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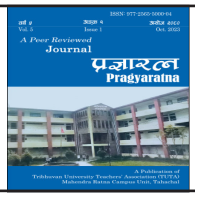

Teacher Belief, Knowledge, and Practice: A Trichotomy of Mathematics Teacher Education

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Abstract

This theoretical paper explored and examined three key concepts: the beliefs, knowledge, and practices of math teachers. In order to derive some pedagogical and research implications, I conducted a literature review on teacher beliefs, knowledge, and practices. Three themes emerged from the reviews: teacher beliefs influence practice but in a supportive environment; teacher beliefs influence teacher change (development); and teacher knowledge encompasses more than the whole of content, pedagogical, and technological knowledge and their potential combinations.

Keywords: *Mathematics teacher beliefs, teacher knowledge, teacher practices, professional goal, epistemic teacher.*

Introduction

De facto teaching

Reflecting on my experiences in mathematics and mathematics education classes, I observe a diversity of teaching and learning practices that vary across different grade levels, contexts, and educational environments. Initial encounters with mathematics classes predominantly centered the pedagogical approach around the teacher, adopting an instrumentalist view (Siswono et al., 2019). A singular method, without any flexibility, directed the mathematical problems entirely (Algani & Eshan, 2023). During my third-grade examination, I exhibited a disregard for both long division and short division methods. Instead, I arrived at the correct answers through direct division, but I received no credit for my efforts. I found myself devoid of the means to articulate my thoughts, lacking justification for my stance, and constrained to comply solely with the directives provided by the instructor (Bray, 2011). I recall that the mathematics problems we tackled often seemed devoid of meaning to us, simply reproducing senseless facts without mastery of underlying concepts and skills (Little, 2009). I acquired knowledge of operations and procedures without delving into the underlying concepts and significance of our actions. The schools did not provide any visual aids or tangible examples of operational mathematics.

Everything we did that also involved solving problems from textbooks through repeated practice. Use of manipulatives and visual materials is important in elementary and middle school mathematics (Furner, 2014). However, throughout my high school years, I assumed the role of an observer, listener, and merely a note-taker within the instrumental classroom, where learning meant following the instructor's cookbook to solve problems without knowing the reasons and meanings (Siswno et al., 2019). This is not to place blame on any individual, but rather to highlight a systemic issue with the teaching and learning of mathematics, which lacks reflective practices, discussions, and open challenges for students. Teaching was a ritual of delivering mathematical facts and algorithms through an opaque method to the students' minds (Aragon et al., 2024). In general, people believed that mathematics was a challenging subject in school, requiring rote memorization (Ernest, 1989 a, b). Ritual drills and practice activities in the classroom enacted this belief throughout my school to university level mathematics classes.

De jure learning

I acquired deeper knowledge of mathematics primarily through self-study or collaboration with peers. During my early years in elementary school, I cannot recall specific lessons imparted by my teacher; however, I distinctly remember the mathematical concepts I constructed through interactions with my peers (Chapman, 2004). I sought to explore mathematics in advance of formal classroom instruction. This provided me with a chance to adequately prepare prior to the commencement of a lesson. I previously engaged with problem examples and formulas from the textbooks. I engaged in discussions about them with my peers. My peers recognized the value of being well-informed prior to the class, and we engaged in a collaborative effort to educate and learn from one another (Jarvela et al., 2016). We would often administer examinations for ourselves, meticulously reviewing one another's answer sheets and assigning scores accordingly. Our instinct to learn independently of the teachers' instructions guided our teamwork. In a sense, it was a self-regulated learning practice (Dure & Okeke, 2021), where a small circle of highly motivated learners helped each other (Biber, 2022). There was a domination of group cohesion over individual competition (Zamecnik et al., 2022). My friends recognized the efficacy of collaborative study, exchanging knowledge, and distributing roles and responsibilities through a reciprocal teaching approach (Aslam et al., 2021). This facilitated the formation of a learning circle with my peers, spanning from elementary education through to my pursuit of a graduate degree (master of education) in mathematics education (Pattison et al., 2017). My companions served as my genuine mentors throughout my educational journey, guiding me from the basics of skip counting to the complexities of operations research, game theory, real and complex analysis, differential geometry, and more with peer acceptance, mutual respect, and collaboration in each other's learning (Klang et al., 2021).

De novo research

This section presents my attempt to systematically delineate and develop my research agenda during graduate school and beyond. My preliminary research experience pertains to my master's thesis on high school educators' perspectives of optional mathematics in Kathmandu. I possess a limited recollection of the research I undertook in alignment with previous thesis studies at Tribhuvan University. I followed the research methods and processes based on the prior theses at the Central Library of Tribhuvan University. I had minimal expertise in research. As a result, I had to rely on basic descriptive statistics to showcase my findings. Later on, during my M.Phil. studies in education at Kathmandu University, I gained a more understanding of research methodologies and procedures in general and autoethnography in particular (Belbase et al., 2008). A subsequent work, utilizing laboratory and field tests on coalbed methane gas co-produced water at the University of Wyoming, offered me a perspective on pure scientific inquiry (Belbase et al., 2013). Another subsequent research project focused on preservice mathematics teachers' beliefs about teaching geometric transformations using Geometer's Sketchpad (Belbase, 2015), culminated in my PhD dissertation. Each activity provided me with a unique opportunity to analyze literature, develop an understanding of the selected discipline, and confront the conflict between my inner exploration and the established corpus of knowledge. The various research domains enhanced my comprehension of interdisciplinary research methodologies and practices in education and the environment (Bom et al., 2023; Dangi et al., 2018), fostering diverse research paradigms, possibilities, and opportunities (Ellis & Berry III, 2005; Kivunja & Kuyini, 2017). I employed a variety of research methods, such as quantitative surveys (Belbase, 2020; Belbase et al., 2020; Belbase, 2018), qualitative autoethnography (Belbase et al., 2008), experimental bench-scale and field experiments (Belbase et al., 2013), task-based clinical interviews (Belbase, 2017), literature reviews (Belbase, 2019; Belbase et al., 2022), and many others (Acharya et al., 2022). Each of these studies created a set of beliefs and ways of doing research across the diverse research paradigms shifting and reorganizing theories, methods, and practices. My classroom practices of teaching mathematics and research methods intricately support these research practices. My experiences in learning, teaching, and diversity in research provided a foundation for understanding the diversity of beliefs, knowledge, and practices and analyzing how they inform and influence each other.

Teacher Beliefs and Practices

Several critical elements influence how an educator engages in mathematics instruction and comprehension within the classroom. One of these elements is the educator's beliefs (Beswick, 2006). These beliefs pertain to the nature and function of mathematics, the learning and teaching of mathematics, and the assessment practices (Berk & Cai, 2019; Purnomo, 2017). A significant factor that impacts one's practice is the teacher's mental schemas, which form a system of beliefs regarding mathematics and the teaching and learning of mathematics (Ernest, 1989a,b). Ernest delineates three fundamental belief components held by mathematics teachers: their understanding of the *essence of mathematics*, their framework for the *nature of mathematics instruction*, and their perspective on the *process of learning mathematics*. He puts forth a framework that elucidates the connections between beliefs and their influence on practical application. This model represents the intricate interconnections between perspectives on the nature of mathematics, as well as the theoretical and practical approaches to learning and teaching mathematics. He posits that “mathematics teachers' beliefs exert a significant influence on the practice of teaching' (Ernest, 1989a, p. 254). He further emphasizes that “values as well as beliefs and philosophies play a key role in determining the underlying images and philosophies embodied in mathematics classroom practice' (Ernest, 1991, p. 13).

There are numerous scholarly investigations on teacher beliefs and classroom practices (e.g., Batista, 1994; Belbase, 2019; Belbase, 2015; Beswick, 2012; Peterson et al., 1989; Ernest, 1989a, b; Foss & Kleinsasser, 1996; Handal & Herrington, 2003; Kagan, 1992; Kane et al., 2002; Lammassaari et al., 2024; Lamichhane & Belbase, 2017; Marshman & Goos, 2019; McGinnis & Parker, 2001; Pajares, 1992; Perry et al., 1996; Prawat, 1990 & 1992; Prnomo, 2017; Richardson, 1996; Schoen & LaVenia, 2019; Stipek et al., 2001; Thompson, 1992; Tillema, 1995; Weinstein, 1990) that examine the beliefs of educators broadly, with a specific focus on their perspectives regarding mathematics and other disciplines. The number of studies and publications on teacher beliefs in general and mathematics teachers in particular is growing rapidly in the last few decades (Wang et al., 2023). I wish to offer an analysis of some scholarly contributions pertaining to teacher beliefs within the realm of mathematics education and how they affect their practices in the classroom.

