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Application of Variable Separation Technique in the Analysis of First-Order Differential Equations

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Abstract

First order differential equations are used to model things that change at a rate that depends on the thing itself. We looked at three models: how a population grows, how radioactive things decay and Newton's Law of Cooling. What we found out is that we can use a technique called separation of variables, on each of these models. This technique lets us transform the models into a form so we can integrate the variables on their own. The population growth solutions we got show that population growth happens fast when the growth rate is always the same. This is like what happens with things they get weaker really fast over time. When things cool down the difference in temperature gets smaller and smaller compared to the temperature around them. These results tell us that the method we split the problems not offers us clear solutions but also helps us comprehend the math behind population expansion and these other physical things, including radioactive chemicals and cooling processes. However, the study also identifies a limitation: the method is effective only when the differential equation can be represented in separable form. Equations involving non-separable terms require other approaches. Despite this restriction, the findings imply that separation of variables gives a systematic and transparent framework for solving a major class of first-order equations and acts as a conceptual bridge into more advanced analytical methods.

Keywords: differential equations, variables, modeling, law of cooling

Introduction

Differential equations is a really big part of modern mathematics. This is because differential equations can be used to understand a lot of things that happen in life. Differential equations help us make sense of things that are always changing like the cooling of a cup of tea or the number of people in a city going up or down or sickness spreading from person to person, like the flu. These things all happen slowly over time. Differential equations are a way to model and understand these kinds of changes. First-order differential equations are pretty simple. They only involve the derivative of the dependent variable. This dependent variable shows us the rate of change of something with respect to something. First-order differential equations are important because they are easy to understand and they are used a lot in life. That is why order differential equations are often used as the basis for mathematical modeling, in science and engineering. We use order differential equations to model things in science and engineering because they are straightforward and they really work (Wirkus & Swift, 2014).

The aim of this article's is not only to understand the mathematical technique but also to appreciate its usefulness in solving practical problems. Through step-by-step examples and relatable applications will show how mathematics helps explain and predict real-world events in a logical and systematic way.

Literature review

This work in this article is focusing on application of variable separation technique in the analysis of first order differential equations. Many books, websites, and others papers were used to complete this article's, which are mentioned in the reference section at the last of this project work. A first order differential equation is an equation that involves the first derivative of an unknown function but no higher-order derivatives. It typically takes the form: $\frac{dy}{dx} = f(x, y)$These equations appear in diverse areas such as motion, population dynamics. First-order equations can be linear or nonlinear, and different solution methods apply depending on the type **(Boyce, DiPrima, & Meade, 2017)**.

Separation of variables is among the oldest and best understood schemes in differential equations and method of separation of variables one of the simplest and most powerful methods to solve first-order ordinary differential equations. This technique is particularly useful for modeling exponential growth, cooling/heating processes, and population dynamics. It allows students and researchers to derive explicit solutions from initially coupled rate equations (Simmons, 2016). Real life applications of first order ODEs solved by separation of variables are widespread. For instance, in population dynamics, the Malthusian growth model can be expressed as $\frac{dP}{dt} = kP$, where P is the population at time t, and k is a constant growth rate. This equation can be solved by separating the variables and integrating both sides (Murray, 2002). Similarly, Newton's Law of Cooling, which models the temperature change of an object, is governed by a first-order differential equation of the form $\frac{dT}{dt} = -k(T - T_{env})$, which also yields to variable separation (Zill, 2018).

Methodology

The methodology of this study is based on the analytical technique of **separation of variables**, which provides exact solutions for a broad class of first-order differential equations. Variables Separation method works when the variables in the equation typically one independent and one dependent variable can be separated onto opposite sides of the equation. Once separated, the equation becomes easier to integrate, making it possible to find the general or particular solution. An equation is called separable if you can group all the terms involving x and dx on one side, and all the terms involving y and dy on the other side. The general form of such an equation is

$$\frac{dy}{dx} = f(x) \longrightarrow (i)$$

where f(x) is a known continuous function, the solution is straightforward. We can find y(x) by integrating both sides of the equation with respect to x. This gives

$$y(x) = \int f(x)\, dx + c \longrightarrow (ii)$$

where c is an arbitrary constant and the integral sign denotes any single antiderivative of f(x). We now see that this procedure can be applied to a broader class of differential equations, known as separable equations, having the form

