Received Date: March 2025 Revised: April 2025 Accepted: June 2025

Convergence and Divergence in Improper Integrals and Their Applications

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Doi: https://doi.org/10.3126/ppj.v5i1.85844

Abstract

This research paper is about the convergence and divergence of improper integrals, or integrals with infinite limits of integration, or integrands with singularities. The purpose of the research is to mathematically rigorously study under what conditions improper integrals converge or diverge and demonstrate the usefulness of these tendencies, particularly in probability theory, physics, and engineering. Detailed analysis and solved examples illustrate the use of convergence and divergence for ascertaining area, computing probabilities, and solving differential equations. The paper also compares different methods of determining convergence and divergence, including the comparison tests, the limit comparison tests, and the Cauchy principal value.

Keywords: Improper integrals, convergence, divergence, cauchy principal value, comparison test, real-world applications, infinite limits.

Introduction

Improper integrals in mathematical analysis are integrals where the integrand is unbounded at one or more points in the interval of integration, or the endpoints of integration are at infinity. Limits are used to define the integrals. For instance, the improper integral is expressed as follows when the upper limit of integration approaches infinity:

$$\int_a^{\infty} f(x)\,\mathrm{dx} = \lim_{b\to\infty} \int_a^b f(x)\,\mathrm{dx}$$

Similarly, the improper integral is written as a limit approaching the point of singularity if the function becomes infinite at a point within the interval:

$$\int_a^b f(x)\,\mathrm{dx} = \lim_{c\to a^+} \int_c^b f(x)\,\mathrm{dx} \text{ (Kathuria, 2023).}$$

Other than that, they are required in the calculation of integrals for most fields of mathematics and applied sciences, especially asymptotic analysis, convergence of infinite series, and the calculation of integral transforms like the Laplace and Fourier transforms.

There are two kinds of improper integrals. Type 1 improper integrals have infinite integration limits, for example:

$$\int_a^\infty f(x)\,\text{dx} \text{ or } \int_{-\infty}^\infty f(x)\,\text{dx} \text{ (Apostol, 1974).}$$

On the other hand, Type 2 improper integrals occur when the integrand contains singularities (points where the function becomes infinite) within the interval of integration. An example of such an integral is:

$$\int_a^b \frac{1}{(x-c)^p}\,\text{dx where } c \in [a, b] \text{ and } p > 0 \text{ (Chatterjee, 2012).}$$

There, careful observation is necessary near the singularity in order to determine whether or not the integral converges.

In mathematical modeling, convergence or divergence of improper integrals is significant. For getting meaningful results in mathematical and physical models, the whole area under the curve must be finite, as indicated by a convergent improper integral. For instance, a convergent integral ensures that the system will possess finite energy in heat conduction or wave propagation models. Divergent integrals, on the other hand, can indicate irrational or infinite values, in which case the model would have to be altered or regularized. It is therefore important to ascertain whether an improper integral converges in order to guarantee that mathematical models accurately describe the physical world.

Improper integrals have a great variety of applications in different fields. Improper integrals are used to find charge distributions in physics, solve wave equations using Fourier analysis, and find total energy in systems extending to infinity. Improper integrals have an important role in signal processing in engineering, particularly with the use of the Fourier and Laplace transforms to examine signals and system responses over infinite intervals of time. In defining probability density functions for continuous random variables in probability theory, especially in distributions such as the normal distribution, improper integrals find crucial application. For instance, the following provides the expected value of a continuous random variable X.

$$E(X) = \int_{-\infty}^\infty \text{x}\, f(x)\,\text{dx (Joyce, 2013).}$$

where $f(x)$ is the probability density function. Convergence of such integrals ensures that the expected value and other statistical measures are finite and sensible.

Therefore, improper integrals are essential to mathematical theory and many practical applications in probability, physics, and engineering, where it is important to know convergence to get finite, physically reasonable results.

Statement of Problem

This paper explores methods for determining the convergence and divergence of improper integrals and demonstrates their application to real-world problems.

Objectives

The main objectives of this research work are:

- To provide a full mathematical analysis of improper integrals, with a focus on convergence and divergence.
- To provide a comparative study of various methods for determining the convergence or divergence of improper integrals, including comparison tests, limit comparison tests, and the Cauchy principal value.
- To solve examples of improper integrals illustrating convergence as well as divergence.

