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Richards Equation and its Solution for Application in Landslide Hazards.

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Abstract

This paper presents the numerical solution of Richards Equation of hydrology for unsaturated soils. To describe the prediction and analysis of infiltration - induced landslide due to the heavy and concentrated rainfall, it needs to develop an accurate model for the system.Richards equation of water hydrology may be used to model part of the landslide hazards system in variably saturated (unsaturated) soils. The major problem with the application of Richards equation is in the linearization of non-linear partial differential equation. In this work, we apply the empirical relationship developed by Haverkamp et al. to linearize the non-linear PDE. The numerical solution is obtained by using different time stepping schemes and used the outcomes for the calculation of safety factor (F_S) prescribed by different infinite slope models. Since safety factor is the index which helps to determine the condition of surface failure situations. From this we can predict the potential landslide area affected by heavy and concentrated rainfall.

1. Introduction

In numerous branches of science and engineering such as soil mechanics, fluid mechanics, agricultural engineering, environmental engineering and ground water hydrology, prediction of fluid movement in unsaturated porous media(soil) becomes an emerging problem. In unsaturated zone the fluid motion is assumed to obey the classical Richards equation. This equation, obtained by applying the mass conservation law and the Darcy law and it can be expressed as different forms either pressure head ψ or moisture content θ as the dependent variable. The constitutive relationship between $\psi = \psi$ (z, t) and θ = θ (z, t) allows the conversion from one another. Equation (1) shows the mixed form of Richards Equation [1].

$$\frac{\partial \theta}{\partial t} - \nabla.K(\psi)\nabla\psi - \frac{\partial K}{\partial z} = S \qquad (1)$$

Where, θ is the volumetric moisture content, ψ is the pressure head K (ψ) is the unsaturated Hydraulic conductivity, describes the ease with which water can move through pore space, and depends on the permeability of the material used and the properties of fluid

[2]. S is the source or sink term . Constitutive relations between θ = θ (z, t) and $\psi = \psi$ (z, t) and between K and ψ are developed appropriately, which consequently gives nonlinear behavior to equation (1).

Reliable approximation of these relations are in general tedious to develop and thus also challenging. To gather the parameters either from field measurements or laboratory experiments are relatively expensive and such relations are limited to particular cases. Perhaps the most widely used empirical constitutive relations for the moisture content and hydraulic

conductivity is due to the work Haverkamp et al. [3]. We use the following popular model from ground water hydrology due to Haverkamp et al. [3], which describes this constitutive relation as the continuous function of ψ.

$$\theta(\psi) = \theta_r + \frac{\alpha(\theta_s - \theta_r)}{\alpha + |\psi|^{\beta}}, \qquad K(\psi) = \frac{K_s A}{A + |\psi|^{\gamma}} \qquad (2)$$

where θ_s and θ_r represent the saturated and residual moisture content respectively, Ks corresponds to the saturated hydraulic conductivity, and A, α, β, ψ are dimensionless soil parameters. Both the function K(ψ) and $\theta(\psi)$ in equation (2) are highly nonlinear since they can dramatically change over a small range of ψ . With these the Richards equation (1) becomes a highly nonlinear partial differential equation and analytical solution is not possible except for some special cases. The flow of water in unsaturated porous media (soil) is complex in nature. Because of its complexity and lack of absolute and reliable analytical solution to calculate the flow in unsaturated zone, the use of numerical methods to solve the problems related to unsaturated flow has grown considerably in recent years. Since the expressions of equation (2) makes the equation (1) highly nonlinear, it is important to utilize efficient and accurate solution procedures. Different numerical methods are used to approximate the solution of equation (1). Different methods yield different accuracy. They are with high computational cost and are not reliable for some cases as well. Therefore, infiltration problems are still considered to be one of the most important topics in groundwater hydrology.

Several numerical schemes based on finite difference, finite element and finite volume spatial discretization techniques to the partial differential equation have been developed to approximate the solution of the Richards equation [4], [5], [6], [7], [8], [9]. Adaptive time-stepping strategies are studied in [10].

