta^5 l_{x-2}}{2l_x}\right) \tag{136}$$

$$\Rightarrow \mu_x = -\frac{1}{l_x}\frac{dl_x}{dx} = -\left(\frac{\Delta l_{x-1} + \Delta l_x}{2l_x}\right) + \left(\frac{\Delta^3 l_{x-1} + \Delta^3 l_{x-2}}{12l_x}\right) - \left(\frac{\Delta^5 l_{x-3} + \Delta^5 l_{x-2}}{60l_x}\right) \tag{137}$$

Let

$$\alpha_1(l) = \Delta l_{x-1} + \Delta l_x \tag{138}$$

then

$$\alpha_1(l) = l_x - l_{x+1} + l_{x+1} - l_x \tag{139}$$

$$\alpha_1(l) = l_{x+1} - l_{x-1} \tag{140}$$

Similarly,

$$\alpha_2(l) = \Delta^3 l_{x-1} + \Delta^3 l_{x-2} \tag{141}$$

$$\alpha_2(l) = l_{x+2} - 3l_{x+1} + 3l_x - l_{x-1} + l_{x+1} - 3l_x + 3l_{x-1} - l_{x-2} \tag{142}$$

$$\alpha_2(l) = l_{x+2} - 2l_{x+1} + 2l_{x+1} - l_{x-2} \tag{143}$$

Following the same pattern,

$$\alpha_3(l) = \Delta^5 l_{x-3} + \Delta^5 l_{x-2} \tag{144}$$

$$\begin{aligned}\alpha_3(l) &= l_{x+2} - 5l_{x+1} + 10l_x - 10l_{x-1} + 5l_{x-2} - l_{x-3} + l_{x+3} - 5l_{x+2} + 10l_{x+1} \\ &\quad -10l_x + 5l_{x-1} - l_{x-2}\end{aligned} \tag{145}$$

$$\alpha_3(l) = l_{x+3} - 4l_{x+2} + 5l_{x+1} - 5l_{x-1} + 4l_{x-2} - l_{x-3} \tag{146}$$

$$\mu_x = -\frac{1}{2l_x}\left[\alpha_1(l)\right] + \frac{1}{12l_x}\left[\alpha_2(l)\right] - \frac{1}{60l_x}\left[\alpha_3(l)\right] \quad (147)$$

$$\mu_x = -\frac{1}{2}\left(l_{x+1} - l_{x-1}\right) + \frac{1}{12l_x}\left(l_{x+2} - 2l_{x+1} + 2l_{x-1} - l_{x-2}\right) - \frac{1}{60l_x}\left(l_{x+3} - 4l_{x+2} + 5l_{x+1} - 5l_{x-1} + 4l_{x-2} - l_{x-3}\right) \quad (148)$$

$$\Rightarrow \mu_x = \frac{-30}{60l_x}\left(l_{x+1} - l_{x-1}\right) + \frac{5}{60l_x}\left(l_{x+2} - 2l_{x+1} + 2l_{x-1} - l_{x-2}\right) - \frac{1}{60l_x}\left(l_{x+3} - 4l_{x+2} + 5l_{x+1} - 5l_{x-1} + 4l_{x-2} - l_{x-3}\right) \quad (149)$$

$$\Rightarrow \mu_x = \frac{-30l_{x+1} + 30l_{x-1} + 5l_{x+2} - 10l_{x+1} + 10l_{x-1} - 5l_{x-2} - l_{x+3} + 4l_{x+2} - 5l_{x+1} + 5l_{x-1} - 4l_{x-2} + l_{x-3}}{60l_x} \quad (150)$$

$$\Rightarrow \mu_x = \frac{-45l_{x+1} + 45l_{x-1} + 9l_{x+2} - 9l_{x-2} - l_{x+3} + l_{x-3}}{60l_x} \quad (151)$$

$$\Rightarrow \mu_{x+t} = \frac{-45l_{x+t+1} + 45l_{x+t-1} + 9l_{x+t+2} - 9l_{x+t-2} - l_{x+t+3} + l_{x+t-3}}{60l_{x+t}} \quad (152)$$

Therefore equation (21) implies

$$\mu_{x+t} = \frac{\left[40l_{x+t-1} - 40l_{x+t+1} + 5l_{x+t+2} - 5l_{x+t-2}\right]}{60l_{x+t}} = \frac{-45l_{x+t+1} + 45l_{x+t-1} + 9l_{x+t+2} - 9l_{x+t-2} - l_{x+t+3} + l_{x+t-3}}{60l_{x+t}} \quad (153)$$

$$\Rightarrow {}_tp_x = e^{-\int_0^t \left\{\frac{\left[-45l_{x+s+1} + 45l_{x+s-1} + 9l_{x+s+2} - 9l_{x+s-2} - l_{x+s+3} + l_{x+s-3}\right]}{60l_{x+s}}\right\} ds} \quad (154)$$