Methodology

This study involves a theoretical analysis of improper integrals, focusing on two main types:

Type 1 improper integrals with infinite limits of integration, such as $\int_a^{\infty} f\,(x)\,\mathrm{dx}$

Type 2 improper integrals, where the integrand has singularities within the interval of integration, such as $\int_a^b \frac{1}{(x-c)^p}\,\mathrm{dx}$ with $c \in [a, b]$ and $p > 0$.

Comparison test, limit comparison test, and Cauchy principal value are the tests used in the study to ascertain convergence. The use of such methods to calculate the behavior of improper integrals in practical situations is shown through examples and worked problems.

Literature Review

One of the focal areas of mathematical analysis has been studying improper integrals, i.e., their convergence or divergence. The study of integrals with infinite limits or singularities was greatly aided by the early foundational work by mathematicians like Cauchy and Riemann. Cauchy's contributions, specifically his explanation of integrals over unbounded sets, laid the groundwork for the proper handling of improper integrals, and his ideas continue to play a major role in the development of integral calculus (Riemann, 1854). His introduction of the principal value as a way to handle singularities revolutionized the studies of divergent integrals and paved the way for techniques still utilized in modern analysis (Cauchy, 1823).

Comparison techniques are usually used to determine the convergence and divergence of improper integrals. By comparing a given integral with a second integral, which is known to be convergent or divergent, the comparison test offers an indirect way of figuring out what the given integral did. The limit comparison test, which tests the limit of a ratio of two integrands, is a more sophisticated technique used for testing convergence. These approaches are presented as the original means of deciding on the behavior of improper integrals with complex behavior at infinity or near singularities, and they are discussed to full extent in real analysis textbooks (Apostol, 1974; Rudin, 1976). These are required to study improper integrals because they utilize theoretical mathematics and applied sciences.

Improper integrals are critical in applied problems in physical and engineering applications. They are, for instance, employed in heat conduction and electromagnetic wave propagation to determine the total energy in systems that approach infinity or have singularities (Arfken & Weber, 2012). In order to find solutions for long-term system behavior, Fourier and Laplace transforms frequently applied in signal processing and systems engineering also depend upon the convergence of improper integrals over infinite intervals (Bracewell, 1999).

In probability theory, improper integrals determine probability density functions, particularly in continuous distributions like the normal distribution. Improper integral convergence is crucial in defining valid probability models since the integral of the probability density function over all possible values must converge so that the total probability is one (Ross, 2014).

Improper integrals cannot be solved by using advances in numerical methods, especially when their analytical solution is not feasible. With improper integrals, methods like adaptive quadrature and Monte Carlo integration have shown promise, especially with integrals in complex or multidimensional spaces (Burden & Faires, 2011). In computational science, engineering simulation, and economic modeling, where precise numerical integration is a requirement to deliver credible results, these numerical approaches have provided the applied usefulness of improper integrals (Bender & Orszag, 1978).

Lastly, it is not possible to ignore the part that improper integrals play in asymptotic analysis and a theory. Improper integrals' behavior lies at the heart of understanding these limits since asymptotic analysis revolves around analyzing functions at infinity or singularities. The convergence of improper integrals can be applied to describe physically meaningful quantities, and divergence can imply the need for regularization or alternative models. This has its serious implications on physics, engineering, and other applied sciences (Edwards & Penney, 2010). The numerical and analytical examination of improper integrals is therefore an area under active investigation with many important implications for many diverse fields**.**

Results and Discussion

Mathematical Action of Improper Integrals: Convergence and Divergence

Improper integrals arise when the interval of integration is infinite or if there exists a singularity in the integrand in the interval of integration. Mathematically speaking, improper integrals can be categorized broadly into two categories:

Type 1: Integration intervals which are infinite.

Type 2: Unbounded integrand at a point of the interval.

In order to examine the convergence or divergence of such integrals, let's decompose the process mathematically for both categories (Kathuria, 2023).