Because of the restricted stability criteria of the Forward Euler schemes for the parabolic partial differential equations, effective cost has not given to develop Forward Euler schemes for the Richards equation and most of research work is devoted in developing Backward Euler schemes. Due to the highly nonlinear nature of the problem, to implement Backward

Euler schemes, no matter which numerical approaches we use, the problem has to be linearized somehow at some stage. As such some iterative approaches need to be applied to tackle this highly nonlinear problem [3, 11]. The Backward Euler schemes are unconditionally stable, but with the iterative process involved they all turn out to be computationally high cost and in certain circumstances, unrealistic. Current trends in computational approach is to develop efficient parallel algorithm for High-performing Computers. To be able to parallelize the numerical procedures, iteration free algorithm should be considered. For this, Forward Euler scheme is the best option. In [12], a linearized Richards equation model is studied. In [8, 13], stability analysis of a Forward Euler scheme for the Richards equation is studied.

In this paper we use Kirchhoff integral transform to reduce the highly nonlinear equation to a functional linear parabolic equation and solve it numerically using Forward Euler, Backward Euler, Crank-Nicolson and a stabilized Runge-Kutta-Legendre super time-stepping with dirichlet boundary condition and compare the performances.

The purpose of this paper is to implement a relatively new numerical approach (super time-stepping to stabilize Forward Euler scheme [14] to transformed Richards equation using dirichlet boundary condition for the prediction of landslide hazards. The work here presented describes and verifies the employment and accuracy of a stabilized Runge-Kutta Legendre super time-stepping strategy

(RKL) to Forward Euler finite difference scheme to simulate flow in unsaturated porous media (soil) with Dirichlet boundary condition. The efficiency of using our scheme is that it is unconditionally stable, easy to implement, can be easily extended to problems in higher dimensions, and it is parallelizable.

This paper embodies different sections. The section two describes the surface failure condition. The section three presents the methodology. The numerical methods are presented based on finite difference schemes (with different time-stepping schemes) and different test cases are solved and the results are compared and finally, the section five embodies the conclusion.

2. Landslide in Unsaturated Soils

Landslides occur especially in hilly area of the Himalaya region around the world. Depending on tectonic activity, varied topography, intense rainfall, and geological instability, the 2400 km long Himalaya Mountain chain is highly susceptible to landslide. These mountainous terrains are the habitat to millions of living beings scattered in Nepal, Bhutan, northern Pakistan, northern India, and the other area of the world. Uneven landscape, changeable geological structures, weak and breakable rocks together with heavy and unusual concentrated precipitation during rainy seasons collectively cause rigorous land sliding problems and related phenomena in the Himalaya mountain region.

The main cause for severe and concentrated rainfall events which are common nowadays is climate change. The effect of heavy and uneven rainfall, infiltration - induced landslides are found everywhere in the Himalaya mountain in the world. Losses of natural and artificial infrastructure, damaging of construction including roads, houses, and fatalities exposed describe that the landslides are dramatically increased nowadays. In variably saturated (unsaturated) soil, infiltration-induced landslide related to the infiltration process generally coincides to the ground surface where the shallow landslide may befall due to the continuous precipitations. Especially shallow landslide occurred within the vadose zone of unsaturated soil by sliding the surface of sloppy areas. Slope failure is the tectonic behavior of soil surface, which depends on the infiltration process in which fluctuation of matric potential in variably saturated (unsaturated) soil plays a vital role and triggers of landslide from rainfall.