Characterization of Teacher Beliefs

The existing research literature classifies beliefs about mathematics learning and problem solving into four distinct categories (Op't Eynde et al., 2002):

(i) Perspectives regarding the essence of mathematics, the process of acquiring mathematical knowledge, and the approach to problem-solving
(ii) Perspectives about oneself regarding the acquisition of mathematical knowledge and the resolution of problems, specifically in terms of motivational beliefs.
(iii) Perspectives on the pedagogy of mathematics, the societal framework surrounding mathematical education, and the intricacies of problem-solving.
(iv) Epistemological beliefs, referring to convictions regarding the essence of knowledge and the mechanisms of understanding.

I assert that the subject matter and methodology, self-assurance, contextual factors (such as class or community), and one's philosophical perspective interconnect these classifications of beliefs. This classification of beliefs undoubtedly aids in organizing the examination of beliefs; however, it does not constitute a comprehensive system of categorization.

Op't Eynde et al. (2002) assert that a collaborative endeavor to create a thorough classification of all students' beliefs related to mathematics is lacking. The discourse explores the fundamental essence and underpinnings of beliefs, which are psychological constructs about the world considered as truths, and the unexamined acceptance of sensory experiences as a result of social existence. The conversation also examines the interplay between belief and knowledge, the significance of core beliefs over peripheral ones, the psychology-driven nature of belief changes, and the role of beliefs in areas such as ego enhancement, self-protection, and personal and social regulation. Students' beliefs regarding mathematics are characterized as: “Students' mathematics-related beliefs are implicitly or explicitly held subjective conceptions students hold to be true about mathematics education, about themselves as mathematicians, and about the mathematics class context' (Op't Eynde et al., 2002, p. 27). To define belief or a belief system is to engage in a profound exploration of the underlying framework. In my view, while definitions can aid in grasping certain facets of beliefs, the notion of belief as subjective interpretations that individuals regard as true presents a significant shortcoming. We can determine the values one holds toward others, not the veracity of a belief. Valuing a concept or an idea does not necessarily imply that we assert its truthfulness.

Furinghetti and Pehkonen (2002) employed nine distinct characterizations of beliefs to examine the agreements and disagreements among 18 mathematics educators, aiming to elucidate their comprehension of beliefs. The nine categories include different ways of judging a group of objects, people's subjective knowledge, ideas connected to larger mental frameworks, beliefs about the existence of entities and different realms, people's undeniable personal realities, their consistent subjective knowledge, which encompasses emotions, their own empathies and sentiments, their conscious or subconscious inclinations related to the field of mathematics, and their long-lasting, ingrained tendencies to react to certain stimuli in certain ways. (Furinghetti & Pehkonen, 2002). The panelists in the study found that several of their disagreements stemmed from terms such as incontrovertible, stable, conception, and the relationship between beliefs and knowledge. Their agreements corresponded to the foundational aspects of beliefs, including context, personal identity, and emotional influence, as well as the impact of these beliefs on individual behavior and responses.

Furinghetti and Pehkonen (2002) further posit that "contextualization and goal-orientation render the characterization an efficient one.' Furenghetti & Pehkonen (2002) assert that "the boundary between affective factors and beliefs is often fuzzy' (McLead, 1992, as cited in p. 53). They additionally highlight the depiction of beliefs as conscious and unexamined beliefs that profoundly influence an individual's expectations, engagement, and contributions within the mathematics classroom. I concur that convictions influence both the cognitive and emotional spheres, and as a result, this impact extends to behaviors.

Leder and Forgasz (2002) examine the prevalent definitions of beliefs as well as the methodologies employed to measure them broadly and specifically within the realm of mathematics education. Despite the elusive nature of universal agreement on the definition of belief, they assert the intrinsic relationship between beliefs, attitudes, and values (Leder & Forgasz, 2002, p. 96). Engaging in a discussion about the summary of methods for measuring attitudes and beliefs is worthwhile (Leder & Forgasz, 2002). The beliefs in mathematics education research include an "integrated system of personalized assumptions about the nature of mathematics, of students, about learning, and about teaching' (Artzt, 1999, as cited in Leder and Forgasz, 2002). Artzt's (1999) research centered on the premise that the knowledge, beliefs, and objectives of educators significantly shape their instructional methodologies (Leder & Forgasz, 2002). While their assertion is not novel, their reasoning gains clarity through the manner in which they intertwine beliefs with knowledge and practice.

A study by Perkkila (2003) looked into the beliefs and ways of teaching mathematics of six primary school teachers. A Likert-scale belief questionnaire asked them about their math beliefs, how they learned and taught math, and how they used math textbooks. Perkkila embraced Ernest's (1989a, b) classifications of perspectives on mathematics—the instrumentalist perspective, the Platonist perspective, and the problem-solving perspective—for the examination of educators' beliefs regarding mathematics, the process of learning mathematics, and the practice of teaching mathematics. It proposed that "teacher education programs ought to place greater emphasis on the students' own thinking and reflection' (Perkkila, 2003, p. 7). The concept of teacher beliefs, as framed by the instrumentalist, Platonist, and problem-solving perspectives, does not comprehensively encompass the beliefs of all educators. It seems imprudent to attempt to classify every educator's beliefs within these rigid frameworks.

Belbase (2019) provided a comprehensive analysis of mathematics teacher beliefs through three distinct lenses: relational, institutional, and praxis. Relational lens focuses on the beliefs of mathematics teachers in relation to their classroom environment and the dynamics of the student-teacher relationship. It emphasizes how teachers' beliefs are shaped by their interactions with students and the specific context of their classrooms (Belbase, 2019). For instance, a teacher's belief in the importance of fostering a supportive and engaging classroom atmosphere can significantly influence their teaching practices and strategies (Belbase, 2019). Institutional lens examines the influence of institutional efforts on shaping teacher beliefs (Belbase, 2019). It considers how schools and educational systems promote either reform-oriented or traditional beliefs among teachers (Belbase, 2019). This includes the impact of institutional policies, professional development programs, and curricular standards on teachers' perspectives (Belbase, 2019). Teachers' beliefs about content knowledge, pedagogical knowledge, and technological knowledge, as well as their application in the classroom, are central to this lens (Belbase, 2019). For example, a school that emphasizes the integration of technology in teaching may shape teachers' beliefs about the importance of using digital tools in mathematics instruction (Belbase, 2019). Praxis lens delves into teachers' beliefs about the theories and practices of mathematics teaching, learning, and assessment (Belbase, 2019). It explores how teachers view and implement theoretical frameworks in their classroom practices (Belbase, 2019). The praxis lens highlights the connection between theory and method, examining how teachers embrace and integrate educational theories into their teaching methods (Belbase, 2019). For instance, a teacher who values constructivist theories may implement student-centered learning activities that encourage exploration and critical thinking (Belbase, 2019). Belbase's (2019) analysis underscores the multifaceted nature of mathematics teacher beliefs and how they are influenced by various contextual, institutional, and theoretical factors. This comprehensive approach provides valuable insights into the complex interplay between teachers' beliefs and their teaching practices (Belbase, 2019, 2018).

The aforementioned studies reveal a multifaceted interplay among beliefs, knowledge, values, perceptions, and practices. These constructs constitute a comprehensive framework of existence, guiding an individual's behavior in specific contexts. I envision these constructs represented in a Venn diagram (Figure 1), where belief, knowledge, and perception intersect, creating a shared core that embodies personal values (Hatfield, 2012, personal communication). The regions that partially overlap form subconstructs, reflecting the intricate interplay of affective, cognitive, social, and cultural dimensions. The overlaps in question are very flexible, except for the core, which may change depending on the person and the situation. This indicates that as we gain more knowledge and teaching experience in math and other subjects, these ideas will evolve over time. Hence, the characterization of mathematics teacher beliefs is a complex process that should take into account the context in which form initial perception of mathematics, teaching and learning mathematics, research on mathematics teaching and learning, and assessment practices. These contextual factors and underlying perceptions together with knowledge and experience, may sediment into mathematics teacher beliefs about the subject, object, and the process. In a long run, these beliefs, perceptions, and knowledge can condense into core values of mathematics teachers. These values seem to be more stable and influential in their thinking, reasoning, and acting within classrooms.