The equation (154) has the following implication on the expected value of the random life time. Let T_x be the random life time of (x), then $\mathbf{E}(T_x)$ is defined as

$$\mathbf{E}(T_x) = \int_0^{\Omega - x} t \times ({}_tp_x) dt \quad (155)$$

$$\mathbf{E}(T_x) = \int_0^{\infty} ({}_tp_x) dt \quad (156)$$

Let $v(t)$ be a continuous function defined within the interval $[0, \Omega - x]$. Consider the partition of the interval

$$t_0 = 0; t_1 = 1;,,,; t_{k-1} = t_{\Omega-x-1} = \Omega - x - 1; t_k = t_{\Omega-x} = \Omega - x \tag{157}$$

$$\mathbf{E}(T_x) = \int_0^\infty \frac{l_{x+t}}{l_x} dt \to \int_0^\infty \nu(t) dt \tag{158}$$

$$\nu(t) = {}_t\, p_x \tag{159}$$

$$\nu(t_0) = \nu(0) = \frac{l_x}{l_x} = {}_0\, p_x = 1 \tag{160}$$

$$\nu(t_1) = \nu(1) = \frac{l_{x+1}}{l_x} = {}_1\, p_x \tag{161}$$

$$\nu(t_2) = \nu(2) = \frac{l_{x+2}}{l_x} = {}_2\, p_x \tag{162}$$

$$\nu(t_3) = \nu(3) = \frac{l_{x+3}}{l_x} = {}_3\, p_x \tag{75}$$

$$\nu(t_4) = \nu(4) = \frac{l_{x+4}}{l_x} = {}_4\, p_x \tag{163}$$

$$\nu(t_{m-1}) = \nu(\Omega - x - 1) = \frac{l_{\Omega-x-1}}{l_x} = {}_{\Omega-x-1}\, p_x \tag{164}$$

$$\nu(t_m) = \nu(\Omega - x) = \frac{l_{\Omega-x}}{l_x} = {}_{\Omega-x}\, p_x \tag{165}$$

Let

$$\Delta = \frac{t_m - t_0}{m} = \frac{\Omega - x - 0}{\Omega - x} = 1 \tag{166}$$

$$\mathbf{E}(T_x) = \int_0^\infty ({}_t\, p_x) dt = \frac{\Delta}{3} \begin{bmatrix} \{\nu(0) + \nu(\Omega - x)\} + 4 \begin{Bmatrix} \nu(1) + \nu(3) + 4\nu(5) + 4\nu(7) + \ldots \\ +4\nu(\Omega - x - 1) \end{Bmatrix} \\ +\ldots 2\{\nu(2) + \nu(4) + \nu(6) + \nu(8) + \ldots\} \end{bmatrix} \tag{167}$$

$$\mathbf{E}(T_x) = \frac{\Delta}{3} \begin{bmatrix} \{{}_0 p_x + {}_{\Omega-x}\, p_x\} + 4 \begin{Bmatrix} {}_1 p_x + {}_3\, p_x + {}_5\, p_x + {}_7\, p_x + \ldots \\ + {}_{\Omega-x-1} p_x \end{Bmatrix} \\ +\ldots 2\{{}_2 p_x + {}_4\, p_x + {}_6\, p_x + {}_8\, p_x + \ldots\} \end{bmatrix} \tag{168}$$

$$\mathbf{E}(T_x) = \int_0^\infty ({}_t\, p_x) dt = \frac{\Delta}{3} \begin{bmatrix} \{{}_0 p_x + {}_{\Omega-x}\, p_x\} + 4({}_1 p_x) + 4({}_3 p_x) + 4({}_5 p_x) + 4({}_7 p_x) + \ldots \\ +4({}_{\Omega-x-1} p_x) + \ldots 2({}_2 p_x) + 2({}_4 p_x) + 2({}_6 p_x) + 2({}_8 p_x) + \ldots \end{bmatrix} \tag{169}$$

$$\mathbf{E}(T_x)=\frac{\Delta}{3}\begin{bmatrix} {}_0p_x+4({}_1p_x)+2({}_2p_x)+4({}_3p_x)+2({}_4p_x)+4({}_5p_x)+2({}_6p_x) \\ +4({}_7p_x)+2({}_8p_x)+\ldots+4({}_{\Omega-x-1}p_x)+{}_{\Omega-x}\,p_x \end{bmatrix} \tag{170}$$