The infiltration - induced landslides in the Himalaya make tremendous damage to lives, property, infrastructure and environment, particularly in the monsoon season. Therefore, it is necessary to discuss about the factors, issues and environmental effects to the infiltration - induced landslide in this region. The main causes of such kind of landslides are surface failure that depends on the nature and connectivity of slope instabilities. Now it is relevant to analyze the behavior of slope stability in variably saturated (unsaturated) soil. The concrete study of soil structure associated to slope model helps to calculate the factor of safety which are conventional to analyze the slope stability of sliding surface along the failure plane. Particular weather conditions such as continuous and concentrated precipitation, rapid snow melting or uneven weather change are the main causes for slope instabilities. These phenomena are directly related with the pore water pressure of soil. Pour water pressure can may up and down in abnormal manner in infiltration and redistribution processes, i.e., during rainfall and after the rainfall. Thereby increasing the pore saturation, the corresponding pore pressure also increases. Continuously repeated these processes the effective stress of the soil may be deducted and it becomes the main cause of slope failure. Richards equation a nonlinear partial differential equation, which describes the soil water flow and can be applied for a quantitative assessment of the landslide hazards. The formulation of new model, connecting to Richards equation with infinite slope stability model and its numerical solution can be a powerful tool to capture the integrity of slope failure corresponding to the available weather information. The efficiency and accuracy of such formulation and procedure depends on the quality of the available soil parameters which are directly connected to weather data and physical feature of the landscape. Therefore, an acceptable mechanism to landslide hazards assessment can accurately be adapted to the potential area considering to account [16,17,18,19].

For the significant assessment of landslide hazards in the potential landslide area we have implemented the solution of Richards equation. For modeling of infiltration-induced landslide in variably saturated(unsaturated) soils, the instability/stability analyses connected to the one dimensional Kirchhoff transformed Richards equation with the variation of moisture content relative to variation of water pressure head and vice-versa is adapted. The characteristic of infiltration-induced landslide in unsaturated soil on the ground surface is seemingly closed to the physical features of shallow landslide that occurs during the period of intense rainfall. This study governs the landslide hazards problem involving infinite slope model for safety factor. Safety factor is the ratio of two forces in which the one prevents the slope from falling and the other adheres the slope to be collapsed. The numerical value of safety factor represent the landslide hazards index, depending on it prediction can be made for the possibility of landslide hazards to the given topography. Generally, index greater than one defines stable condition and less than one defines unstable condition. Different types of slope model are developed to calculate the safety factor, the following infinite slope model is used here for its easiest and reliable calculation [14,15,16].

$$Fs = \frac{C}{zy_t \sin\delta\cos\delta} - \frac{hy_w\theta\tan\beta}{zy_t \sin\delta\cos\delta} + \frac{\tan\beta}{\tan\delta} \qquad (3)$$

where C is the effective cohesion, θ is the moisturity, y_t and y_w are the unit weight of the soil and water respectively, β is internal frictional angle, z is soil thickness, δ is the slope of the inclined surface. The combination of the infinite slope model (3) with Richards equation, explore the landslide hazard calculation directly from weather data. In particular, the values of θ obtain from solution of Richards equation are used to get information about safety factor in the given infinite slope model. In collaborating the weather data with this moisture θ, we can obtain the potential landslides area.

3. Methodology

3.1 Simplified one-dimensional Richards Equation [17,18]

We consider Richards equation (1) in one space dimension with no sink and source term.

$$\frac{\partial\theta}{\partial t} = \frac{\partial}{\partial z}\left(K(\psi)\frac{\partial\psi}{\partial z}\right) - \frac{\partial K}{\partial z} \qquad (4)$$

Richards equation (4) is typically used to simulate infiltration experiments. These experiments begin with either wetting or dry soil on top of the ground surface and water is poured on top of the ground surface in case of dry soil, showing a clear connection with the Darcys law. We assume that the infiltration with known pressure head at the top and bottom of the soil column. That is we use the following initial and boundary conditions.

$$\begin{aligned} &\psi(z,0) = \psi_0(z), 0 < z < L, \\ &K(\psi) - K(\psi)\frac{\partial\psi}{\partial z} = q(t), z > 0, t > 0, \qquad (5) \\ &\psi(L,t) = \beta(t), t > 0. \end{aligned}$$