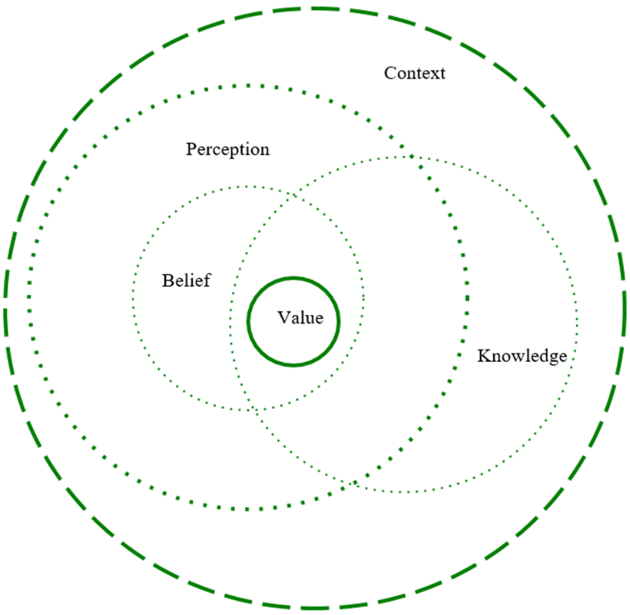

Figure 1: Interrelationship between belief, knowledge, perception, and value

Sources of Mathematics Teacher Beliefs

Handal (2003) examined the mathematical beliefs held by teachers, positing that these beliefs stem from their educational experiences in schools, which subsequently manifest in their instructional practices. Handal further asserted that the beliefs held by teachers could serve as a lens through which they navigate their decision-making processes, rather than solely depending on their pedagogical expertise or curricular directives. Through their interactions with their peers and other instructors, educators may develop these convictions. He articulates his apprehensions regarding teacher education programs, emphasizing the need for a greater focus on pedagogical knowledge while noting the insufficient attention given to the transformation of underlying beliefs. According to Handal (2003), "teacher education programs are not particularly effective in cultivating educators whose beliefs align with curriculum innovation and research' (Kennedy, 1991, as cited in Handal, 2003, p. 49). He asserts that our educational system fails to effect changes in teachers' beliefs and subsequently in their practices, functioning instead as a "vehicle to reproduce traditional behaviorist mathematical beliefs' (Kennedy, 1991, p. 50). He further explores the mathematical convictions held by pre-service educators. Kennedy (1991) asserts that many pre-service mathematics teachers view mathematics as a field based on memorized rules and procedures, believing in a single optimal method to solve any mathematical problem and categorizing mathematical problem-solving as either entirely correct or entirely incorrect.

Yates (2006) undertook a study involving one hundred and twenty-seven primary educators, employing a survey instrument to assess their beliefs regarding mathematics and pedagogy. Yates (2006) conducted this study using quantitative methods, including statistical measures of central tendency and correlational and factor analyses. Yates (2006) further observed that there was no statistically significant correlation between teachers' beliefs about the essence of mathematics and their beliefs about the pedagogy and acquisition of mathematics. He asserted that "teachers' beliefs were not associated with their age, qualifications, or duration of mathematics teaching experience, indicating that these beliefs were likely shaped through an apprenticeship of observation derived from their own experiences as students in mathematics classrooms (Fang, 1996, as cited in Yates, 2006). This merely conveys a fragment of the truth regarding the formation of teacher beliefs and their influence on teaching practices. The quantitative examination of beliefs and practices could yield outcomes that ostensibly reflect a context that is, in reality, nonexistent. The act of averaging teaching contexts represents a significant shortcoming in the realm of numerical analysis and mathematical formulations.

Next, I would like to delve into three significant themes that emerged from the comprehensive review of literature on mathematics teacher beliefs, knowledge, and practices. Firstly, I explore how the beliefs held by mathematics teachers can profoundly influence their classroom practices. This theme highlights the connection between teachers' personal convictions and their instructional methods, demonstrating how these beliefs shape their approach to teaching mathematics. Secondly, I examine how these beliefs can evolve, leading to changes in teachers' perspectives and actions. This theme addresses the dynamic nature of teacher beliefs, showing how they can shift towards either reform-oriented approaches or traditional, classical deductive teaching methods. It underscores the impact of evolving beliefs on teaching practices and educational outcomes. Finally, I elaborate on the third theme, which focuses on the knowledge possessed by mathematics teachers and the inherent complexities associated with it. This theme delves into the multifaceted nature of teacher knowledge, considering how it encompasses not only content knowledge but also pedagogical, contextual, and social skills and the ability to navigate the intricate landscape of educational practices.

Theme 1: Teacher beliefs influence practices

Stipek et al. (2001) posited that the beliefs and values held by prospective teachers significantly shape their classroom practices. They contended that influencing their beliefs is crucial for effecting changes in classroom practices. The researchers examined educators' convictions regarding the instruction and comprehension of mathematics, as well as the connections between these convictions and their pedagogical practices. They discovered a correlation between educators' beliefs and their corresponding classroom practices, aligning with the anticipated direction. Stipek et al. (2001, p. 223), asserted that "more traditional beliefs were associated with more traditional practices.' "Educators adhering to more conventional beliefs also afforded students comparatively less autonomy and upheld a social environment that shunned errors' (Stipek et al., 2001, p. 223). This research distinctly illustrates the connection between educators' beliefs and their instructional methodologies within the classroom setting. In an ideal autonomous environment, the interplay between teacher beliefs and classroom practices is characterized by a dynamic mutual influence.

There are instances where institutional demands, the capacity of educators to navigate circumstances, and limitations in time and resources, along with testing requirements, can lead to a certain level of incongruity between one's pedagogical beliefs and the practices employed in the classroom (Brosnan et al., 1996; Taylor, 1990; van Zoest et al., 1994). Op't Eynde et al. (2002) articulate a framework concerning students' convictions regarding the acquisition of mathematical knowledge and the process of problem solving.

In another study, Schoen and LaVenia (2019) explored teachers' beliefs about mathematics teaching and learning and their influence on classroom practices. They examined mathematics teachers' beliefs within the cognitive constructivist view and the transmissionist view of teaching and learning mathematics as two contrasting continuums. They outlined mathematics teachers' beliefs, categorized as "facts first' and "fixed instructional plan,' in their data collection tool. Their data comes from 206 elementary mathematics teachers from Florida. They assessed their mathematics and teaching-learning beliefs first and validated the instrument from the perspective of a fixed instructional plan or facts. In another study, Marshman and Goos (2019) examined mathematics teacher educators' beliefs about mathematics, teaching mathematics, and learning and how these beliefs might impact their practices. They found that the sampled teacher educators had a problem-solving view aligned with the method of teaching. They seemed to believe that mathematics is a growing field of study, not a finished product but open for revision, and that it is not distinct from other fields but interrelated (Marshman & Goos, 2019). These educators also believed that mathematics teachers should allow their students to struggle and let them justify the mathematical statements. Most educators who taught both pedagogy and mathematics content held the belief that investigating mathematical problems in small groups was an effective instructional practice. However, the majority of educators who solely taught mathematics content without any pedagogical courses appeared to hold the belief that teaching should follow a correct sequence, provide direct instruction, and use an expository method that aligns with an instrumentalist perspective (Marshman & Goos, 2019). Therefore, mathematics teachers' beliefs influence how they plan for teaching, how they present materials, engage students, and assess student learning. These actions can be linked to the changes they may support or constrain in the educational landscape of mathematics education.

Theme 2: Teacher beliefs impart change

I have previously examined the interconnections among belief, knowledge, perception, and values. It appears that these constructs are mutually constitutive and exert influence upon one another. Consequently, the examination of the evolution of beliefs, knowledge, and practices reveals a dynamic interplay in which each element is interdependent within a given context. I am endeavoring to examine the literature to understand the role these constructs play in effecting change in one's practice broadly and how shifts in belief specifically influence action. Researchers in mathematics education believe that the beliefs of mathematics teachers mediate the relationship between their mathematical knowledge and pedagogy (Clark et al., 2014). This perspective is consistent with the findings of Lammassaari et al. (2024), who found that teachers who adhere to the knowledge transmission theory of teaching often provide direct instructions, while those who adhere to the reflective collaborative theory typically employ a constructivist teaching approach. However, the influence of cultural context was significant. For example, teachers in Finland mostly believed reflective-collaborative theory compared to transmission theory, whereas in Taiwan there was a balance of both theories in teacher beliefs (Lammassaari et al., 2024).