$$\frac{dy}{dx} = \frac{f(x)}{g(y)} \longrightarrow (iii)$$

Clearly, any separable equation reduces to the simpler form in Eq. (i). To solve a separable differential equation, we rewrite it as

$$g(y)\, dy = f(x)\, dx \longrightarrow (iv)$$

At this point, we can observe that the left-hand side of the equation involves only the variable y and its derivative, while the right-hand side involves only x, hence the name “separated equation.” We can now solve the separated equation by simply integrating each side of the equation with respect to x, getting

$$\int g(y)\, dy = \int f(x)\, dx + c$$

in order to emphasize the constant of integration c (Farlow, 2006)**.**

Example 3.1: Solve

$$(1 + y^2)dx + y(1 + x^2)dy = 0$$

Solution: Given

$$(1 + y^2)dx + y(1 + x^2)dy = 0$$

$$or, \quad y(1 + x^2)dy = -x(1 + y^2)dx$$

$$or, \quad \frac{y}{1 + y^2}dy = -\frac{x}{1 + x^2}dx$$

$$or, \quad \frac{y}{1 + y^2}dy + \frac{x}{1 + x^2}dx = 0$$

Integrating, we get

$$\int \frac{y}{1 + y^2}dy + \int \frac{x}{1 + x^2}dx = 0$$

or, $\log(1+y^2)+\log(1+x^2)=\log c$

or, $\log(1+y^2)(1+x^2)=\log c$

$(1+y^2)(1+x^2)=c$

Which is the required solution.

Applications

First order equations are often used to model real life situations involving rates of change, like speed, population growth or decay, heat flow, and fluid movement. These models help us understand how things change over time in simple or more detailed ways.

Population Growth and Decay Phenomenon

A very common observation about 'population growth' is that unless constrained by environmental or other limitations, populations (human beings, bacteria, plants and so on) tend to grow at a rate which is proportional to the size of the population. Larger the population, higher is the growth rate.

To convert it into a mathematical problem, let us assume that $x = x(t)$ is the population at time t. Also, let us assume that at time $t = 0$, population is $x = x_0$ At any point of time, the rate of increase of the population with respect to time is dx/dt. According to the given situation,

$$\frac{dx}{dt} \propto x$$

So that

$$\frac{dx}{dt} = kx \text{ (growth equation)}$$

$$\frac{dx}{dt} = -kx \text{ (decay equation)}$$

where k is a constant of proportionality. Since the direction of change is positive (as larger the population, higher is the growth), k is positive. Usually it can be determined experimentally. Thus, we are left with $\frac{dx}{dt} = kx,\ x(0) = x_0$ (Sharma, 2010).

Solving the Growth and Decay Equations

There are two main ways we can solve growth and decay equations: one is by using something called the integrating factor method, and the other is by using the method of separation of variables. We carry out the following steps

Using growth equation

$$\frac{dx}{dt} = kx$$

Solve the differential equation

$y' = kx$

$or, \frac{y'}{x} = k$

$or, \int \frac{y'}{x}\,dt = \int k\,dt$

$or, \int \frac{1}{x} dx = \int k dt$

$or, ln\, x = kt + c_1$

$or, x(t) = e^{kt} e^{c_1}$

$\therefore x(t) = Ce^{kt} \longrightarrow (2)$

The general solution $x(t) = Ce^{kt}$ of the growth equation is called the exponential growth curve. One can obtain the solution of the decay equation using the same steps, getting the exponential decay curve $x(t) = Ce^{-kt}$

Newton's Law of Cooling

Heat may be transferred between and within bodies by conduction, convection or radiation. The case of conduction typifies the more general type of transport problem in which matters, energy or charge is transported across a region of space in some well-defined manner. A simple example of heat transfer is provided by Newton's law of cooling, which states that the rate at which an object cools is proportional to the difference between the temperature at the surface of the body, and the ambient air temperature. Thus, if is the surface temperature at time and T_a is the ambient temperature, then

$$\frac{dT}{dt} = -k(T - T_a) \longrightarrow (1) \quad T(0) = T_0$$

where $k > 0$ is some experimentally determined constant of proportionality, and T_o is the initial temperature. The equation (1) is separable and separating variables gives us

$$\frac{1}{T - T_a} dT = -k\, dt$$

Integrating both sides, we get

$$ln(T - T_a) = -kt + C$$

$or, \; T = C_1 e^{-kt} + T_a$, where $C_1 = e^C$

Applying the initial condition implies that

$$T_0 = C_1 + T_a \Rightarrow \; C_1 = T_0 - T_a$$

Therefore, the solution of the IVP is

$$T = (T_0 - T_a)e^{-kt} + T_a \longrightarrow (2)$$

In many real situations such a simple assumption for dT/dt must be modified to more complicated expressions, leading to more difficult differential equations (Cox, 1996).