2.1. Kirchhoff integral transform. We apply Kirchhoff integral transformation to equation (4), letting $h = \psi - z$ and define

$$\phi(h) = \int_0^h \bar{K}(\lambda)d\lambda \ . \qquad (6)$$

Since K(h) > 0 from (6), the function $\phi(h)$ is strictly increasing with $\overline{K}$(h) = K(ψ). Taking derivative of both sides of the transformation, we obtain

$$\frac{\partial \phi}{\partial z} = \frac{\partial \phi}{\partial h}\frac{\partial h}{\partial z} = \overline{K}\,(\mathrm{h})\frac{\partial(\psi - z)}{\partial z}$$

$$= K(\psi)\left(\frac{\partial \psi}{\partial z} - 1\right) = K(\psi)\frac{\partial \psi}{\partial z} - K(\psi) \qquad (7)$$

Again taking derivative of equation (7),

$$\frac{\partial^2 \phi}{\partial z^2} = \frac{\partial\left(K(\psi)(\frac{\partial \psi}{\partial z})\right)}{\partial z} - \frac{\partial}{\partial z}K(\psi)\,. \qquad (8)$$

Using the equation (8), the Richards equation (4) takes the form

$$\frac{\partial \overline{\theta}}{\partial t} = \frac{\partial^2 \phi}{\partial z^2}, \qquad (9)$$

with $\overline{\theta}$(ϕ) = θ(h). The corresponding initial and boundary conditions for the transformed equation (9) takes the form below:

$$\begin{aligned} &\phi(z,0) = \phi_0(z), 0 < z < L, \\ &\frac{\partial \phi}{\partial z} = \bar{q}(t), z = 0, t > 0, \\ &\phi(L,t) = \overline{\beta}(t), t > 0 \end{aligned} \qquad (10)$$

The Kirchhoff transformation transformed the doubly nonlinear equation (4) to a nonlinear parabolic problem (9). Also we note that the Kirchhoff transformation preserves the uniqueness result for the transformed problem.

3. Numerical Method [17,18]

To solve the transformed equation (8) numerically with the prescribed initial and boundary conditions (9), it is more convenient to have a single state variable. For this, we assume θ and ϕ are single valued continuous functions of one another and rearrange to obtain

$$\frac{\partial\theta}{\partial t}=\frac{\partial\theta}{\partial\phi}\frac{\partial\phi}{\partial t}=\left(\frac{1}{(\partial\phi/\partial\theta)}\right)\frac{\partial\phi}{\partial t},\qquad \frac{\partial\phi}{\partial\theta}=\frac{\partial\phi}{\partial h}\frac{\partial h}{\partial\theta} \tag{11}$$

Differentiating (3) and (10) with respect to h, we get

$$\frac{\partial\theta}{\partial h}=\alpha(\theta_s-\theta_r)\left(\alpha+|h|^{\beta}\right)^{-2}.\beta\,|h|^{\beta-1},\qquad \frac{\partial\phi}{\partial\theta}=\overline{K}(h)=K(\psi) \tag{12}$$

Using (11) and (12), the transformed Richards equation (9) takes the form

$$c(\phi)\frac{\partial\phi}{\partial t}=\frac{\partial^2\phi}{\partial z^2}, \tag{13}$$

Where the functional coefficient c depends on ϕ through h as

$$c(\phi(h))=\frac{\alpha\beta(\theta_s-\theta_r)\,|h|^{\beta-1}}{\overline{K}(h)\left(\alpha+|h|^{\beta}\right)^2} \tag{14}$$

3.1. Finite difference discretization. Let $\Delta z=L/M$ and $\Delta t=T/N$. We construct a grid (z_j, t_n), with $z_j = j\,\Delta z$, j = 0,1,2,… ,M and $t_n = n\,\Delta t$, n = 0,1, 2,…,N. Let ϕ_j^n denote ϕ (z_j, t_n). The partial differential equation (12) can be approximated using forward difference in time and central difference in space as:

$$\frac{\partial\phi}{\partial t}\Big|_{(z_j,t_n)}\approx\frac{\phi_j^{n+1}-\phi_j^n}{\Delta t},\qquad \frac{\partial^2\phi}{\partial z^2}\Big|_{(z_j,t_n)}\approx\frac{\phi_{j-1}^n-2\phi_j^n+\phi_{j+1}^n}{\Delta z^2} \tag{15}$$