2.2 Presentation of results

Table 1: Sterling Mortality Table

x	$l_x(M)$	$l_x(F)$	$\mu_x(M)$	$\mu_x(F)$	$l_x\mu_x(M)$	$l_x\mu_x(F)$	$d_x(M)$	$d_x(F)$
0	100000	100000	0.000333	0.000269	33.300000	26.900000	587	495
1	99413	99505	0.000200	0.000141	19.882600	14.030205	40	35
2	99373	99470	0.000162	0.000116	16.098426	11.538520	27	23
3	99346	99447	0.000151	0.000121	15.001246	12.033087	23	17
4	99323	99430	0.000139	0.000122	13.805897	12.130460	18	13
5	99306	99418	0.000140	0.000117	13.902840	11.631906	16	12
6	99290	99406	0.000116	0.000104	11.517640	10.338224	15	12
7	99276	99394	0.000093	0.000100	9.232668	9.939400	14	12
8	99262	99382	0.000085	0.000100	8.437270	9.938200	12	11
9	99249	99371	0.000112	0.000101	11.115888	10.036471	10	10
10	99239	99361	0.000193	0.000132	19.153127	13.115652	9	10
11	99230	99351	0.000323	0.000185	32.051290	18.379935	10	10
12	99221	99341	0.000478	0.000247	47.427638	24.537227	15	12
13	99206	99330	0.000638	0.000309	63.293428	30.692970	25	16
14	99181	99314	0.000780	0.000360	77.361180	35.753040	40	21
15	99141	99293	0.000903	0.000398	89.524323	39.518614	56	28
16	99086	99265	0.000996	0.000404	98.689656	40.103060	70	33
17	99015	99232	0.001078	0.000403	106.738170	39.990496	84	38
18	98932	99194	0.001178	0.000401	116.541896	39.776794	94	40
19	98837	99154	0.001244	0.000414	122.953228	41.049756	103	40
20	98734	99114	0.001253	0.000425	123.7137025	42.123450	112	40
21	98623	99074	0.001221	0.000428	120.418683	42.403672	120	41
22	98503	99074	0.001162	0.000438	114.460486	43.394412	124	41
23	98379	98992	0.001104	0.000454	108.610416	44.942368	123	42
24	98257	98950	0.001071	0.000474	105.233247	46.902300	118	43
25	98139	98907	0.001052	0.000498	103.242228	49.255686	111	44

26	98028	98863	0.001083	0.000537	106.164324	53.089431	107	46
27	97921	98817	0.001135	0.000570	111.140335	56.325690	104	48
28	97817	98769	0.001178	0.000616	115.228426	60.841704	105	51
29	97713	98718	0.001240	0.000673	121.164120	66.437214	108	55
30	97604	98663	0.001326	0.000730	129.422904	72.023990	113	59
31	97491	98605	0.001417	0.000817	138.144747	80.560285	118	64
32	97373	98541	0.001533	0.000897	149.272809	88.391277	125	69
33	97248	98472	0.001664	0.000981	161.820672	96.601032	134	76
34	97114	98396	0.001804	0.001089	175.193656	107.153244	144	84
35	96971	98311	0.001962	0.001185	190.257102	116.498535	155	93
36	96815	98219	0.002135	0.001281	206.700025	125.818539	168	102
37	96647	98117	0.002328	0.001389	224.994216	136.284513	183	111
38	96464	98005	0.002523	0.001510	243.378672	147.987550	198	121
39	96266	97884	0.002728	0.001632	262.613648	159.746688	216	131
40	96050	97753	0.002961	0.001741	284.404050	170.187973	234	142
41	95816	97611	0.003203	0.001839	306.898648	179.506629	253	154
42	95563	97457	0.003452	0.001899	329.883476	185.070843	274	165
43	95290	97292	0.003744	0.002373	356.765760	230.873916	295	175
44	94994	97117	0.004020	0.000659	381.875880	64.000103	318	184
45	94676	96933	0.004241	0.002343	401.520916	227.114019	343	195
46	94333	96739	0.004432	0.004061	418.083856	392.857079	370	207
47	93963	96531	0.004619	0.002377	434.015097	229.454187	392	221
48	93571	96511	0.004841	0.002930	452.977211	282.777230	410	235
49	93161	96076	0.005134	0.003151	478.288574	302.735476	426	251
50	92735	95825	0.005507	0.003450	510.691645	330.596250	443	269
51	92292	95556	0.005968	0.003808	550.798656	363.877248	465	291
52	91827	95265	0.006514	0.004208	598.161078	400.875120	494	316
53	91333	94949	0.007121	0.004653	650.382293	441.797697	530	347
54	90803	94602	0.007778	0.005149	706.265734	487.105698	574	382
55	90229	94220	0.008498	0.005676	766.766042	534.792720	624	421
56	89605	93799	0.009282	0.006255	831.713610	586.712745	678	464
57	88927	93335	0.010135	0.006893	901.275145	643.358155	736	511