Wilson and Cooney (2002) discuss the effects of educators' beliefs on their capacity and inclination to adapt. They contend that "when the focus of research transitions to a sense-making perspective, the distinctions between knowing and believing become indistinct as we endeavor to comprehend the phenomena of teacher change and the factors that propel that change' (Wilson & Cooney, 2002, p. 131). They assert that "teacher change encompasses not only alterations in classroom behavior but also involves a relativistic conceptualization of the act of teaching itself' (p. 132). Wilson and Cooney (2002) juxtapose the dualistic approach to mathematics instruction, which prioritizes outcomes devoid of contextual significance, against a relativistic perspective that highlights the importance of context and the processes involved in learning mathematics. Their argument posits that the concept of teacher transformation concerning their beliefs and practices reflects a progression from a dualistic perspective to a relativistic understanding. This alteration considers the context, emphasizing the importance of grounding instruction in students' existing knowledge. Consequently, teaching transforms into an adaptive process rather than adhering to a fixed sequence of instructional strategies (Wilson & Cooney, 2002). In this context, I would like to assert that the individual's capacity for skepticism, introspection, and reconstruction fundamentally anchors the portrayal of a reform-oriented educator.

Wilson and Cooney (2002) conducted an analysis of various studies that examined the beliefs of teachers and the dynamics of change, referencing works such as Even (1999), Lloyd (1999), Lloyd and Frykholm (2000), Lloyd and Wilson (1998), Sowder et al. (1998), and Wood and Sellers (1997). They discerned and tackled three fundamental themes: teacher reflection, educators' capacity to engage with students' comprehension, and the interplay between content and pedagogy. They asserted that those studies demonstrated the connection between educators' beliefs and their cognitive processes and behaviors. The studies revealed that "teacher reflection about student understanding is particularly powerful in terms of helping teachers connect their instruction to the work students are doing' (Wilson & Cooney, 2002, p. 143). Ultimately, they examined educators' emphasis on the teaching and learning process concerning both content and pedagogy. The report by Lloyd and Wilson (1998) revealed certain discrepancies between pedagogical beliefs and the realities of classroom practices. As previously mentioned, the discrepancies may arise from an adverse educational atmosphere, insufficient parental support, and a scarcity of resources.

Hart (2002) conducted a longitudinal study over four years examining the beliefs of teachers subsequent to their involvement in a teacher enhancement initiative. This investigation explored the convictions educators possess regarding their personal transformation journey. Hart asserts that "a critical component of developing this body of knowledge (about teacher change) is the teacher's voice' (p. 167). Hart (2002) also says, "If educators change their beliefs about teaching and learning in a way that fits with the principles of a certain change model, it becomes essential to not only look closely at the core of that change but also to articulate and recognize the factors that support it' (Hart, 2002). Hart reiterates that the development of effective teacher education programs necessitates acknowledging the existence of change and understanding educators' beliefs about it.

In 2002, Chapman explored the beliefs of two secondary mathematics educators regarding their transition from a primarily teacher-centered approach to a more student-centered methodology in high school settings. Chapman (2002) classified the mathematics educators into three distinct groups: those who independently modify their instructional methods, those who adapt their teaching with external assistance, and those who remain resistant to change despite participating in professional development initiatives. However, Chapman (2002) remains uncertain about the fundamental pattern of response to change, although he considers the relationship between thought and action to be a noteworthy factor in this dynamic. Chapman (2002) asserts that there has been insufficient focus on "the structure of beliefs and their potential impact on the teaching practices of experienced teachers' (p. 179). The research concentrated on four seasoned high school mathematics educators hailing from various institutions. Chapman (2002) employed a humanistic methodology grounded in phenomenology for the purposes of data collection and analysis. Chapman (2002) employed interviews, role-play, and class observations as methodologies for data collection. "The interviews centered on paradigmatic and narrative explorations of the educators' historical, contemporary, and prospective pedagogical practices, as well as their cognitive processes concerning word problems, problem solving, and mathematics broadly' (Chapman, 2002, p. 180). The findings elucidated the progression that two educators underwent as they navigated different phases of development. This included recognizing the difference between teaching mathematics and engaging in mathematical practice, encountering dilemmas and tensions, perceiving problem-solving as a game, establishing connections, reflecting on their own cognitive processes, and fostering a sense of creative engagement (Chapman, 2002). Chapman (2002) articulated a particular dilemma: "determining which alternative experiences one ought to encounter to shape all relevant beliefs' (p. 192). The narrative accounts detailing the lived and ongoing mathematical experiences of the participating teachers undoubtedly enhanced the robustness of the study. The integration of class observation and the role-play method enhanced the reliability of the interview narratives, thereby augmenting the trustworthiness of the data collected (Chapman, 2002).

In another study, Philippou and Christou (2002) investigated the efficacy beliefs held by 157 primary educators in Cyprus concerning the instruction of mathematics. The study employed a five-point Likert scale with twenty-eight items. The participants in this study indicated "a considerable degree of teacher self-assurance in the instruction of mathematics, despite their inability to effectively manage students' learning' (Philippou & Christou, 2002, p. 221). The results indicated a significant level of positive personal self-efficacy beliefs and an aptitude for assisting non-motivated students; however, there was a notable deficiency in effectively managing the school climate and valuing the pre-service program.

The aforementioned studies distinctly elucidated the intricate interrelationship between beliefs and change as mediated by teacher development and education programs. Undoubtedly, beliefs influence change; however, it is crucial to acknowledge the pivotal role that knowledge plays in this process. Should one hold the conviction that action research within the classroom fosters teacher development and subsequently induces change, it is imperative to possess the requisite knowledge for planning, organizing, and evaluating such research endeavors. Consequently, an evolving interplay among conviction, understanding, and action results in transformation contingent upon context or surroundings. The interplay of belief, knowledge, and practice is crucial in the evolution of teaching methodologies (Luft & Roehrig, 2007). According to Lammassaari et al. (2024), most belief researchers suggest that teachers should alter their beliefs to effect changes in their classroom practices.

A flexible school environment facilitates the conceptualization of teacher change more readily than a rigid and structured one. The adaptable setting or context in schools lets teachers use adaptable strategies that encourage students to take new actions, think deeply about their actions, and do creative activities. This helps students grow in ways that don't have clear boundaries (Fig. 2b). An environment or context that isn't flexible can make it harder for teachers to do creative and reflective work, which can limit the chance for big changes to happen in a set and rigid framework (Fig. 2a).