Let $0\le\lambda\le1$. Using a weighted average of the derivative ($\partial^2\phi/\partial z^2$) at two time levels, t_n and t_{n+1}, the equation (13) can be discretized as

$$\phi_j^{n+1}=\phi_j^n+\sigma_j^n\left[\lambda\left(\phi_{j-1}^{n+1}-2\phi_j^{n+1}+\phi_{j+1}^{n+1}\right)+(1-\lambda)\left(\phi_{j-1}^n-2\phi_j^n+\phi_{j+1}^n\right)\right] \tag{16}$$

Where $\sigma_j^n=\Delta t/(c_j^n\,\Delta z^2)$.

Equation (16) is used to update the values of ϕ_j^{n+1} for the internal nodes. Using the constant water pressure head at the upper and lower boundary, we get

$$\begin{aligned} \phi_0^{n+1} &= \overline{\beta_1}(t_{n+1}) \\ \phi_M^{n+1} &= \overline{\beta_2}(t_{n+1}) \end{aligned} \qquad (17)$$

The numerical scheme (16) and (17) represents a forward in time central in space (Forward Euler), backward in time central in space (Backward Euler) and Crank-Nicolson (CN) schemes for $\lambda = 0$, $\lambda = 1$ and $\lambda = 1/2$ respectively [15]. The error associated with this approximation is $O(\Delta z^2 + \Delta t)$ for all $\lambda \neq (1/2)$. In the case of Crank-Nicolson, it is $O(\Delta z^2 + \Delta t^2)$, second order accurate in both space and time.
We can express the above numerical procedure in a tridiagonal matrix system as

$$\begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ -\lambda\sigma_1^n & 1+2\lambda\sigma_1^n & -\lambda\sigma_1^n & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & -\lambda\sigma_{M-1}^n & 1+2\lambda\sigma_{M-1}^n & -\lambda\sigma_{M-1}^n \\ 0 & 0 & 0 & \dots & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \phi_0^{n+1} \\ \phi_1^{n+1} \\ \dots \\ \dots \\ \phi_{M-1}^{n+1} \\ \phi_M^{n+1} \end{bmatrix}$$

$$= \begin{bmatrix} \overline{\beta}_1(t_{n+1}) \\ (1-\lambda)\sigma_1^n\phi_0^n + (1-2(1-\lambda)\sigma_1^n)\phi_1^n + (1-\lambda)\sigma_1^n\phi_2^n \\ \dots \\ \dots \\ \dots \\ (1-\lambda)\sigma_{M-1}^n\phi_{M-2}^n + (1-2(1-\lambda)\sigma_{M-1}^n)\phi_{M-1}^n + (1-\lambda)\sigma_{M-1}^n\phi_M^n \\ \overline{\beta}_2(t_{n+1}) \end{bmatrix} \qquad (18)$$

The numerical scheme (18) can be used to update the transformed variable ϕ_j^n to its value in the next time level ϕ_j^{n+1}. However, we cannot advance the algorithm to the next time level ϕ_j^{n+2} without evaluating the function $c(\phi_j^{n+1})$ which requires computing the intermediate variable h_j^{n+1}. For this, we apply equation (13) which can be approximated as

$$h_j^{n+1} = h_j^n + \frac{\phi_j^{n+1} - \phi_j^n}{\overline{K}(h_j^n)} \qquad (19)$$

4. Results and Discussion

4.1 Experimental setup

To demonstrate the approximate solution of the prescribed model, the approximating process carried out in the above section is written in python and executed. We define a specific infiltration experiment in unsaturated soil and observe the corresponding outcomes. At first the setup consisted on a landslide area having dimension 70cm x 100cm. The physical constituent of the landslide area is sandy soil. The corresponding soil parameters and consecutive relationship between $\theta(\psi)$ and K(ψ) are interpolated from the work of Haverkamp et al. The simulation process is started with a flux of θ = 0.1cm^3/cm^3 maintaining water pressure head ψ = -61.5cm at the bottom of the annulus z=Z_{bot} as lower boundary. At z=Z_{top}, which coincides to the soil surface, a constant flux q(t)= 13.69 cm/hr for t <0.7 hr and zero normal flux for t > 0.7 hr maintained as upper boundary. For curved surface of the annulus, we set zero flux as an boundary.