58	88191	92824	0.011091	0.007582	978.126381	703.791568	799	561
59	87392	92264	0.012166	0.008363	1063.211072	771.603832	866	614
60	86526	91649	0.013359	0.009219	1155.900834	844.912131	939	673
61	85587	90976	0.014664	0.010132	1255.047768	921.768832	1021	738
62	84567	90239	0.016116	0.011149	1362.881772	1006.074611	1109	808
63	83458	89431	0.017703	0.012289	1477.456974	1099.017559	1205	883
64	82253	88548	0.019396	0.013492	1595.379188	1194.689616	1308	964
65	80945	87585	0.021176	0.014755	1714.091320	1292.316675	1419	1052
66	79525	86533	0.023036	0.016071	1831.937900	1390.671843	1536	1147
67	77989	85386	0.025081	0.017530	1956.042109	1496.816580	1655	1243
68	76334	84143	0.027408	0.019211	2092.162272	1616.471173	1773	1341
69	74561	82801	0.029871	0.021003	2227.211631	1739.069403	1893	1442
70	72668	81359	0.032322	0.022842	2348.775096	1858.402278	2024	1556
71	70645	79803	0.034818	0.024750	2459.717610	1975.124250	2161	1678
72	68484	78125	0.037563	0.026880	2572.464492	2100.000000	2289	1799
73	66195	76326	0.040753	0.029387	2697.644835	2242.992162	2405	1917
74	63790	74409	0.044377	0.032227	2830.808830	2397.978843	2515	2036
75	61275	72373	0.048335	0.035317	2961.727125	2555.997241	2634	2170
76	58641	70203	0.052666	0.038701	3088.386906	2716.926303	2765	2320
77	55877	67883	0.057445	0.042532	3209.854265	2887.199756	2896	2477
78	52980	65406	0.062667	0.046916	3320.097660	3068.587896	3025	2636
79	49955	62770	0.068359	0.051720	3414.873845	3246.464400	3150	2801
80	46805	59969	0.074489	0.056799	3486.457645	3406.179231	3266	2,977
81	43539	56992	0.080967	0.062237	3525.222213	3547.011104	3369	3,158
82	40170	53833	0.087723	0.068112	3523.832910	3666.673296	3453	3,328
83	36717	50505	0.094574	0.074472	3472.473558	3761.208360	3509	3,479
84	33208	47027	0.101399	0.081330	3367.257992	3824.705910	3528	3,609
85	29680	43418	0.108146	0.088552	3209.773280	3844.750736	3502	3,716
86	26177	39702	0.114586	0.096028	2999.517722	3812.503656	3425	3,796
87	22753	35906	0.120612	0.103528	2744.284836	3717.276368	3293	3,839
88	19460	32067	0.126132	0.110846	2454.528720	3554.498682	3108	3,834
89	16351	28234	0.130908	0.117803	2140.476708	3326.049902	2875	3,770

90	13476	24463	0.134872	0.124157	1817.535072	3037.252691	2602	3,642
91	10874	20822	0.137993	0.129784	1500.535882	2702.362448	2299	3,446
92	8575	17376	0.139728	0.134222	1198.167600	2332.241472	1980	3,186
93	6596	14190	0.139729	0.137109	921.652484	1945.576710	1658	2,873
94	4938	11317	0.138362	0.138504	683.231556	1567.449768	1348	2,520
95	3590	8797	0.136017	0.138636	488.301030	1219.580892	1057	2,139
96	2533	6658	0.132820	0.137561	336.433060	915.881138	799	1,755
97	1734	4903	0.129075	0.135978	223.816050	666.700134	582	1,390
98	1152	3513	0.126201	0.134742	145.383552	473.348646	409	1,064
99	743	2449	0.123262	0.132959	91.583666	325.616591	277	786
100	466	1663	0.118526	0.129966	55.233116	216.133458	182	566
101	284	1097	0.115082	0.126664	32.683288	138.950408	117	396
102	167	701	0.107086	0.122753	17.883362	86.049853	72	268
103	95	433	0.104912	0.117475	9.966640	50.866675	43	176
104	52	258	0.100321	0.110594	5.216692	28.533252	25	111
105	27	147	0.077778	0.102154	2.100006	15.016638	14	67
106	14	80	0.113095	0.093333	1.583330	7.466640	7	39
107	6	41	0.066667	0.075203	0.400002	3.083323	4	21
108	3	20	0.033333	0.061667	0.099999	1.233340	2	11
109	1	9	0.016667	0.116667	0.016667	1.050003	1	5
110	0	4	0.000000	0.129167	0.000000	0.516668	0	2
111	0	2	0.000000	0.058333	0.000000	0.116666	0	1
112	0	1	0.000000	0.016667	0.000000	0.016667	0	0

3.0 Discussion of results

Figures 1 and 5 exhibit the principle of rectangularisation which has the following consequences. The ageing population may lead to increased demand for healthcare services particularly for age related conditions. The rectangularisation of the mortality curve may lead to an increase in age-related diseases such as Alzheimer's diseases and frailty. The ageing population may lead to increased caregiving burdens on families and social support systems. The rectangularisation of the mortality curve may have significant economic and social implications including changes in pension systems, healthcare financing and social security programs. In Figure 1, the trajectory of the survival curve forms an asymptote at 96 for the males whereas the trajectory of the survival curve forms an asymptote at age 98 for the females. The survival curve approaches a non-zero value as age increases indicating that a proportion of the population will never experience the event of interest.