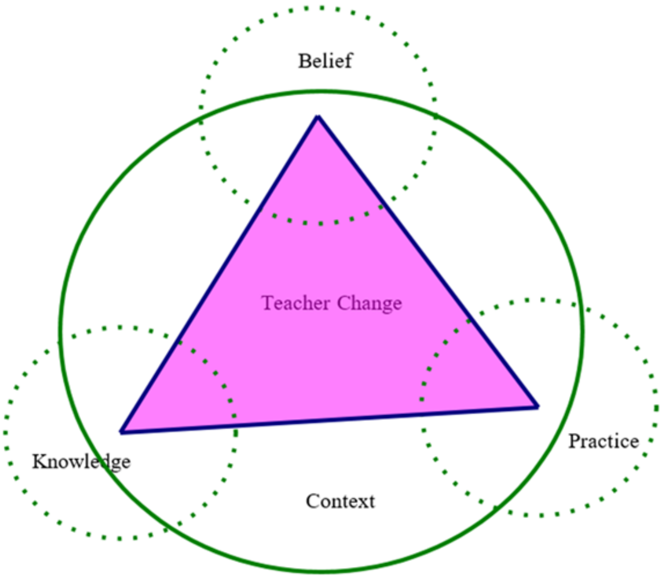

Figure 2a. Rigid Change in Rigid Environment

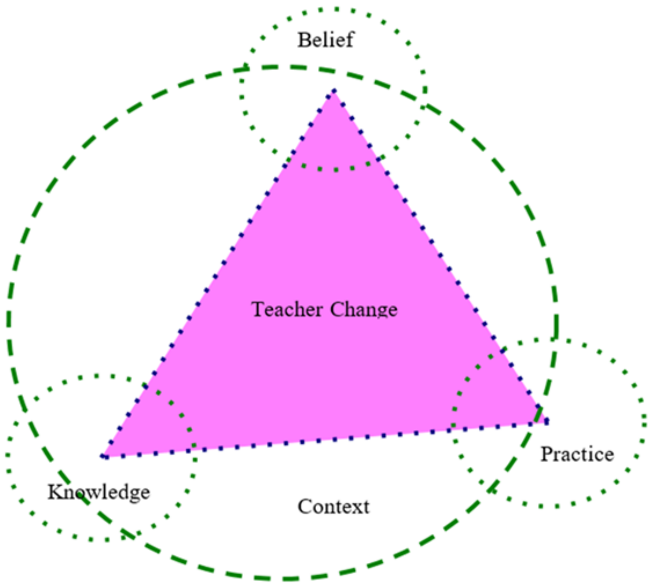

Figure 2b. Flexible Change in a Flexible Environment

Theme 3: Mathematics teachers' knowledge

Numerous studies have examined mathematics teachers' understanding of content, pedagogy, and learning (e.g., Ball & Bass, 2003; Calderhead, 1996; Grossman, 1990; Grossman et al., 1989; Hill & Ball, 2004; Hill et al., 2004; Mewborn, 2001; Shulman, 1986, 1987; van Driel et al., 1997; Wiens et al., 2022). These investigations on content knowledge, student understanding, pedagogical methods, and classroom context have established a basis for teacher expertise. Mewborn (2001) asserts that extensive research has examined the crucial forms of knowledge for teaching mathematics at the primary level. Mewborn also reminds us of Dewey's assertion that knowledge for teaching and knowledge for practice within a discipline are distinct from one another.

Different areas of concern regarding teacher knowledge exist. The Council of Chief State School Officers (CCSSO) has established standards for teacher knowledge and practice in teaching. The CCSSO's Interstate Teacher Assessment and Support Consortium (InTASC) developed standards to assess teacher knowledge in four different areas: the learner and learning, content, instructional practice, and professional responsibilities

(Wiens et al., 2022). The Council of Chief State School Officers (2013) has nested these four areas into ten different InTASC standards for mathematics and other teacher knowledge:

1. Learner development: This standard emphasizes teachers' knowledge about learners and their development, as well as their ability to design and implement developmentally appropriate pedagogy.
2. This standard focuses on teachers' knowledge and practice of differentiated instruction based on learners' individual differences.
3. Learning environments: This standard emphasizes teachers' understanding of classroom and school environments to foster positive, engaging, motivating, and collaborative learning environments.
4. Content knowledge—This standard focuses teacher knowledge of concepts, procedures, applications, tools, and methods for meaningful learning of the contents.
5. Application of content—This standard considers teacher knowledge in the areas of connecting concepts, procedures, and application of contents through creativity and critical thinking in a collaborative approach to problem solving with diversity of perspectives.
6. Assessment—This standard outlines teacher knowledge of a variety of assessment tools and techniques that are engaging, helping and monitoring student growth, and appropriate decision-making.
7. Planning for instruction: This standard emphasizes teacher knowledge of instructional plans that integrate content, curriculum, assessment, and pedagogy in the sociocultural context.
8. Instructional strategies—This standard outlines teacher knowledge of different teaching and learning methods to integrate contents according to learners' needs and necessary skills.
9. Professional learning and ethical practice—This standard focuses on teacher knowledge of their professional skills, self-reflective judgments, and ability to adapt with the changes, keeping learners at the center.
10. This standard focuses on the leadership roles that teachers play in their communities of practice with the aim of fostering collaboration and professional growth. (CCSSO, 2013)

Therefore, teacher knowledge is a complex domain that encompasses various aspects, such as learner characteristics, learning styles, the learning environment, the integration of content into contexts, assessments, planning, teaching strategies, professional and ethical practices, and leadership roles. The Stanford Center for Assessment, Learning, and Equity (SCALE) has also incorporated these domains into the EdTPA. The edTPA tasks prioritize lesson planning, instructional methods, assessment and tools, teaching analysis, and academic language as crucial areas of teacher knowledge for observation, assessment, and evidence (Stanford Center for Assessment, Learning, and Equity, 2019).

Mewborn (2001) categorizes the literature on teachers' expertise into five primary study genres corresponding to the decades of the 1960s, 1970s, 1980s, 1990s, and 2000s. "Previous studies predominantly consisted of quantitative studies aimed at establishing a correlation between educators' expertise and learners' performance' (Mewborn, 2002, p. 29). The researchers were unable to demonstrate a robust correlation between teachers' expertise and pupils' achievement. Subsequent research in the 1970s and 1980s sought to delineate the strengths and shortcomings in instructors' understanding of certain curriculum areas, including fractions and geometry. In the 1990s and 2000s, researchers employed qualitative analysis to explore the complex relationship between knowledge and teaching practice. Mewborn (2001) further asserts that "numerous elementary educators indeed lack a conceptual comprehension of the mathematics they are required to instruct' (p. 30). The evidence insufficiently indicates that these educators have the opportunity to acquire conceptual knowledge of mathematics during their teacher preparation programs (Mewborn, 2001). Mewborn criticizes teachers' misconceptions or insufficient conceptual understanding in the areas of quotitive division, formulating word problems with whole numbers divided by fractions, ratios, and proportions, as well as area and perimeter, among others. "While these educators had some understanding of mathematics, they lacked sufficient knowledge of mathematics as a discipline and/or pedagogical content knowledge necessary to instruct mathematics in alignment with contemporary reform initiatives,' Mewborn (2001, p. 31) further asserts.

I would like to examine teacher knowledge in mathematics education via the lenses of content knowledge (CK), pedagogical knowledge (PK), and pedagogical content knowledge (PCK).

Content Knowledge (CK)

Is it possible to instruct mathematics without possessing content knowledge? Is it possible to teach mathematics with solely content knowledge? It is inconceivable for an individual to teach mathematics without possessing adequate content knowledge. If I must instruct on a topic, such as the addition of improper fractions, but lack the requisite content understanding, how can I effectively teach students that subject matter? However, the subsequent question is more contemplative regarding the instruction of mathematics when one possesses simply subject knowledge. Understanding the fundamentals of teaching and the necessary activities for teaching mathematics, along with having a solid understanding of the subject matter, makes the process of teaching math enjoyable, fulfilling, and significant. Thus, possessing content knowledge is essential but not sufficient for teaching mathematics. Consequently, subject knowledge may be a necessary, yet insufficient, requirement for the addition of improper fractions (or any mathematical topic).

Ball et al. (2008) have made significant advancements regarding the characteristics of subject knowledge for teaching. Shulman and his collaborators significantly reframed the analysis of teacher knowledge to emphasize the importance of content in teaching. They further underline that the subject matter was only contextual. Shulman and his colleagues subsequently characterized content understanding as a distinct form of technical knowledge essential to the teaching profession (Ball et al., 2008). Content knowledge refers to the understanding of a subject and its associated material, such as the principles and structure of mathematics (Shulman, 1986). Shulman (1986) emphasizes that the 1875 California Teachers Examination primarily focused on subject matter. The 10-item subtest on Theory and Practice of Teaching received only 50 points out of a possible 1,000. This indicates that the educators' examinations during that period were entirely content-driven. The examinations encompassed written arithmetic, mental arithmetic, and algebra, among others. The test did not include geometry, but it featured industrial drawings, which may have encompassed geometric elements. Regardless of the test's subject matter, "ninety to ninety-five percent of the test pertained to the content, the subject matter intended for instruction, or at least the knowledge base presumed necessary for educators, irrespective of whether it was explicitly taught' (p. 5).

Shulman (1986) asserts that the emphasis on teaching the subject matter markedly contrasts with the emerging regulations of the 1980s around teacher assessment. He moreover states that "policy makers examine the research on literature instruction and discover it abundant with references to direct instruction, time on task, wait time, structured turns, lower-order questions, and similar concepts' (Shulman, 1986, p. 5). "They discover minimal or no citations regarding the subject matter, resulting in standards or mandates that lack any reference to the content dimensions of teaching' (Shulman, 1986, p. 5). Shulman (1986) further asserts that the research community has overlooked the significance of content, identifying it as a neglected paradigm in teacher development during that period. He also emphasizes the largely overlooked pedagogy of the 1870s and the notable absence of content in the 1980s. Shulman (1986, p. 6) inquires: "Has a division occurred between the two? Is it consistently said that one either knows content and pedagogy is secondary or that one knows pedagogy and is not responsible for content?'