1. Type 1: Improper Integrals over Infinite Interval

Definition:

An improper integral is referred to as a Type 1 improper integral if the upper or lower limit of the integral is infinity. The integral of a function $f\ (x)$ from a to ∞ is defined as the limit:

The convergence or divergence of improper integrals is based on the behavior of the integrand close to points of singularity or as the limits of integration approach infinity. In this section, we provide explicit examples to demonstrate the study of improper integrals (Jyoti. 2019),

Example 1: Convergence of an Improper Integral

Consider the improper integral:
$I = \int_1^\infty \frac{1}{x^2}\ dx$
We evaluate this by finding the limit:
$I = \lim_{b\to\infty} \int_1^b \frac{1}{x^2}\ dx = \lim_{b\to\infty} [-\frac{1}{x}]_1^b = \lim_{b\to\infty} (-\frac{1}{x}+1) = 1$
Since the result is finite, the integral converges. (Narayan, & Raisinghania, 2007).
Example 2: Divergence of an Improper Integral
Consider the improper integral:
$I = \int_1^\infty \frac{1}{x} dx$
We evaluate this as: $I = \lim_{b\to\infty} \int_1^b \frac{1}{x}$dx $= \lim_{b\to\infty} [\frac{1}{x}]_1^b = \lim_{b\to\infty}$ [ln x]$_1^b$ $= \lim_{b\to\infty}$ (ln (b) − ln (1)) = $\lim_{b\to\infty}$ ln (b)
Since ln (b) increases without bound as $b \to \infty$, the integral diverges.
2. Type 2: Improper Integrals with Unbounded Integrands
Definition:
An improper integral is called a Type 2 improper integral when the integrand becomes unbounded (infinite) at a point within the interval of integration. Suppose $f\ (x)$ is continuous on
$[a, b)$, but $f\ (x)$ becomes unbounded as $x \to b$. The improper integral is defined as:
$\int_a^b f\ (x)$ dx = $\lim_{t\to b^-} \int_a^b f\ (x)$ dx

Example 3: (Convergent Case):
Consider the integral: $\int_0^1 \frac{1}{\sqrt{x}}$ dx
We calculate this as a limit:
$\int_0^1 \frac{1}{\sqrt{x}}$ dx = $\lim_{t\to 0^+} \int_t^1 \frac{1}{\sqrt{x}}$ dx
The antiderivative of $\frac{1}{\sqrt{x}}$ is $2\sqrt{x}$, so:

$\int_t^1 \frac{1}{\sqrt{x}}$ dx = 2 $\sqrt{1}$ − 2 $\sqrt{t}$ = 2 − 2 $\sqrt{t}$

Taking the limit as $t \to 0^+$:

$\lim_{t\to 0^+} 2 - 2\sqrt{t} = 2$

Thus, the integral converges to 2:

$\int_0^1 \frac{1}{\sqrt{x}}$ dx = 2

Example 4: (Divergent Case):

Consider the integral: $\int_0^1 \frac{1}{x}$ dx

We calculate this as a limit: $\lim_{t\to 0^+} \int_t^1 \frac{1}{x}$ dx

The antiderivative of $\frac{1}{x}$ is ln (x), so:

$\lim_{t\to 0^+} \int_t^1 \frac{1}{x}$ dx = $\lim_{t\to 0^+}$ (ln (1) – ln (t)) = $\lim_{t\to 0^+}$ (- ln (t)) = $\lim_{t\to 0^+}$ (In $\frac{1}{t}$) = ∞

Thus, the integral diverges: $\int_0^1 \frac{1}{x}$ dx = ∞ (Malik, & Arora, 1992).

Corollaries:

Corollary 1:

p-Test for Improper Integrals: The integral $\int_1^\infty \frac{1}{x^p}$ dx converges if p >1 and diverges if $p \le 1$.

Corollary 2:

Behavior Near Singularities: The integral $\int_0^1 \frac{1}{x^p}$ dx converges if $p < 1$ and diverges if $p \ge 1$.

This provides a general context for studying convergence or divergence of improper integrals in any mathematical context (Apostol, 1974).

Comparison Test

The comparison test is a valued method for determining convergence or divergence. For example, if

$f(x) \ge g(x) \ge 0$ and $\int_a^\infty f(x)\, dx$ converges, then $\int_a^\infty$ g $(x)\, dx$ also converges (Longo, & Valori, 2006).

