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Law of Exponential Change and its Application in some Physical Phenomena

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Abstract

Exponential functions increase or decrease very rapidly with small change in the independent variable. They describe the physical phenomena like growth or decay in a wide range with wide variety of physical, chemical, biological and environmental processes. The different types of mathematical models based on these functions partly accounts for their importance. In real world situations, we are interested in a quantity y *(velocity, temperature, electric current, population, compound interest, capacitance of the capacitor, whatever) that increases or decreases at a rate that at any time* t *is proportional to the amount present. If we also know the amount present at the time* $t = 0$, *i.e.,* y, *then we can find* y *which is a function of* t *by solving the differential equation* $\frac{dy}{dt} = ky$ *with initial condition* $y = y_0$. When $t = 0$, *we have* y *as function of* t, *if* y *is positive and increasing, then* k *is positive and* $\frac{dy}{dt} = ky$ *is used to say that the rate of growth is proportional to what has already been accumulated. If* y *is positive and decreasing, then* k *is negative and* $\frac{dy}{dt} = ky$ is used to say that the rate of decay is proportional to the amount still left. The solution of the equation $\frac{dy}{dt} = ky$ when $y = y_0$ at $t = 0$ is $y = y_0 e^{kt}$ *that gives growth for* $k > 0$ *and decay for* $k < 0$. *The number* k *is the rate of constant in the equation. This paper attempts to describe and analyze on the Law of exponential change and its application in growth and decay, like, population growth, compound interest, radioactive decay, heat transformed etc.*

Keywords: Exponential change, Growth and Decay, Initial condition, Differential equation.

Introduction

Numbers that are the solutions of polynomial equation with rational coefficients are called algebraic numbers. -1 is an algebraic number because it satisfies the equation $x + 1 = 0$ and $\sqrt{5}$ is also an algebraic number because it satisfies the equation $x^2 - 5 = 0$. Numbers that are not algebraic are called transcendental, e and π are the transcendental numbers.

A function $y = f(x)$, is called an algebraic function if it is satisfied the equation of the form $p_n y^n + p_{n-1} y^{n-1} + ... + ... + p_1 + p_0 = 0$ in which the p_n s are polynomials in x with rational coefficients. The function $y = \frac{1}{\sqrt{x+1}}$ is algebraic because it satisfies the equation $(x+1)y^2 - 1 = 0$. Here the polynomials are $p_2 = x+1$, $p_1 = 0$ and $p_0 = -1$. Polynomials and rational functions with rational coefficients are algebraic, as all sums, products, quotients rational powers and rational roots of algebraic functions. Functions that are not algebraic are called transcendental functions. The six basic trigonometric functions with their inverse are transcendental functions. The exponential and logarithmic functions are also transcendental functions (Apposto1, 1974). Ordinary differential equations are the important tools to describe natural phenomena in which these functions are directly connected. The solution of those equations is generally in the form of exponential and logarithmic functions. Thus, law of exponential change plays a crucial role to describe those natural phenomena in mathematical form.

Whatever we have a quantity y whose rate of change over time t is proportional to the amount of y present, we have a differential equation which satisfies the above condition as

$$\frac{dy}{dt} = ky$$

If, in addition, $y = y_0$ when $t = 0$, the solution of the equation is in the exponential form i.e., $y = y_0 e^{kt}$.

Exponential functions increases or decreases very rapidly with changes in the independent variable. They can describe physical phenomena like, growth and decay in a wide range of natural and industrial situations. The variety of models based on these function partially accounts for their importance.

In modeling many real-world situations, a quantity y increases or decreases at a rate proportional to its size at a given time t, examples of such quantities like as, include the amount of decaying radioactive materials, funds earning interest in a bank account, the size of population, charging and discharging of a capacitance, growth of current in a circuit containing a resistance and inductance, the temperature difference between a hot cup of coffee and the room in which it sits and for others physical phenomenon. Such quantities change accounting to the law of exponential change (Binmore 1982, Simmons 2007).