To analyze the infiltration-induced landslide, we have interpolated the moisturity variation obtained from the above experiment to the hydrological model to obtain the stability analysis. We have used the solution obtained from equation (1) to the equation (3).

4.2 Results

In general, Richards Equation is a highly nonlinear degenerate PDE. The nonlinear behavior appears from the use of constitutive relationship between θ and ψ also K and ψ. The numerical solutions of the Richards Equation is computed using four different schemes namely Forward Euler, Backward Euler, Crank--Nicolson and RKL. First, we observed that Forward Euler scheme is conditionally stable where as Backward Euler, Crank--Nicolson and RKL schemes are unconditionally stable. Thus, to create a long simulation with the explicit scheme, it is necessary to fix the CFL--satisfying time step--size. Since we do not have accurate solution to estimate the errors, we use the numerical solution obtained from the fully explicit numerical scheme with a fine mesh (Δz= 1$ cm, and Δ t=0.002sec) as the reference solution. Figure 1 shows the water flow movement in unsaturated zone. The moisture content appears in different time level with the depth of the soil surface. The figure depicts the natural way of flowing of moisture in the unsaturated zone. Figure 2 and 3 depict discrepancy between the time stepping schemes for the solution of 1D Richards equation. Figure 4 and 5 show the corresponding safety factor obtained from the solution of Richards Equation incorporating to the infinite slope model. According to the figure we can predict that when moisturity increases then there is a possibility of surface failure.

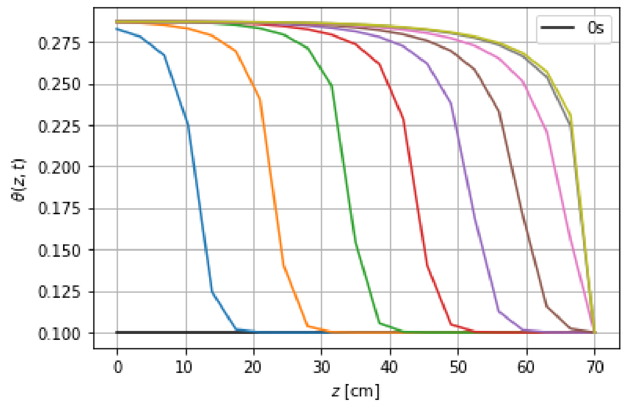

Figure 1: Flow phenomena in unsaturated zone by Richards Equation.

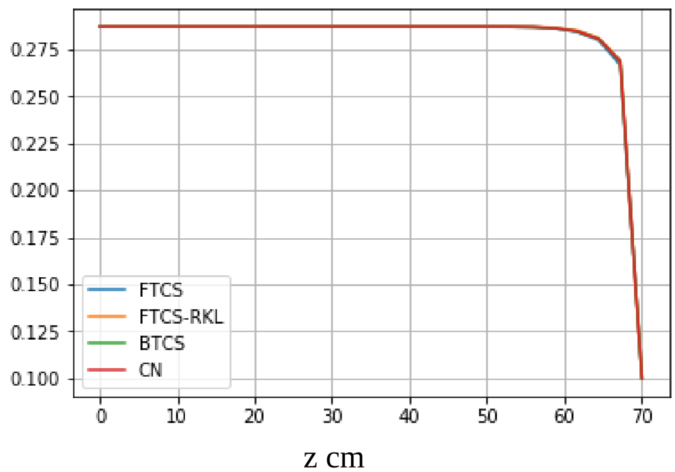

Figure 2: Comparison of time stepping schemes with super step 20.