Radioactive Decay

Radioactive decay is a process in which unstable atomic nuclei transform into more stable forms by emitting particles or energy. This decay happens naturally over time and follows a predictable pattern. A key characteristic of radioactive decay is that the rate at which a substance decays depends directly on how much of the substance is currently present. This relationship is an example of a first order differential equation and can be solved using the method of variable separation (Boyce, DiPrima, & Meade, 2017).

Mathematical Modeling

If N(t) represents the quantity of radioactive material at time t, then the rate of change of N over time can be described by

$$\frac{dN}{dt} = -kN$$

Here, k is a positive constant that represents the decay rate specific to the material. The negative sign indicates that the substance is decreasing. This is a first-order linear differential equation and can be solved by the method of variable separation

$$\frac{dN}{N} = -kdt$$

$$\text{or,} \quad \int \frac{dN}{N} = -k \int dt$$

$$\text{or,} \quad \ln N = -kt + C$$

Solving for N, we get the general solution

$$N(t) = N_0 e^{-kt}$$

where N_0 is the initial quantity at $t = 0$, and e is the base of the natural logarithm. This model reveals that the decay of radioactive substances follows an exponential decline. Moreover, the half life $T_{1/2}$ the time it takes for half of the substance to decay relates to the decay constant k by

$$T_{1/2} = \frac{\ln 2}{k}$$

This relationship is crucial in calculating how long a substance will remain significantly radioactive, which has applications in nuclear medicine, radioactive waste management, and carbon dating (Kreyszig, 2010).

Results and Discussion

In this section, we present the results obtained through the application of the variable separation technique to different categories of first-order differential equations. For each model, the solution process is outlined, graphical behavior is described, and the implications of the findings are discussed in the context of existing studies.

Example 5.1: (Population Growth)

During the early stages of the COVID-19 outbreak in Nepal, the number of infected individuals was observed to double every 7 days. If there were 150 infected individuals on a certain day, how many infected individuals would there be after 21 days, assuming the number grows exponentially?

Solution: Let

$x(t)$ be the number of infected individuals at time t days after the initial time.

$x(0) = 150$

$x(7) = 300$ (since it doubles in 7 days)

The growth is proportional to the current population

$$\frac{dx}{dt} = kx \longrightarrow (1)$$

Solve the differential equation

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$$y' = kx$$

$$\text{or,} \quad \frac{y'}{x} = k$$

$$\text{or,} \quad \int \frac{y'}{x} dt = \int k\, dt$$

$$\text{or,} \quad \int \frac{1}{x} dx = \int k dt$$

$$\text{or,} \quad \ln x = kt + c_1$$

$$\text{or,} \quad x(t) = e^{kt} e^{c_1}$$

$$\therefore x(t) = Ce^{kt} \longrightarrow (2)$$

When $x(0) = 150, t = 0$

From equation (2), now we can write,

$$150 = Ce^0 \Longrightarrow C = 150$$

$$x(t) = 150e^{kt}$$

When $x(7) = 300, t = 7$

$$300 = 150e^{kt} \Longrightarrow 2 = e^{7k}$$

$$\text{or,} \quad \ln(2) = 7k$$

$$k = \frac{\ln(2)}{7} \approx 0.099$$

So the solution is

$$x(t) = 150e^{0.099t}$$

Now we find the number of cases after 21 days

$$x(21) = 150e^{0.099\times21} = 150e^{2.079} \approx 1200$$

After 21 days, the number of COVID-19 infected people in Nepal would be approximately **1,200** if the growth rate continued unchanged. Number of infected people during a time interval.