If the amount present at time $t = 0$, is y_0, then we can find y as a function of t by solving the following initial value problem,

Differential equation: $\frac{dy}{dx} = ky$ (1)

Initial condition: $y = y_0$ when $t = 0$

If y is positive and increasing, then k is positive, and we use equation (1) to say that the rate of growth is proportional to what has already been accumulated. If y is positive and decreasing, then k is negative, and we use equation (1) to say that the rate of decay is proportional to the amount still left.

We see that the constant function $y = 0$, is a solution of equation (1) if $y_0 = 0$. To find the non-zero solution, we have

$$\frac{dy}{dt} = k$$

$$\frac{1}{y}\frac{dy}{dt} = k$$

$$\int \frac{1}{y}\frac{dy}{dt}\,dt = \int k\,dt$$

$$\int \frac{1}{y}dy = \int k\,dt$$

$$\log|y| = kt + c$$

$$|y| = e^{kt+c}$$

$$|y| = e^{c}e^{kt}$$

$$y = \pm e^{c}e^{kt}$$

$$y = Ae^{kt}$$

By allowing A to take on the value 0 in addition to all possible values $\pm e^{c}$. We can include the solution $y = 0$ in the formula. We find the value of A for the initial value problem by solving for A when $y = y_0$ and $t = 0$.

$$y_0 = Ae^{k0}$$

$$y_0 = A$$

The solution of the initial value problem is therefore $y = y_0 e^{kt}$. Quantities changing in this way are said to undergo exponential growth if $k > 0$ and exponential decay if $k < 0$. Thus the solution of the equation $\frac{dy}{dt} = ky, y = y_0$ when t = 0, is $y = y_0 e^{kt}$, that gives growth for $k > 0$ and decay for $k < 0$. The number k is the rate of constant of the equation. This equation is the law of exponential change (Thomas 2018, Rossl 2018).

Several physical phenomena related with population growth, continuously compound interest, radioactive decay (formation of new elements), heat transfer (Newton's law of cooling) follow the law of exponential change.

The main aim of this study is to obtain the application of exponential function in the case of physical phenomena like growth and decay. For this we use it to determine the population growth (people, plants, animals, microbes etc.), to determine the continuously compound interest, to find the radioactive decay of elements (carbon-14, polonium – 210, radon -222 etc.), to determine the transfer of heat of an object to the surrounding medium. The organization of the paper is as follows: In section 2 some physical phenomena are explored. In section 3, results with some experimental data are presented and the conclusion is in section 4.

1. Physical Phenomena with law of exponential change

Law of exponential change is a primary factor to determine the change of initial physical quantities in a discrete or continuous manner governing by an ordinary differential equation. In which the change is depends on the quantity which present at the first and there is no any input and output quantity in the system. Here we want to describe how does the process enhance and how can we use the law of exponential change in this situation (Simmons 2007, Murray 1972, Coddington, 1994, Subrahmanyam 1988).

2.1 For Population Growth, Compound amount: The number of individuals in a population (of people, plants, foxes, bacteria) is a discontinuous function of time because it takes on discrete values. However, as the number of individuals become large enough it can safely be described with a continuous function or a differential equation. If we assume that the proportion of reproducing individuals remains constant and assume a constant fertility, these at any instant the birth rate is proportional to the number of population $y(t)$ of individuals present. It further we neglect departures, arrivals, and deaths. the growth rate $\frac{dy}{dt}$ has been same as the birth rate ky. In other words, in the form of differential equation we have $\frac{dy}{dt} = ky$, so that $y = y_0 e^{kt}$. This expression is used to obtain the population growth after t years. Similarly, for continuously compound interest, $A(t) = A_0 e^{rt}$ is used to obtain the amount of money after t years in the continuous interest rate r.