Wavy kinks kinematics and the impact on instantaneous mortality rate intensities

The survival curve becomes asymptotic to a horizontal line, indicating that the mortality rate or hazard rate decreases to zero as age increases. Consequently, the property implies that there is a subset of the population that will survive indefinitely or at least for a very long time. Both mortality rates in Figs. 2 and 6 exhibit randomness around advanced age 108. Furthermore, the distribution of lives in accordance to duration is represented by the frequency curves in Figs. 3 and 7. The maximum age x_M at death can be estimated to read 84 in Figs. 3 and 7 respectively for humans. In the mortality table, this defines the age at death where the curve of death $l_x \mu_x$ is the highest. Where the current age x of the insured is $x < x_M$, then the most likely duration of life is the difference between the current age x and x_M. Using equation (154), the duration at which a life has an even probability of survival is given by

$$e^{-\int_0^t \left\{ \frac{[-45l_{x+s+1}+45l_{x+s-1}+9l_{x+s+2}-9l_{x+s-2}-l_{x+s+3}+l_{x+s-3}]}{60l_{x+s}} \right\} ds} = 0.5 \quad (155)$$

$$e^{-\int_0^t \left\{ \frac{[-45l_{x+s+1}+45l_{x+s-1}+9l_{x+s+2}-9l_{x+s-2}-l_{x+s+3}+l_{x+s-3}]}{60l_{x+s}} \right\} ds} = 0.5 \quad (156)$$

$$\int_0^t \left\{ \frac{[45l_{x+s+1}+45l_{x+s-1}+9l_{x+s+2}+9l_{x+s-2}+l_{x+s+3}-l_{x+s-3}]}{60l_{x+s}} \right\} ds = \log_e 0.5 \quad (157)$$

In Fig. 4, the number of deaths reach the maximum 3500 at age 84 while in Fig. 8, the number of deaths reach the maximum 3750 at age 87. Suppose the frequency curve drawn is such that the height of the ordinate corresponding to any age is proportional to the number of death d_x, then d_x defines the area included between the ordinates for ages x and $x+1$. Since

$$l_x = \sum_{y=x}^{\Omega-1} d_y = \int_0^{\Omega-x} l_{x+s}\mu_{x+s} ds \quad (158)$$

Then l_x will be the total area between the ordinate for the age x and the limit of the mortality curve.

In Table 1, the Male mortality rate is higher than the Female mortality rate because males are exposed to hazards. In Fig. 2 within $105 \le x \le 114$ and Fig. 6 within $105 \le x \le 119$, the trajectories exhibit sudden jumps or discontinuities in the mortality data, particularly between the affected age intervals. The performance of the interpolation is observed to be sensitive to outliers or unusual data points in the mortality rate dataset, potentially skewing results. The wavy kinks observed in the Figs. 2 and 6 could lead to increased longevity potentially resulting in a larger and older population. The irregular mortality pattern could lead to changes in healthcare costs potentially resulting in increased costs at certain ages but decreased at others. The wavy kinks could impact social security and pension systems potentially leading to changes in benefit structures and funding strategies. The irregular mortality pattern could lead to changes in insurance products and pricing potentially resulting in more complex and nuanced pricing strategies.