Statement:

Given two functions $f(x)$ and $g(x)$, if $f(x) \ge g(x) \ge 0$ for all $x \ge a$, and the improper integral $\int_a^\infty f(x)$ dx converges, then the improper integral $\int_a^\infty$ g $(x)\, dx$ also converges.

Proof:

Step 1: Setup and Assumptions

We are given two non-negative functions $f(x) \ge g(x) \ge 0$ for all $x \ge a$.

We know that $\int_a^\infty f(x)$ dx converges to some finite value, say L_f, i.e.,

$\int_a^\infty f(x)$ dx = $L_f < \infty$

Step 2: Properties of Integrals

Since $f(x) \ge g(x) \ge 0$, the integral of $g(x)$ must be smaller than or equal to the integral of

$f(x)$ over the same interval:

$\int_a^\infty g(x)\,dx \le \int_a^\infty f(x)\,dx = L_f$

This inequality follows from the fact that integrating a smaller or equal function will result in a smaller or equal integral, provided both functions are non-negative.

Step 3: Comparison Test

Since $\int_a^\infty f(x)\,dx$ is finite (i.e., it converges), and $g(x) \le f(x)$, the integral $\int_a^\infty g(x)\,dx$ must also be bounded above by the finite value L_f.

By the comparison test, if $f(x) \ge g(x) \ge 0$ and $\int_a^\infty f(x)\,dx$ converges, then $\int_a^\infty g(x)\,dx$ must also converge.

Therefore, $\int_a^\infty g(x)\,dx$ converges.

Example 5:

Let's illustrate this with a concrete example of two functions where $f(x) \ge g(x) \ge 0$.

Let $f(x) = \frac{1}{x^2}$ and $g(x) = \frac{1}{x^3}$, and consider the integrals from $a = 1$ to ∞.

Step 1: Compute $\int_1^\infty f(x)dx$:

$\int_1^\infty f(x)dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^2}\,dx = \lim_{b\to\infty} \left[-\frac{1}{x}\right]_1^b = \lim_{b\to\infty} \left(-\frac{1}{x}+1\right) = 1$

Thus, $\int_1^\infty \frac{1}{x^2}\,dx$ converges to 1.

Step 2: Compute $\int_1^\infty g(x)dx$:

$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^3}\,dx = \lim_{b\to\infty} \left[-\frac{1}{2x^2}\right]_1^b = \lim_{b\to\infty} \left(\frac{1}{2b^2}+\frac{1}{2}\right) = \frac{1}{2}$

Thus, $\int_1^\infty \frac{1}{x^3}\,dx$ converges to $\frac{1}{2}$.

In this example, $f(x) = \frac{1}{x^2} \ge g(x) = \frac{1}{x^2} \ge 0$, and both integrals converge, which is consistent with the result of the comparison test.

Corollaries:

Corollary 3: If $f(x) \ge g(x) \ge 0$ for $x \ge a$, and if $\int_a^\infty f(x)\,dx$ diverges, then $\int_a^\infty g(x)\,dx$ may or may not diverge. This corollary points out that divergence of $f(x)$ does not necessarily imply the divergence of $g(x)$.

Corollary 4: If $g(x) \le h(x) \le f(x)$ for $x \ge a$ and if both $\int_a^\infty f(x)\,dx$ and $\int_a^\infty g(x)\,dx$ converge, then $\int_a^\infty h(x)\,dx$ also converges. This corollary is an extension of the comparison test because it establishes that if a function is sandwiched between two convergent integrals, then the integral of the function must be convergent.

Corollary 5: If $f(x) \ge g(x) \ge 0$ for $x \ge a$, and $\int_a^\infty g(x)\,dx$ diverges, then $\int_a^\infty f(x)\,dx$ must also diverge. This corollary highlight that divergence of the smaller function implies the divergence of the superior function (Apostol, 1974).

Limit Comparison Test

Theorem: Limit Comparison Test for Improper Integrals

Let $f\ (x)$ and $g\ (x)$ be two continuous functions defined on $[a, \infty)$, and suppose that:

$\lim_{x\to\infty} \frac{f\ (x)}{g\ (x)} = L$

where L is a positive, finite constant. Then, both improper integrals:

$\int_a^\infty f\ (x)$ dx and $\int_a^\infty$ g (x) dx either both converge or both diverge (Ash, 2012), (Chatterjee, 2012).