According to Shulman (1986), there are several representations of content knowledge, including Bloom's cognitive taxonomy, Gagne's classifications of learning, Schwab's differentiation between substantive and syntactic structures of knowledge, and Peter's concepts, which align with Schwab's. Shulman (1986) contends that content knowledge extends beyond mere familiarity with the arguments or concepts of a discipline. It necessitates comprehending the structures of the subject matter, encompassing both substantive and syntactic frameworks. The substantive structures symbolize the various ways in which the field arranges its fundamental concepts and principles to encompass its realities. A discipline's syntactic structure determines truth or falsity and validity or invalidity through various methods (Shulman, 1986).

Goldschmidt and Phelps (2007) contend that teacher educators and stakeholders invested in teacher excellence and preparation acknowledge the necessity for teachers to possess a fundamental grasp of material. They further propose that state and federal policies should incorporate concerns about subject matter preparations. The No Child Left Behind (NCLB) Act of 2001 mandates that instructors must pass a state examination to demonstrate their subject matter proficiency in mathematics and other disciplines. This necessitates that educators grasp the subject matter during delivery and assimilation in mathematics classes.

Shulman (1986) analyzes the emphasis on content in the 1870s and pedagogy in the 1980s, illustrating a complete paradigm shift over a century; yet, the topic of achieving an appropriate balance remains unresolved. The NCLB Act (2001) emphasizes the significance of content knowledge in conjunction with pedagogical expertise for effective teaching in schools. The National Council of Teachers of Mathematics (2000) has delineated five fundamental strands as subject requirements for grades kindergarten through twelve in its

Principles and Standards for School Mathematics. The strands are (i) number and operations, (ii) algebra, (iii) geometry, (iv) measurement, and (v) data analysis and probability. NCTM (2000) delineates six principles: equity, curriculum, teaching, learning, assessment, and technology for teaching math. These principles emphasize the need for mathematical content to be appropriately challenging, which in turn promotes students' acquisition of increasingly complex mathematical concepts and allows teachers to apply their expertise with pedagogical flexibility. The content knowledge for mathematics teachers may vary across grade levels. For instance, preservice and inservice teachers in elementary grades should possess a solid foundation in concepts, procedures, and applications related to numbers and operations, algebra, geometry, measurement, statistics, and probability (National Governors Association Center for Best Practices and Council of Chief State School Administrators, 2010; NCTM, 2000).

Pedagogical Knowledge (PK)

As previously mentioned by Shulman (1986), there has been a transition from the dominance of content in the 1870s to the prominence of pedagogy in the 1980s. At the onset of the reform period, pedagogy became the focal point, resulting in an overemphasis on pedagogical knowledge. The National Council of Teachers of Mathematics (2000) outlined five process standards: problem solving, communication, connections, reasoning and proof, and representation. In my view, these process standards highlight the importance of teachers' pedagogical knowledge. To meet the expectations of mathematics education, mathematics teachers must equip themselves to instruct on content and process standards, which serve as guidelines for the pedagogical approaches they should adopt. The standard for problem solving indicates that addressing a problem is not just the objective of learning mathematics but also a significant aspect of the process itself. The standards for reasoning and proof indicate that mathematical meaning and proof provide a robust method for developing and articulating insights regarding a diverse array of phenomena. The standard emphasizes that by investigating phenomena, justifying results, and applying mathematical assumptions across various content areas and levels of complexity at all grades, students must grasp the logical nature of mathematics. The communication standard highlights that communication allows ideas to be the focus of reflection, refinement, discussion, and amendment. Despite the common practice of dividing and presenting mathematics in this way, the connection standard emphasizes that we should not view it as a series of isolated strands or standards. This standard reflects NCTM's perspective that mathematics is an integrated field of study. Ultimately, the representation standard suggests that we can express mathematical concepts in a variety of ways, including images, tangible materials, tables, graphs, numerical values, letter symbols, spreadsheet formats, and more. The document further explains that the representation of mathematical ideas is essential to how individuals comprehend and apply those concepts. Despite not explicitly mentioning the term ‘pedagogy', the process standards implicitly underscore the importance of pedagogical knowledge for mathematics teachers. It seems that the term ‘pedagogy' carries political implications, and NCTM aims for teachers to have autonomy in their pedagogical choices. Alternatively, ‘pedagogy' may represent individual philosophical and political beliefs, which NCTM seeks to remain neutral about.

Pedagogical Content Knowledge (PCK)

Shulman (1986) asserts that pedagogical content knowledge encompasses the most effective ways to represent ideas, the strongest connections, and illustrations. Explanations and proofs serve as the most effective methods for demonstrating and articulating the subject, ensuring it is logical for others. He further states:

> Pedagogical content knowledge encompasses an understanding of the factors that influence the ease or difficulty of learning specific topics. This includes the conceptions and preconceptions that students of various ages and backgrounds bring to the learning of commonly taught subjects and lessons. (Shulman, 1986, as cited in Ball et al., 2008, p. 392)

PCK consists of three elements: content knowledge, curriculum knowledge, and teaching knowledge (An et al., 2004; Turnuklu & Yesildere, 2007). An et al. (2004) assert that teaching knowledge is the fundamental element of pedagogical content knowledge. “PCK encompasses valuable representations, cohesive concepts, elucidating examples and counterexamples, beneficial analogies, significant relationships, and interconnections among mathematical ideas' (Grouws & Schultz, 1996, p. 443). Mathematical content knowledge and pedagogical content knowledge are essential components of effective mathematics instruction and the development of mathematical concepts in students' minds (Shulman, 1986; Turnuklu & Yesildere, 2007). PCK relates to how teachers convey their subject matter knowledge to their pedagogical knowledge, enhancing their pedagogical thinking, reasoning, and conceptual understanding of mathematics (Cochran et al., 1993; Kahan et al., 2003).

Turnuklu and Yesildere (2007) found in their study that possessing a deep understanding of mathematical knowledge is essential yet insufficient for teaching mathematics. They highlighted the connection between understanding mathematics and the ability to teach it. Every mathematics teacher should receive education that encompasses both mathematical and pedagogical content knowledge. According to Strawhecker (2005), the teacher and the quality of teacher preparation have an impact on student achievement and attitudes toward mathematics. She determined, based on her research, that field experience and various elements of mathematics teacher preparation impact preservice teachers' pedagogical content knowledge in mathematics.

PCK represents a blend of mathematical knowledge, content knowledge in mathematics, pedagogical knowledge, and an understanding of the relationships within pedagogy. To me, the pedagogical relationship is a central aspect of PCK.

According to van Manen (1994), there are three types of pedagogical relationships: personal, intentional, and interpretive. If a teacher possesses mathematical, content, and pedagogical knowledge, it does not necessarily imply that he or she has pedagogical sensibilities regarding their relationship with students. The connection between students and teachers fosters mutual understanding. It is essential for a teacher to demonstrate pedagogical care. I concur that a teacher has a purposeful connection with students aimed at assisting them in realizing their full potential for intellectual growth, such as in mathematics and language skills. At the interpretive level of the pedagogical relationship, a teacher must interpret and understand the current situation and experiences of the child while also anticipating the moments when the child is fully able to engage in the learning culture (van Manen, 1994). Therefore, I hold the belief that PCK surpasses the simple amalgamation of CK and PK, as in these relationships, the entirety consistently surpasses the mere sum of its parts.

Relationship among CK, PK, and PCK

Shulman (1986) observed that mathematics education in the 1880s was predominantly content driven, whereas by the 1980s, it had shifted to being pedagogy driven. I believe our current position lies between these two continuums. The National Council of Teachers of Mathematics (2000) has made an effort to balance content and pedagogy by outlining five content standards and five process standards, although it does not explicitly address pedagogy.

Ball (2000) highlights two key aspects essential to the practice and reality of teaching: the ability to break down one's own knowledge into simpler, more conclusive forms, where the critical elements are clear and distinguishable. Ball states that understanding for teaching necessitates going beyond the implicit knowledge that defines and suffices for individual comprehension and execution. Indeed, it relates to the pedagogy of caring (van Manen, 1994) and understanding students' mistakes or valuing their uniquely expressed insights (Ball, 2000), which necessitates an exceptional ability to unravel one's own deeply condensed comprehension. Thus, content knowledge, pedagogical knowledge, and pedagogical content knowledge merge to form a comprehensive understanding of teachers' expertise. Additionally, they may complement each other, although this understanding may not be entirely exclusive or comprehensive. I believe there is an ambiguous distinction between CK, PK, and PCK. The boundary is indistinct, allowing for the flow of each type of knowledge (CK, PK, and PCK) crossing the border and overlapping on each other to form a complex of teacher knowledge of mathematics in the context of teaching, learning, and assessing.