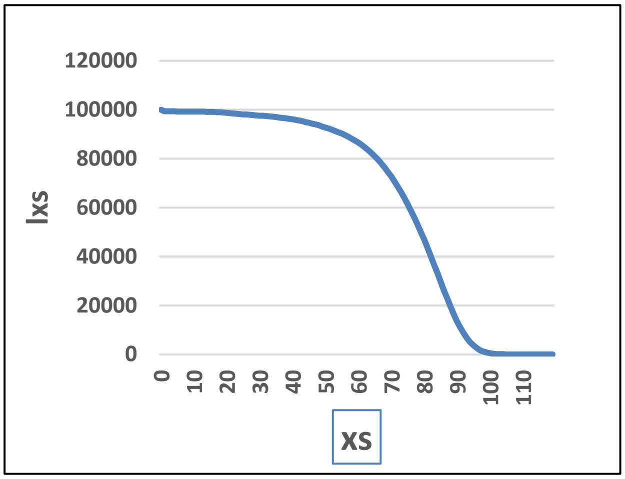

Figure 1: Male's Survival Curve

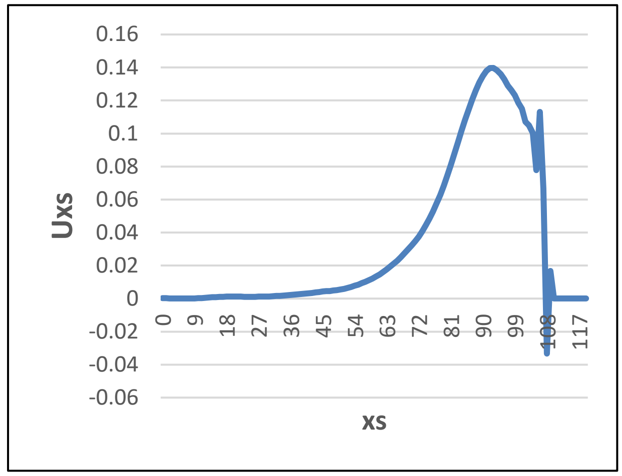

Figure 2: Male's mortality curve

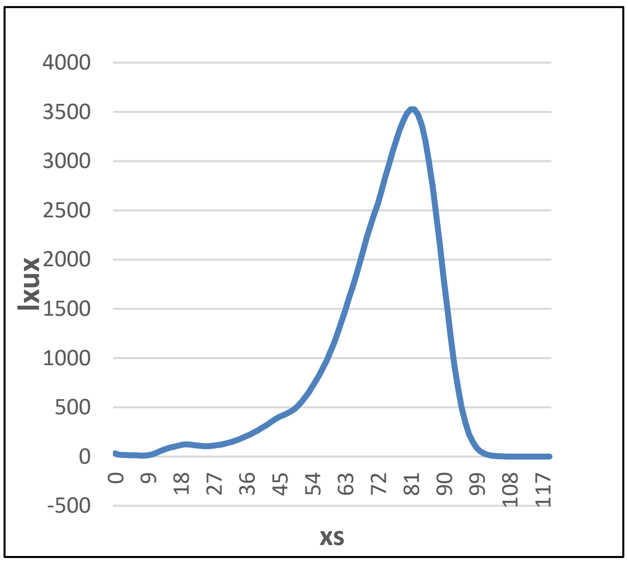

Figure 3: Male's Curve of Death

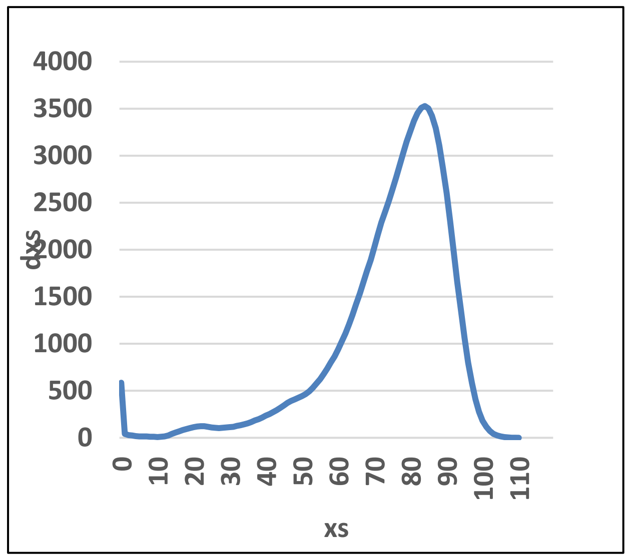

Figure 4: Male's Number of Death

In Table 1, the interpolated value for the year shows a significant increase between ages 9 and above or decrease from age 5 and 8 compared to surrounding years, this could indicate trends in the population's health such as an improvement or worsening in health conditions, access to healthcare or public health interventions. The rate of change between data points especially using a smooth curve can give insights into how mortality has changed over time. Although, comparing interpolated mortality rates across different periods may be challenging when the underlying data structures shift significantly over time. Continuous refinements and updates to mortality data are necessary to ensure the interpolation reflects current trends and realities accurately. The interpolated rates showing steep rise in the frequency curves could indicate a health crisis or worsening conditions. Where the interpolated values show a steady decline could suggest improvements in healthcare, disease prevention or public health policies.