2.2 For radioactivity: When an atom exists some of its mass as radiation, the remainder of the atom re-forms to make an atom of some new elements. This process of radiation and change is called radioactive decay and an element whose atoms go spontaneously through this process is called radioactive. Thus radioactive carbon–14 decays in nitrogen, radium through a number of intermediate radioactive steps, decays in to lead. Experiments have shown that at any given time the rate at which a radioactive element decays is approximately proportional to the number of radioactive nuclei present. Thus the decay of a radioactive element is describe by the equation $\frac{dy}{dt} = -ky, k > 0$. If y_0 is the number of radioactive nuclei present at time $t = 0$, the number still present at any later time t will be $y = y_0 e^{-kt}, k > 0$.

2.3 For heat transform: Soup left in a tin cup cools to the surrounding air. A hot silver ingot immersed in water cools to the temperature of the surrounding water. In situations like these, the rate at which an object's temperature is changing at any given time is roughly proportional to the difference between its temperature and the temperature at the surrounding medium, this observation is Newton's law of cooling and there is an equation for it as :

$$\frac{dT}{dt} = -k(T - T_s),$$

Where T is the temperature of the object at time t, and T_s is the surrounding temperature. If we substitute y for $T - T_s$, then in terms of y this equation can be written as $\frac{dy}{dt} = -ky$. Thus the solution of this differential equation is

$y = y_0 e^{-kt}$. Hence the above equation becomes $T - T_s = (T_0 - T_s)e^{-kt}$, where T_0 is the value of T at time t = 0.

2.4 Capacitance: The property of capacitor to store electricity is called capacitance. As we may measure the capacity of a water tank, not by the total mass or by volume of water it can hold, but by the mass in kg of water required to raise its level by one meter. Similarly, the capacitance of the capacitor is defined as the amount of charge required to create unit potential difference between its plates. Suppose we give Q coulomb of charge to one of the two plates of capacitor and if a p. d. of V volt is established between the two, then its capacitance is

$$C = \frac{Q}{V} = \frac{charge}{p.d.}$$

Hence, capacitance is the charge required per unit potential difference. And the capacitance of a given capacitor is determined by charging and discharging through resistor with the law of exponential change.

3 Result and Discussion

We have discussed here with some phenomena related to physical, chemical, and biological point of view. In those phenomena we have related the law of exponential change and want to know the consequence of the result to the previous one.

3.1 Population Growth : We have modeled population growth with the law of exponential change $\frac{dp}{dt} = kp, p(0) = p_0$, where p is the population at time t, k > 0 is a constant growth rate, and p_0 is the size of the population at time t = 0 the solution of the model $\frac{dp}{dt} = kp$ is $p = p_0 e^{kt}$. However an issue to be addressed is ‘’ how good is the model’’? To begin an assessment of the model, notice that the exponential growth equation

$$\frac{\frac{dp}{dt}}{p} = k. \qquad (2)$$

is constant. This rate is called the relative growth rate. The table depicts the world population at midyear for the years 2001 to 2023. Taking dt = 1 and $dp = \Delta p$. We see from the table that the relative growth rate in equation (2) is approximately the constant .01157. Thus based on the tabulated data with t= 0 representing 2000, t= 1, representing 2001 and so forth, the world population could be modeled by the differential equation: $\frac{dp}{dt} = 0.01157\,p$, with initial condition : p(0) = 6148.

The solution to this initial value problem gives the population function $p = 6148e^{0.01157t}$. In year 2023, so t= 23, the solution predicts the world population in mid year to be about 8022 million or 8.02 billion, which is just below the actual population 8.04 billion given by the U.S. Bureau of the census. It also forecast the world population in the mid year 2030 about 8688 i.e. 8.68 billion.