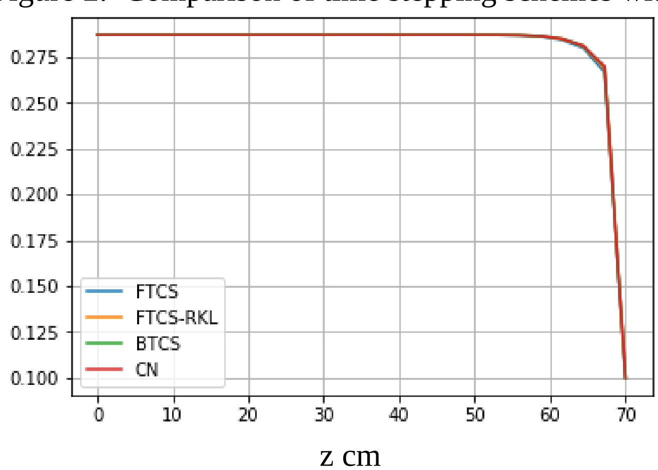

Figure 3: Comparison of time stepping schemes with super step 35.

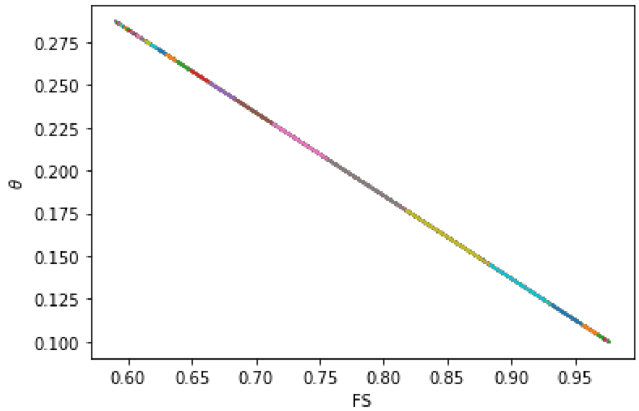

Figure 4: Variation of safety factor with moisture content at depth 10 m.

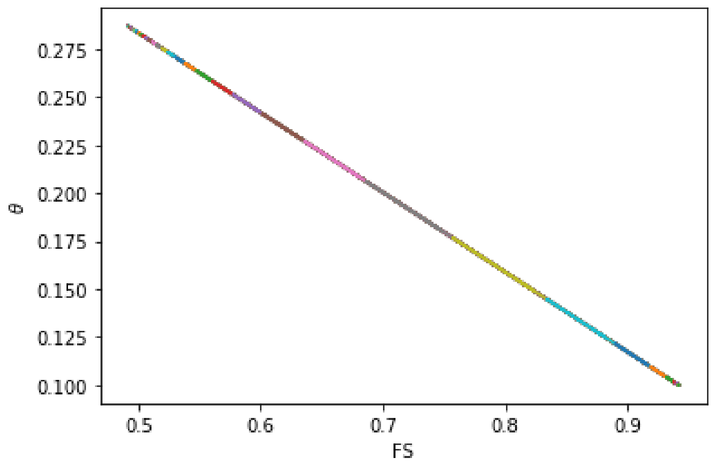

Figure 4: Variation of safety factor with moisture content at depth 15 m.

5. Conclusion

This paper has presented one dimensional Kirchhoff transformed Richards equation. We solved this equation numerically using forward in time and central in space, backward in time and central in space and Crank-Nicolson method, which are based on finite difference schemes. Because of more realistic and consistent behavior, we have used the solution obtained from Crank-Nicolson method in infinite slope stability model for the analysis of landslide hazards. We have used hydrological infinite slope stability model (3) to calculate the factor of safety. We obtained values of moisture content θ the most necessary quantity for evaluating safety factor in the model (3) and interconnected these values to model (3). The other parameters used in the model were obtained from available weather data. The result, we got after interconnecting the values of θ to the infinite slope equation shows that the prediction of landslide is possible for the available parameters depending on the constituent of the soil and nature of the land. We conclude that with the help of Richards equation we can evaluate the landslide hazards directly from the weather data also. This is the wide application of Richards equation from the point of geological view. This work can be extended to achieve accurate prediction of massive landslides to unsaturated heterogeneous soils with abruptly changing wetness conditions.

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