Proof:

In order to provide the proof of the Limit Comparison Test, we will split the proof into two general cases: when the integral converges, and when the integral diverges.

Convergence Case:

Assume $\int_a^\infty$ g (x) dx converges. We want to prove that $\int_a^\infty f\ (x)$ dx also converges.

Since $\lim_{x\to\infty} \frac{f\ (x)}{g\ (x)} = L$, where L is a positive finite constant, this means that for large values of x,

$f\ (x)$ behaves similarly to $g\ (x)$. More formally, for any small positive ϵ, there exists some $N > a$ such that for all $x > N$:

$(L - \epsilon)\ g\ (x) \le f\ (x) \le (L + \epsilon)\ g\ (x)$

Since $\int_a^\infty$ g (x) dx converges, we know that:

$\int_N^\infty\ (L\ - \epsilon)\ g\ (x)\ dx$ and $\int_N^\infty\ (L\ + \epsilon)\ g\ (x)\ dx$ also converge. Therefore, by the comparison theorem for improper integrals,

$\int_a^\infty$ f (x) dx must converge as well.

Divergence Case:

Now, assume that $\int_a^\infty$ g (x) dx diverges. We want to prove that $\int_a^\infty$ f (x) dx also diverges.

Again, since $\lim_{x\to\infty} \frac{f\ (x)}{g\ (x)} = L$, for large values of x,

$f\ (x)$ behaves similarly to $g\ (x)$. There exists some $N > a$ such that for all $x > N$:

$(L - \epsilon)\ g\ (x) \le f\ (x) \le (L + \epsilon)\ g\ (x)$

Since $\int_N^\infty$ g (x) dx diverges, we know that $\int_N^\infty\ (L\ - \epsilon)\ g\ (x)\ dx$ must also diverge. By the comparison theorem for improper integrals, $\int_a^\infty$ f (x) dx must diverge as well.

Thus, if $\int_a^\infty$ f (x) dx converges, then $\int_a^\infty$ g (x) dx also converges, and if $\int_a^\infty$ g (x) dx diverges, then

$\int_a^\infty$ f (x) dx also diverges.

This completes the proof.

Example 6:
Let's apply the Limit Comparison Test to the following functions:
$f(x) = \frac{1}{x^2+1}$ and $g(x) = \frac{1}{x^2}$
We wish to evaluate the convergence of the improper integrals:
$\int_1^\infty \frac{1}{x^2+1}$ dx and $\int_1^\infty \frac{1}{x^2}$ dx
Step 1: Compute the limit:

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{\frac{1}{x^2+1}}{\frac{1}{x^2}} = \lim_{x\to\infty} \frac{x^2}{x^2+1}$$

Dividing both the numerator and denominator by x^2, we get:

$$\lim_{x\to\infty} \frac{1}{1+\frac{1}{x^2}} = 1$$

Since $L = 1$ is a positive finite constant, we can apply the Limit Comparison Test.
Step 2: Determine the convergence of $\int_1^\infty$ g (x) dx:
This is a standard improper integral. The antiderivative of $\frac{1}{x^2}$ is $-\frac{1}{x}$ so:

$$\int_1^\infty \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^\infty = \left(-\frac{1}{\infty}\right) - (-1) = 0 - (-1) = 1$$

Therefore, $\int_1^\infty \frac{1}{x^2}$ dx converges.
Step 3: Apply the Limit Comparison Test:
Since $\int_1^\infty \frac{1}{x^2}$ dx converges and $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$, by the Limit Comparison Test, the integral $\int_1^\infty \frac{1}{x^2+1}$ also converges.

Corollaries:

Corollary 6: If $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$ and $\int_a^\infty$ g (x) dx converges, then $\int_a^\infty$ f (x) dx converges.

Corollary 7: If $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty$ and $\int_a^\infty$ g (x) dx diverges, then $\int_a^\infty$ f (x) dx diverges

Corollary 8: If $\lim_{x\to\infty} \frac{f(x)}{g(x)} = L > 0$, then both integrals $\int_a^\infty$ f (x) dx and $\int_a^\infty$ g (x) dx will converge or diverge together.