Table 5.1

Growth of Infected Population Over Time

Period of time	Number of infected people
0	150
7	300
14	600
21	1,200

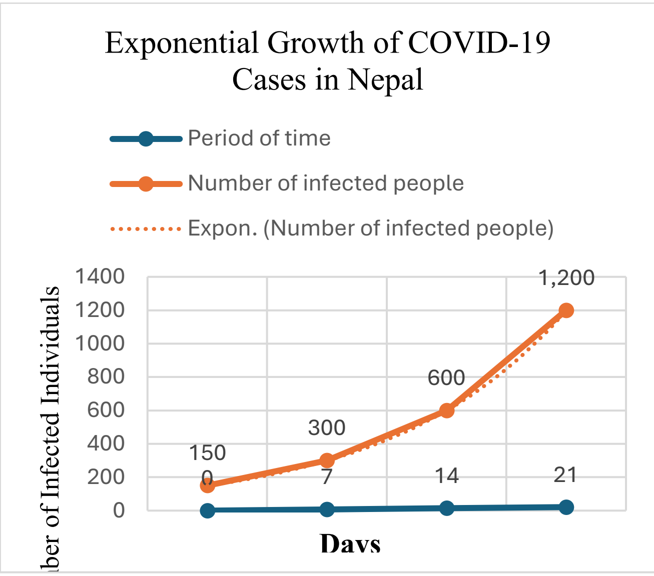


Figure 5.1: Exponential Growth of COVID-19 Cases in Nepal

The graph shows how the number of COVID-19 infections increased in Nepal during the early stage of the outbreak. You can see that the number of cases started at 150 and doubled every 7 days. After 21 days, the number reached about 1,200, which is marked clearly on the graph.

Example 5.2: (Newton's Law of Cooling)

Suppose that when Miss Scarlett discovered Mr. Boddy's body in the conservatory at noon, its temperature was 82 °F Two hours later, the temperature of the corpse was 72 °F. If the temperature of the conservatory was 65 °F what was the approximate time of Mr. Boddy's death?

Solution: Let T(t) denote the temperature of the body at time t, where T(0) represents the temperature of the body when it is discovered and T(2) represents the temperature of the body 2h after it is discovered. In this case, we have

$\boldsymbol{T_{room}}$ (ambient temperature) = 65°F

$\boldsymbol{T_1}$ (temperature at discovery at noon) = 82°F

$\boldsymbol{T_2}$ (temperature two hours later) = 72°F

Time between T_1 and T_2 = 2 hours

Normal body temperature = 98.6°F

Now using Newton's law of Cooling formula and substituting these values into

$$T(t) = T_{room} + (T_{initial} - T_{room})e^{-kt}$$

$$or, T(t) = 65 + (82 - 65)e^{-kt} = 17e^{-kt} + 65$$

Using $T(2) = 72$, at $t = 2$ we solve the equation

$T(2) = 17e^{-2k} + 65 = 72$

$or, 72 - 65 = e^{-2k} \Longrightarrow 7 = 17e^{-2k}$

$or, e^{-2k} = \frac{7}{17}$

$or, -2k = \ln\left(\frac{7}{17}\right)$

$or, k = -\frac{1}{2} \times \ln\left(\frac{7}{17}\right) = -\frac{1}{2}\ (-0{,}8873)$

$\therefore k = 0.4437$

Now use the same formula to find how long before noon Mr. Boddy died (temperature was 98.6°F then)

$82 = 65 + (98.6 - 65)e^{-kt} \Longrightarrow 17 = 33.6\ e^{-kt} \Rightarrow e^{-kt} = \frac{17}{33.6}$