Turnuklu and Yesildere (2007) asserted that a deep comprehension of mathematics is essential but not adequate for teaching the subject effectively. They emphasize that teaching mathematics effectively requires a solid foundation in pedagogical knowledge as well. Their assertion distinctly illustrates a fundamental connection between content knowledge and pedagogical knowledge. Shulman (1987) contends that the behaviors of policymakers and teacher educators historically align with the idea that teaching necessitates fundamental skills, content knowledge, and overarching pedagogical abilities. He also asserts that people trivialize teaching, overlook its complexity, and minimize its demands. "Educators often struggle to express their knowledge and the processes through which they acquire it' (Shulman, 1987, p. 5). Shulman presents a model of pedagogical reasoning and action that includes comprehension, transformation, representation, selection, adaptation, instruction, evaluation, reflection, and new comprehension. The model effectively integrates content knowledge and pedagogical content knowledge. "A proper understanding of the knowledge base of teaching, the sources for that knowledge, and the complexities of the pedagogical process will make the emergence of such teachers more likely' (Shulman 1987, p. 20). This is a highly impactful idea of integrating the three components (CK, PK, and PCK) into a model of pedagogical excellence that ought to serve as a foundation for reforms in mathematics education.

Technology: A New Dimension in Teacher Education

Technology has impacted every aspect of life: personal and communal, private and public, as well as nearly all types of professions. In this context, today's educators face the challenge of understanding technological tools to effectively incorporate them into the teaching of mathematics in the classroom. In its technology theme, the National Council of Teachers of Mathematics (NCTM, 2000) highlights the crucial role of technology in the teaching and learning of mathematics, emphasizing its influence and enhancement of students' learning. NCTM articulates its stance regarding the role of technology in the teaching and learning of mathematics, emphasizing the importance of technological tools in facilitating effective mathematics education. Teachers today may align their mathematics lessons with the Common Core State Standards for Mathematics (CCSSM), which require the integration of technology in the teaching and learning of mathematics for visualization, computation, modeling, presentation, and communication.

Researchers such as Hatfield (1984), Heid (2005), and Olive (2011) emphasize the significance of technology in mathematics education. The researchers assert that the awareness, knowledge, and experiences of preservice and inservice teachers regarding various technological tools, along with their ability to integrate these tools into the teaching and learning of mathematics, can greatly influence students' learning outcomes in mathematics. The beliefs, attitudes, and values of pre-service or in-service teachers regarding technology in the teaching and learning of mathematics can influence the effective use of these tools. Numerous studies (e.g., McGinnis et al., 1996; Tharp et al., 1997) examined the beliefs of pre-service and in-service teachers regarding the integration of technology in mathematics instruction. Olive (2011) explores research regarding the impact of technology on learning and teaching outcomes. Research on learning and teaching with technology should prioritize exploring the various impacts of technology on learning and teaching outcomes, as it is a critical consideration in design research (Olive, 2011, p. 82). New teaching and learning guidelines clearly expect the integration of technology in mathematics education.

Carlson et al. (2002) contend that in an era defined by technology and information, complex systems are essential subjects of inquiry in their own right. This is not only due to their significance in affecting the lives of everyday individuals, but also because misconceptions surrounding these systems contribute to numerous beliefs, principles, policies, and processes that are crucial for fostering informed citizenship and ethical conduct in intricate societies. They further clarify that "the kind of' systemic understanding highlighted in the context of complex systems is also involved in the development of many other types of concepts, skills, principles, beliefs, attitudes, and problem-solving processes that are not as obviously systemic in nature' (Carlson et al., 2002, p. 46). Technology tools and applications play a vital role in addressing the complexities present in the systemic environment, more so in mathematics teaching and leanring. These tools and applications aid in comprehending the highly dynamic and chaotic nature of teaching and learning and the interplay between teaching and learning in mathematics. I describe these activities as dynamic since both educators and students progress from one level of understanding to the next each time they engage in mathematical pedagogical activities facilitated by artifacts and technology. Additionally, they exhibit a level of chaos, as it can often be quite intricate and even unfeasible to foresee what may occur next. An overlooked incident in teaching and learning mathematics can significantly influence one's perspective on life and the world around them.

Lampert and Ball (1998) address the challenges faced by pre-service elementary mathematics teacher education programs at two state universities. They link their experiences as educators, teacher trainers, and subsequently as researchers in the field of teacher education. Two universities experimented with the impact of technology, specifically the multimedia environment, in elementary mathematics teacher education. They sought to create a model of teaching and learning that emphasized teaching for understanding by integrating their classroom practices into the realm of formal, university-based teacher education, which was under significant scrutiny at that time. They involved pre-service mathematics teachers in a technological environment primarily focused on computers and videos, creating unique opportunities for instructors to explore various aspects of pedagogical practices by incorporating their experiences as educators, recognizing the shortcomings of teacher preparation, understanding practical applications, and valuing prospective teachers as learners. Lampert and Ball (1998) realized that their own teaching provided a framework for their reflections on how to engage in that type of teaching. Lampert and Ball (1998) further explored the dilemmas they encountered frequently and analyzed the teacher's role along with the social complexities of group work in classrooms, considering the implications for teacher education.

Nordin et al. (2010) explored the educational effectiveness of a digital module prototype that incorporated dynamic geometry software, Geometer's Sketchpad (GSP), in the context of mathematics instruction. The criteria for their pedagogical application encompassed student control, student actions, objective orientation, functionality, added value, enthusiasm, knowledge value, flexibility, and responsiveness. They found that the sample digital modules satisfied the pedagogical usability criteria and facilitated the integration of GSP in mathematics teaching. They proposed a study on the use of GSP in mathematics instruction to enhance higher-order thinking skills in high school students.

Hoong (2003) examined the extent of GSP usage in secondary schools and the manner in which teachers implement the tool in their geometry lessons. He utilized the angle properties of polygons, points and lines, and circles, along with concepts of congruency, coordinate geometry, geometrical constructions, locus, mensuration, and the properties of special quadrilaterals and triangles. Hoong (2003) also applied the Pythagorean theorem, principles of similarity, symmetry, transformations, trigonometric rules and formulas, trigonometric graphs, vectors, and more to explore their relevance to textbooks, teaching, and learning. Hoong (2003) discovered that 80% of teachers (in the study sample) incorporated the GSP into their mathematics instruction across various geometry topics. Hoong (2003) discovered that most teachers used the GSP to instruct transformations, not other aspects of geometry.

Stigler and Hiebert (1999) suggest that "one other approach to understanding the difficulties of integrating information technology (IT) in the classroom stems from seeing teaching as a complex cultural activity' (p. 97), and this cultural complexity can occasionally hinder progress in teaching and learning. A wealth of literature exists regarding the application of the GSP in the teaching and learning of geometry within both schools and higher education, including works by Bennett (1994, 1995), Boehm (1997), Olive (1998), and Belbase (2018) among others. The dynamic geometry tools, such as GeoGebra and GSP's evolving characteristics may revolutionize the use of this tool in mathematics classrooms.

Leatham (2002) examines the beliefs of pre-service mathematics teachers and the evolution of those beliefs and practices regarding the teaching and learning of mathematics with technology during a mathematics methods course. By employing the qualitative grounded theory constant comparative method, he analyzed and discussed four cases concerning beliefs about mathematics, teaching mathematics, learning mathematics, and teaching with technology. His findings revealed a direct connection between the pre-service teachers' beliefs about teaching with technology and their beliefs about learning with technology. Leatham (2002) discerned the essence of mathematics, the process of teaching it, the learning involved, and the role of technology through the examination of these cases. Leatham presents Ben's beliefs about mathematics, emphasizing that it extends beyond a mere set of rules. He (Ben) believes in the importance of problem-solving tools that involve logical thinking and reasoning. He also believes that utilizing the principles of mathematical problem solving in real-world situations that require decision-making showcases the effectiveness of the tools in his mathematical toolbox (Leatham, 2002). Ben firmly holds the view that a conceptual understanding of mathematics is essential, despite his greater confidence in procedural methods. Regarding Ben's beliefs about mathematics teaching, Leatham (2002) further asserts that the teacher's role in creating an environment that motivates students to learn is a highly influential one. Leatham (2002) could employ a range of teaching methods, adapting to the overall classroom environment and the specific needs of each student. Ben exhibited a positive attitude and demonstrated a strong understanding of how to use Geometer's Sketchpad in geometry class. Nevertheless, he did not think that any of his high school experiences, such as calculus, represented teaching with technology. Leatham (2002) asserts that Ben's participation in the teacher education program enabled him to grasp the various functions of technology, such as promoting exploration and visualization.