Table 1: World Population (Mid year)

Year	Population(millions)	$\frac{\Delta P}{P}$
2000	6148	$82/6148 \approx 0.0130$
2001	6230	$82/6230 \approx 0.0131$
2002	6312	$82/6312 \approx 0.0129$
2003	6393	81/6393 $\approx$ 0.0126
2004	6475	82/6475 $\approx$ 0.0126
2005	6558	83/6558 $\approx$ 0.0126
2006	6641	83/ 6641 $\approx$ 0.0124
2007	6725	84/6725 $\approx$ 0.0124
2008	6811	86/6811 $\approx$ 0.0126
2009	6898	87/6898 $\approx$ 0.0126
2010	6885	87/6985 $\approx$ 0.0124
2011	7073	88/7073 $\approx$ 0.0124
2012	7161	88/7161 $\approx$ 0.0122
2013	7250	89/7250 $\approx$ 0.0122
2014	7339	89/7339 $\approx$ 0.0121
2015	7426	87/7426 $\approx$ 0.0117
2016	7513	87/7513 $\approx$ 0.0115
2017	7599	86/7599 $\approx$ 0.0113
2018	7683	84/7683 $\approx$ 0.0109
2019	7764	81/7764 $\approx$ 0.0104
2020	7840	76/7840 $\approx$ 0.00969
2021	7909	69/7909 $\approx$ 0.00872

2022	7975	66/6975 $\approx$ 0.00827
2023	8045	70/8045 $\approx$ 0.00870

Sources: U.S. Bureau of the census (2023)

3.2 Cooling a hard Boiled egg: We have observed an example in which how did law of exponential change relate to the heat transfer to a boiled egg. For this we have taken a hard boiled egg at $98^\circ c$ and put it in a sink of water having. $18^\circ c$. After 5 minutes the egg's temperature became $38^\circ c$ in which water has not warmed appreciably and we observed it until the temperature of boiled egg became $20^\circ c$. We found that it is 13 minute to reach the observed temperature. Since it took 5 minute to reach $38^\circ c$., we have seen that additional 8 minute is required to reach $20^\circ c$.

3.3 Reducing the cases of an infectious disease: We have discussed here considering an example of population model to look at how the number of individuals infected by a disease within a given population decrease as the disease is appropriately treated. One model for the way of disease dies out when properly treated is the rate $\frac{dy}{dt}$ at which the number of infected people changes is proportional to the number y. The number of people cured is proportional to the number that has the disease. Suppose that in the course of any given year the number of cases of disease is reduced by 20% and if there are 10,000 cases arise in present and it is found that it takes about 10 years to reduce the infected population in 1000.

3.4 Carbon – 14, Dating : The decay of radioactive elements can sometimes be used to date events from the earth's past. In living organisms, the ratio of radioactive carbon, carbon – 14 to ordinary carbon stays fairly constant during the life time of the organism, being approximately equal to the ratio in the organism's surrounding at the time. After the organism's death, however no new carbon is ingested and the proportion of carbon – 14 in the organisms remain decreases as the carbon – 14 decays. People who do carbon-14 dating with a figure of 5700 years or its half life to find the age of that sample in which 10% of the radioactive nuclei originally present have decayed, we have used the decay equation depending on the law of exponential change and found that the sample is about 866 years old.

3.5 Capacitance of the Capacitor: We have observed the capacitance of a given capacitor by charging and discharging through resister in which the law of exponential change is directly applied. Here we connected a capacitor in a circuit with a battery as a stable source of current and external resistance R, with the emf of the source E with negligible internal resistance. The table below depicts the observation

Table 2. Least Count of Clock

S.No.	Time(t) sec	Current I		Resistor(R) Ohm
		Charging	Discharging	
1	0	60	90	R = 22 KΩ
2	5	40	60	
3	10	30	42	
4	15	24	30	
5	20	22	24	
6	25	18	15	
7	30	14	12	
8	35	10	10	
9	40	8	8	
10	45	6	6	

Table 3. Least Count of Clock

S.No.	Time(t) sec	Current I		Resistor(R) Ohm
		Charging	Discharging	
1	0	28	40	R = 47 KΩ
2	5	22	33	
3	10	18	24	
4	15	16	21	
5	20	16	18	
6	25	14	16	
7	30	12	12	
8	35	11	10	
9	40	10.5	9	
10	45	10	8.5	