$or, -kt = \left(\frac{17}{33.6}\right)$

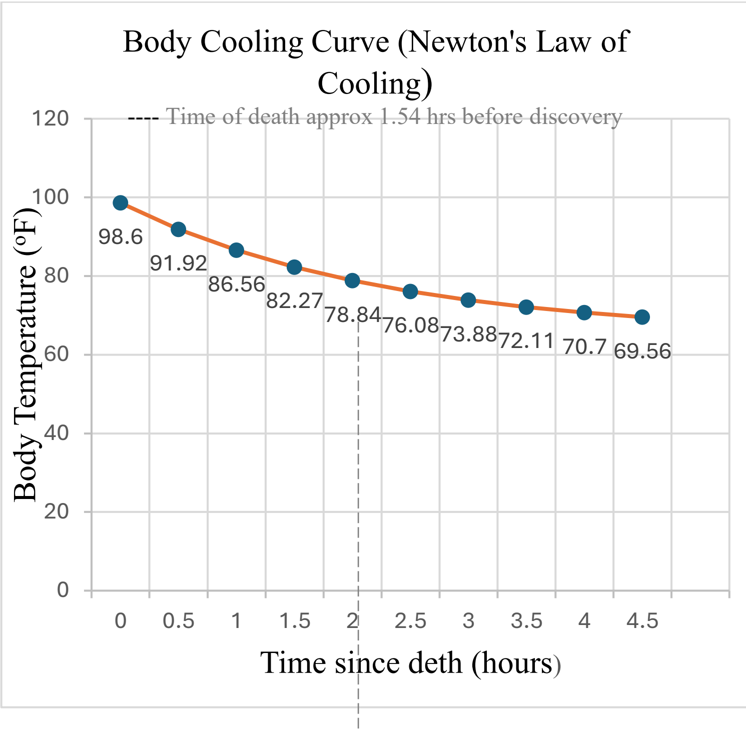
$\therefore t = -\frac{1}{k} \times \ln\left(\frac{17}{33.6}\right) = -\frac{1}{0.4437} \times \ln(0.50595) = 1.54$

Mr. Boddy likely died approximately 1.54 hours before noon, i.e., around 10:28 AM.

Table 5.2

Estimated Body Temperature vs. Time Since Death

Time since death (hours)	Temperature (°F)
0.0	98.60
0.5	91.92
1.0	86.56
1.5	82.27
2.0	78.84
2.5	76.08
3.0	73.88
3.5	72.11
4.0	70.70
4.5	69.56



The graph demonstrates that Mr. Boddy's body temperature dropped from a normal 98.6°F to 82°F over roughly 1.5 hours, allowing investigators to estimate that his death occurred around 10:28 AM.

Example 5.3: (Radioactive Decay)

A sample contains 100 grams of Carbon-14. Given that the half-life of Carbon-14 is 5730 years, determine how much remains after 10,000 years. Also, plot a graph of the decay over a time span of 0 to 10,000 years.

Solution: Given,

Initial amount (N_0) = 100 grams

Half-life ($t_{1/2}$) = 5730 years

Time elapsed (t) = 10,000 years

We start with the differential equation for radioactive decay

$$\frac{dN}{dt} = -kN$$

The is a first-order linear differential equation and can be solved by the method of variable separation

$$\frac{dN}{N} = -kdt$$

$$or, \quad \int \frac{dN}{N} = -k \int dt$$

$$or, \quad \ln N = -kt + C$$

Solving for N, we get the general solution

$$N(t) = N_0 e^{-kt}$$

Use Half-life to Find Decay Constant k

$$T_{1/2} = \frac{ln\,2}{k}$$

$$or, \qquad k = \frac{ln\,2}{T_{1/2}} = \frac{ln\,2}{5730} \approx 0.00012097$$

Compute Remaining Amount After 10,000 Years, Now

$$\mathrm{N(t)} = \mathrm{N_0 e^{-0.00012097 \times 10000}} \approx 29.7 \text{ grams}$$

After 10,000 years, approximately 29.7 grams of Carbon-14 remains from the original 100 grams.

Discussion

The results obtained from these examples demonstrate the versatility of the variable separation technique in solving first-order differential equations across multiple domains. All solutions exhibit exponential behavior, either growth or decay, which matches theoretical predictions and aligns with findings from classical and contemporary studies in applied mathematics, physics, and engineering. The comparison with existing literature confirms that the method of separation of variables provides exact analytical solutions where other techniques (such as numerical methods) might only yield approximations. However, the limitation lies in its applicability only to equations that can be expressed in separable form.

Conclusions

This study demonstrates that variable separation technique is an effective and reliable method for solving first-order differential equations in various real-world applications, such as population growth, radioactive decay, and heat transfer. The method provides exact analytical solutions, offering clear insights into system behavior and confirming expected exponential growth or decay patterns. Its simplicity and accuracy make it preferable to purely numerical approaches, allowing deeper understanding and practical modeling. Future research could explore the application of variable separation to coupled or higher-order differential equations, stochastic systems, or hybrid analytical-numerical methods, further expanding its utility in both theoretical and applied contexts.

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