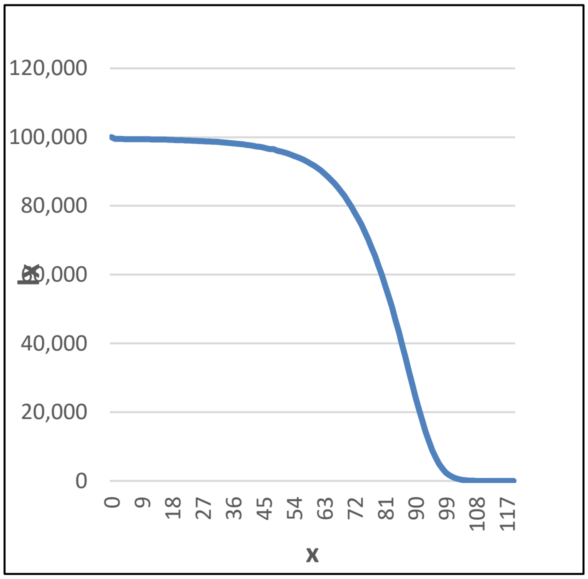

Figure 5: Female's Survival Curve

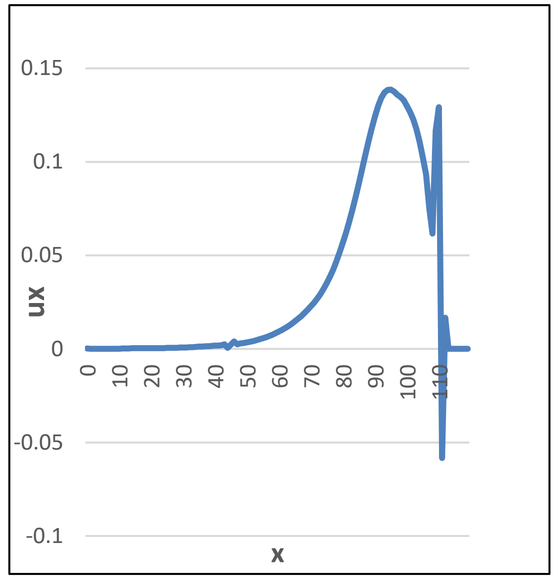

Figure 6: Female's Mortality curve

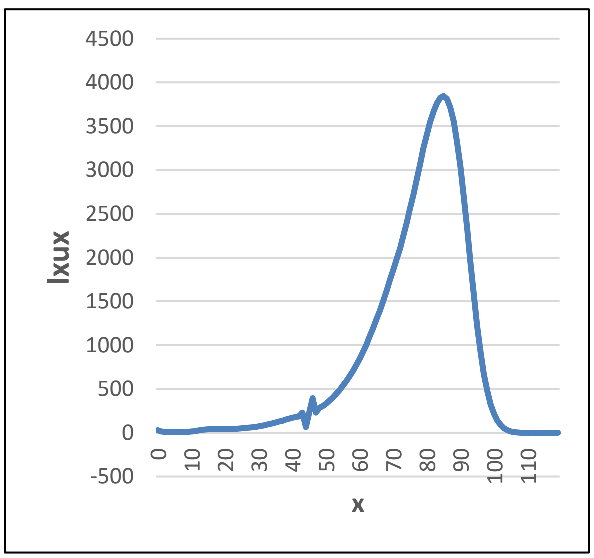

Figure 7: Female's Curve of Death

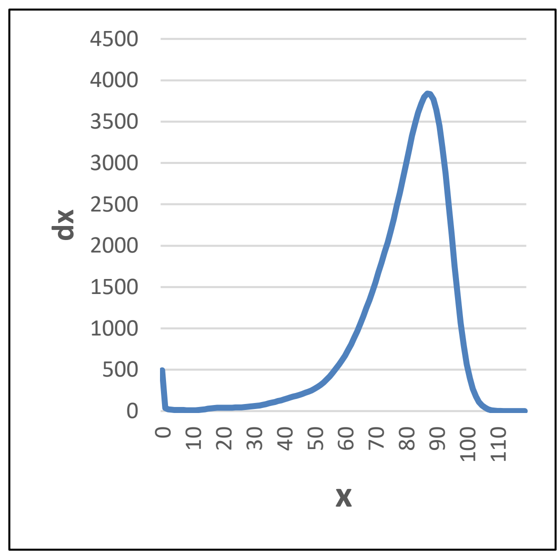

Figure 8: Female's Number of Death

The mortality interpolation can hence be used to forecast future mortality rates to inform public health decisions, resource allocation and policy initiatives aimed at reducing mortality. The regulators and life insurance offices can make informed underwriting decisions based on interpolated mortality rates which could represent the actual risk or prevalence of mortality in certain populations. The interpolation is capable of estimating mortality rates at intermediate ages or time points based on a set of observed mortality rates. This allows for the creation of a mortality model which can be used to predict future mortality rates. The results from the frequency curves show that the interpolation has a smoothing effect on the mortality rates which helps to reduce the impact of random fluctuations in the data. From the analysis, it is clear that the interpolation can handle sparse data which is common in mortality modelling. This means that the approach can be deployed to estimate mortality rates even when there is limited data available. As a result, the interpolation is robust to outliers which can help to curb the impact of unusual or irregular data points on the mortality model.

4.0 Conclusion

Mortality modelling based on Stirling interpolation is an innovative and promising approach that has the potential to revolutionize the field of mortality modelling. By using the interpolation to estimate mortality rates at intermediate age points, this approach can provide accurate estimates of mortality rates as traditional mortality models. The key benefits of mortality modelling based on the interpolation include improved accuracy, increased robustness, flexibility and simplicity. This approach can handle sparse data, it is robust to outliers and can be used to estimate mortality rates at any age point. Additionally, it can capture non-linear relationships between mortality rates and age which can lead to more accurate predictions of future mortality trends.