The interplay among content knowledge, pedagogical knowledge, and technological knowledge establishes a new epistemological foundation: pedagogical content knowledge, pedagogical technological knowledge, and technological content knowledge. The overlap extends to form a zone of technological-pedagogical-content knowledge (TPCK) (Koehler, 2011, Fig. 3). The intricate relationship between beliefs about mathematics content (CK), mathematics teaching and learning (PK), and the integration of technology in teaching and learning (TPCK) could significantly influence the practices of pre-service teachers in teaching mathematics. This study aims to explore the interplay between the beliefs and practices of pre-service elementary and secondary mathematics teachers as they transition from methods and geometry classes to their residential practice. Figure 3 illustrates the conceptual model of teacher knowledge, highlighting the intricate relationships among CK, PK, TK, PCK, TPK, TCK, and TPCK within a specific context. Mishra and Koehler (2006) posit, “In practice, this means that we need to look at each of these parts separately, but also in pairs: pedagogical content knowledge (PCK), technological content knowledge (TCK), technological pedagogical knowledge (TPK), and all three together as technological pedagogical content knowledge (TPCK), which is what teachers know' (p. 1026).

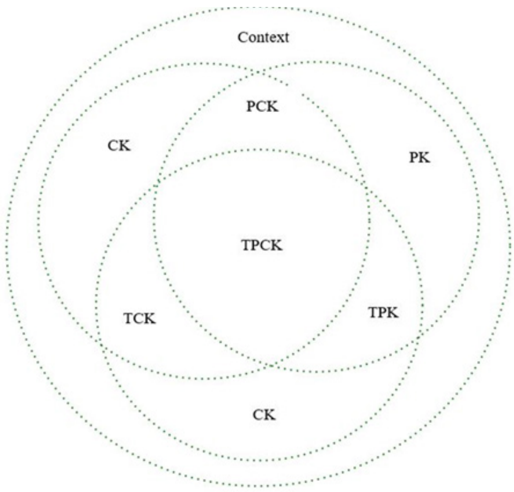

Figure 3: Conceptual Model of Teacher Knowledge (adapted from http://tpack.org/)

New Directions and the Future

Beliefs shape the concepts of ‘de facto teaching', ‘de jure learning', and ‘de novo research'. As an aspiring teacher educator in mathematics education, I find inspiration in exploring issues that resonate deeply with me. My initial aspiration for a study on “Beliefs, Knowledge, and Practices in Mathematics Education' provided me with a sense of being in an epistemic place, though it still feels like a vast field to me. Numerous studies concentrate on a single aspect, whether it be beliefs, knowledge, practices, or, at most, a combination of two of these elements. At this point, I intended to integrate all three elements to provide a comprehensive understanding of how beliefs connect with and mediate in the interrelationship of knowledge and practices among pre-service or in-service mathematics teachers. Upon reviewing the literature, I discovered that belief, knowledge, and practice are crucial constructs that impact each other and the overall teaching and learning process in classrooms; each contributes to and receives information from others (Askew et al., 1997). Askew et al. also note that the interrelationship of these constructs significantly influences teacher effectiveness. Askew et al. (1997) say that understanding effective teachers starts with a model of how they run their classrooms. Two interconnected parts make up this model: a set of beliefs, a body of knowledge (including subject knowledge), and insights teachers have about how to teach math, which we call pedagogical content knowledge.

In this context, today's educators face the challenge of understanding technological tools to effectively incorporate them into mathematics instruction in the classroom. We expect them to possess a wide range of tools, and they should demonstrate proficiency in using various technological resources when teaching mathematics. The National Council of Teachers of Mathematics (NCTM, 2000) included technology as one of the six overarching themes in their Principles and Standards for School Mathematics. The technology theme asserts that “technology is essential in teaching and learning mathematics; it influences the mathematics that is taught and enhances students' learning' (n.p.). NCTM articulates its stance regarding the role of technology in the teaching and learning of mathematics, emphasizing the importance of technological tools in facilitating effective mathematics instruction. This approach may not only enhance proficiency in mathematics but also broadens access to the subject.

In the 21st century, technology serves as a vital resource for learning mathematics, and it is imperative that all schools provide their students with access to this technology. Skilled educators harness technology to enhance students' comprehension, ignite their curiosity, and boost their skills in mathematics. Thoughtful use of technology can provide access to mathematics for all students (NCTM, 2000). Educational institutions at the federal, state, and local levels have placed significant emphasis on incorporating technology into the teaching and learning of mathematics. Schools are working to integrate technology into mathematics and science classes to enhance teaching and learning. Schools, colleges, and faculties of education must ensure that their teacher preparation programs equip preservice teachers to effectively integrate technology into mathematics instruction (Abbitt & Klett, 2007). Abbitt and Klett (2007) highlight significant differences in the nature and practices of technology integration within mathematics teacher education programs across various higher education institutions.

The knowledge and experiences of mathematics teachers regarding technology tools, as well as their ability to integrate these tools into the teaching and learning of mathematics, can greatly influence students' understanding and learning of the subject. Previous research on pre-service teachers' beliefs and practices regarding mathematics teaching and learning with technology has primarily focused on static analysis. Studies that have examined the evolution of beliefs, knowledge, and practices throughout the learning experiences, from class content and methods to residential practices, are rare. This review aimed to explore teachers' knowledge and beliefs regarding the nature of mathematics, teaching, and learning, alongside their use of technological tools. Additionally, it explores the dynamic nature of mathematics teacher beliefs, knowledge, and practices as key elements of their teaching actions and experiences.

Professor Larry Hatfield instituted The Wyoming Institute for the Study and Development of Mathematical Education (WISDOMe), which aimed to conduct research focused on three key identities: Quantitative Reasoning and Mathematical Modeling (QRaMM), Technological Tools and Applications in Mathematics Education (TTAME), and Developing Investigations in Mathematical Experiences (DIME). There was a belief that these three domains of mathematics education would collaborate, support each other, and promote mathematics education in the state and beyond. The establishment of the virtual institution under WISDOM^e was a genuine endeavor that took place between 2010 and 2015, spanning nearly a half-decade. During this time, several close-group conferences at the University of Wyoming, the University of Georgia, and Georgia Southern University yielded promising results, leading to the publication of four volumes of monographs on mathematics education research.

This initiative sparked renewed optimism in the realm of mathematics education research, opening up new possibilities. This online institute sought to encourage and facilitate collaborative research and interactions among a worldwide network of participating researchers (Hatfield, 2011). In this context, Steffe's (2010) definition of an epistemic student makes complete sense. "Epistemic students are interiorized others—the dynamic organizations of schemes of actions and operations in one's mental life' (Steffe, 2010, p. 22). We aimed to understand and reframe the interiorized other (epistemic teacher) with the objective of examining beliefs, knowledge, practices, and the evolution of teachers in mathematics education. We have a significant journey ahead, and achieving this is possible through both personal and institutional collaboration. To change one's actions in mathematics, first we need to change beliefs (Beswick, 2012). The WISDOM^e initiative aimed to transform knowledge and practices in mathematics teaching, learning, and research by challenging existing beliefs. These beliefs emerge from experiences that aim to transform the landscape of "De Facto Teaching, De Jure Learning, and De Novo Research' through progressive beliefs of new possibilities, a firm determination to learn and construct new knowledge, and a transformation of classroom teaching, learning, and assessment practices through collaboration and mutual conscience to enhance peace, humanity, and lasting prosperity. I hope Troy University will be such a venue for fostering such collaboration and instill a hope of new directions of teacher education and development through transformative teacher beliefs, knowledge and practices.

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I would like to express my gratitude to Professor Larry Hatfield for his invaluable support during my graduate studies at the University of Wyoming, Laramie. This article originated as a class assignment in my doctoral program and has since undergone significant revisions to update the material. However, it retains its original title and some subtitles to preserve the initial conceptual framework.

Ethical Declaration

I would like to declare that this article was a class assignment during my doctoral study at the University of Wyoming in 2012. The Education Resources and Information Center within the Institute of Education Sciences has posted a preliminary version of this article (ERIC Number: ED530017). Link: https://eric.ed.gov/?id=ED530017

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