Table 4 Charging and Discharging Calculation

Charging					Discharging						
S.N	$\frac{I_0}{I}$	ln $\frac{I_0}{I}$	C(MF)	slope	$\frac{I_0}{I}$	ln $\frac{I_0}{I}$	C(MF)	slope	Mean C(MF)	R	RC= $\frac{1}{slope}$
1	1	0			1	0					
2	1.5	0.405	561.17		1.5	0.405	561.17		561.17		
3	2	0.7	649.35		2.14	0.76	598.1		623.75		
4	2.5	0.92	741.1		3	1.1	619.83		680.46		
5	2.73	1	909.1	0.05	3.75	1.32	688.70	0.073	798.9	R =22 KΩ	
6	3.33	1.2	947		5	1.61	705.5		826.4		16.22
7	4.3	1.64	934		7.5	2.01	678.4		806.2		
8	6	1.79	878.78		9	2.2	723.1		805.9		

9	7.5	2	909.1		11.25	2.42	751.3		830.2		
10	10	2.3	889.33		15	2.7	757.6		823.46		

Charging and Discharging Calculation

Table 5

Charging					Discharging						
S.N	$\frac{I_0}{I}$	ln $\frac{I_0}{I}$	C(MF)	slope	$\frac{I_0}{I}$	ln $\frac{I_0}{I}$	C(MF)	slope	Mean C(MF)	R	RC= $\frac{1}{slope}$
1	1	0			1	0					
2	1.27	0.24	443.3		1.21	0.19	560		501.65		
3	1.56	0.44	483.6		1.67	0.5	425.5		454.55		
4	1.75	0.56	569.9		1.9	0.64	498.63		534.25		
5	1.75	0.56	759.9	0.028	2.22	0.8	532.0	0.04	675.45	R =47 KΩ	
6	2	0.7	759.9		2.5	0.9	591		646		29.41
7	2.33	0.84	760		3.3	1.2	532		670.15		
8	2.54	0.93	800.78		4	1.38	539.6		717.9		
9	2.67	0.98	868.4		4.44	1.5	567.4		713.63		
10	2.8	1.03	929.56		4.7	1.55	617.7		773.6		

From the above table we observed that when a capacitor is charged through a resister, it shows maximum current first then decreases continuously due to the law of exponential change. Hence the required capacitance has found by charging and discharging the capacitor through a resister.

3.6 Continuously Compound Interest: For continuously compound interest, the resulting formula for the amount of money in a bank account after t years is $A = A_0 e^{rt}$. Interest paid according to this formula is said to be compounded continuously. The amount of money after t years is calculated with the law of exponential change given as $A = A_0 e^{rt}$. With this a bank might decide it would be worth "we compound interest every second, night, and day – better yet we compound the interest continuously''. A suitable example is concluded that when a certain amount as Rs. 621 is deposited in a bank that pays 6% compounded continuously, in 8 years of time duration, i.e., the amount Rs. 1003.58, correspondingly had the bank paid interest quarterly (k=4), the amount would be Rs 1000.01. Thus, the effect of continuous compounding, as compared with quarterly compounding has been an addition of Rs 3.57. A bank might decide it would be worth this additional amount to be able to advertise as "we compound interest every second, night, and day – better yet we compound the interest continuously".

2. Conclusion

An analysis on different physical phenomena like as population growth, compound interest, radioactive decay, heat transformed in sourrounding, current in the L-R circuit, capacitance of the capacitor in charging and discharging, exponential increases or decreases of the quantity measure the rapidity with the small changes on the independent variable relate to those. Ordinary differential equations are the important tools to describe natural phenomena in which these functions are directly connected. The solution of those equations is generally in the form of exponential and logarithmic functions. Hence law of exponential change plays a crucial role to describe those natural phenomena in mathematical form. Customarily, the law of exponential change gives the significant prediction for the natural and industrial situations. Thus, the law of exponential change can be applied to those physical, chemical and biological phenomena to describe their nature in the current situations.

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