The insurance applications of mortality modelling based on the interpolation are numerous and diverse. It can be used in life insurance, annuities, pension funds and public health to estimate mortality rates and predict future mortality trends. This can help to inform policy and resource allocation and can lead to more accurate and reliable estimates of mortality rates. While there are limitations to this approach, such as the assumption of smoothness and sensitivity to the choice of parameters, these can be mitigated through careful selection of parameters and consideration of the data. Overall, mortality modelling based on Stirling interpolation is a powerful and flexible approach that has the potential to provide more accurate and robust estimates of mortality rates than traditional mortality models.

Future research should focus on exploring the applications of kernel smoothing which uses a weighted average of neighbouring data points to smooth out the mortality curve. Again, spline smoothing in the form of piecewise function can be used to smooth out the mortality curves and capture the wavy kinks. Actuaries should embark on data cleaning and preprocessing to ensure that mortality data is accurate, complete and consistent which helps to reduce the impact of kinks.

References

Amaku, M. Coutinho, F. A. B., Eboli, O., & Massad, E. (2021). Revista Brasileira de Ensinode Fisica. 43, e202110132-1- e202110132-9. https://doi.org/10.1590/1806-9126-RBEF-2021-0132

Dickson, D. C., Hardy, M. R., & Waters H. R. (2013). *Actuarial mathematics for life contingent risks*. 2nd ed, Cambridge University Press.

Elnour, K. A. E. A. A. (2018). On the calculus of Dirac delta function with some applications. *International Journal of Mathematics Trends and Technology. 56*(4), 258-270.

Elandt-Johnson, R., & Johnson, N. (1980). *Survival models and data analysis*. John Wiley.

Hossain, M. S., Islam, M. R., & Ali, M. K. (2023). Estimation and analysis of adult mortality and construction of life table for male population of Bangladesh. *Human Biology Review, 13*(1), 1-17.

Hudec, S. (2017). Modelling the force of mortality using local polynomial method in R. In *Proceedings of the 20th International Scientific Conference AMSE: Applications of Mathematics and Statistics in Economics* (pp. 217–226). Szklarska Poreba, Poland.

Gerber, H.U, Leung, B. P. K., & Shiu, E. S. W. (2003). Indicator function and Hattendorff theorem. *North American Actuarial Journal, 7*(1), 38-47.

Kostaki A., & Panousis V. (2001). Expanding an abridged life table. *Demographic Research, 5*(1), 1-22. https://doi.org/1040.4054/DemRes.2001.5.1

Kovacevic, M. S, Jovanovic, M. R., & Milosevic, M. (2021). On the calculus of Dirac delta function with some applications in classical electrodynamics. *Revista Mexicana de Fisica E 18*, 020205, 1-7. https://doi.org/10.31349/RevMexFisE.18.020205

Legua, M. P., Morales, I., & Ruiz, L. M. S. (2006). The Heaviside function and Laplace transforms. In *Proceedings of the 10th WSEAS International Conference on Applied Mathematics* (pp. 445–450), Dallas, Texas, USA, November 1–3, 2006.

McCutcheon, J. J. (1983). On estimating the force of mortality, *Transaction of the Faculty of actuaries 38*, 407-417.

Meghwal, R. S. (2020). Fundamental study of dirac delta function. *International Journal for Research in Applied Science & Engineering Technology. 8*(7), 918-924. https://doi.org/10.22214/ijraset.2020.32665

Neil, A. (1979). *Life contingencies*, *The Institute of Actuaries and Faculty of Actuaries*, special ed, London.

Ndu R, Nwuju K., & Bunonyo W. K. (2019). A comparative study of interpolation using the concept of mathematical norm with a proposed model. *International Journal of Scientific and Research Publications*, *8*(4), 37-53. https://doi.org/10.29322/IJSRP.9.04.2019.p8809

Ogungbenle G. M., & Adeyele J. S. (2020). Analytical model construction of optimal mortality intensities using polynomial estimation. *Nigerian Journal of Technology 39*(1), 25-35.

Ogungbenle G. M. (2023). Does Mccutcheon's mortality polynomial matrix actually account for mortality decline at ten years?. *Kathmandu University Journal of Science, Engineering and Technology, 17*(2), 1-8.

Pavlov, V., & Mihova, V. (2018). Application of survival model in insurance. *AIP Conference Proceedings, 2025*(1), 1–12. https://doi.org/10.1063/1.5064883

Putra, D. A., Fitriyati, N., & Mahmudi, M. (2019). Fit of the 2011 Indonesian Mortality Table to Gompertz's and Makeham's Law of using maximum likelihood estimation. *Indonesian Journal of Pure and Applied Mathematics*, *1*(2), 68-76. https://doi.org/10.15408/inprime.v1i2.13276

Rabbi A. M. F., & Karmaker S. C. (2013). Estimation of force of mortality from third degree polynomial of l_x . *European Scientific Journal, 9*(12), 233-240.

Venetis, J. (2014). An analytic exact form of the unit step function. *Mathematics and Statistics 2*(7), 235-237. https://doi.org/10.13189/ms.2014.020702