This theorem and its corollaries are useful in talking about the convergence of improper integrals by making the comparison with easier or known integrals.

Statement of Cauchy Principal Value

Cauchy Principal Value (P.V.) is a method of assigning a finite value to otherwise divergent integral due to singularities or infinite limits of integration. Mathematically, for an integral of a function $f(x)$ from the interval $[a, b]$ where the function may have a singularity at some point $c \in [a, b]$, the Cauchy Principal Value is expressed as

P.V. $\int_a^b$ f (x) dx = $\lim_{\epsilon\to 0^+} \int_a^{c-\epsilon}$ f (x) dx + $\int_{c+\epsilon}^b$ f (x) dx (Guiggiani, 1991).

Similarly, for the improper integral with the upper limit of integration extended to infinity, the Cauchy Principal Value is defined as:

P.V. $\int_{-\infty}^{\infty}$ f (x) dx = $\lim_{R\to\infty}\int_{-R}^{R}$ f (x) dx

This definition allows us to compute a finite approximate value for certain divergent integrals by cutting off the singularities symmetrically or dealing with the infinite limits. (Nyíri, & Barany, 2000).

Proof of Cauchy Principal Value

Consider the improper integral:

$I = \int_{-\infty}^{\infty} \frac{dx}{x}$

This integral converges at x = 0, but we may apply the Cauchy Principal Value method and assign to it a finite value. Begin by decomposing the integral into two parts, avoiding the singularity at x = 0:

$I_{P.V.} = \lim_{\epsilon\to 0}\int_{-\infty}^{-\in} \frac{dx}{x} + \int_{\in}^{\infty} \frac{dx}{x}$

Both integrals can be evaluated as logarithms:

$\int_{-\infty}^{-\in} \frac{dx}{x} = \lim_{R\to\infty} \mid \frac{-R}{-\in} \mid = \log\frac{R}{\in}$

$\int_{\in}^{\infty} \frac{dx}{x} = \lim_{R\to\infty} \frac{R}{\in}$

Now, combine both terms:

$I_{P.V.} = \lim_{\epsilon\to 0}$ (log $\frac{R}{\in}$ - log $\frac{R}{\in}$) = 0

Thus, the Cauchy Principal Value of this otherwise divergent integral is zero.

Example 7:

Consider the integral:

P.V. $\int_{-1}^{1} \frac{dx}{x}$

This integral has a singularity at $x = 0$. Using the Cauchy Principal Value, we compute it as:

P.V. $\int_{-1}^{1} \frac{dx}{x} = \lim_{\epsilon\to 0}\int_{-1}^{-\in} \frac{dx}{x} + \int_{\in}^{1} \frac{dx}{x}$

Evaluating both integrals:

$\int_{-1}^{-\in} \frac{dx}{x}$ = log $\mid \frac{\in}{1} \mid$

$\int_{\in}^{1} \frac{dx}{x} = -\log\epsilon$

Now, combine both terms:

P.V. $\int_{-1}^{1} \frac{dx}{x}$ = log 1 – log 1 = 0

Thus, the Cauchy Principal Value of this integral is 0. (Rudin, 1976).

Corollary 9: Symmetry of Function (Chatterjee, 2012).

If the integrand $f\ (x)$ is an odd function, i.e.,

$f\ (-x) = -\ f\ (x)$, then:

P.V. $\int_{-a}^{a}$ f (x) dx = 0

Corollary 10: Principal Value of Rational Functions

For a rational function $f\ (x) = \frac{P\ (x)}{Q\ (x)}$, where $Q\ (x) = (\mathrm{x} - \mathrm{c})^{\ n}$ and n > 0, the principal value may occasionally yield a finite value when the standard integral is divergent at $x = c$.

These examples and corollaries illustrate the powerful usefulness of Cauchy Principal Value in the approximation of integrals otherwise divergent due to infinite limits or singularities.

Applications

Physics:

In solving wave equations, examining charge distribution in electromagnetic fields, and finding energy in infinite systems, improper integrals are utilized (Kreyszig, 2011; Folland, 2007).

Engineering:

Engineers utilize improper integrals to analyze system responses and signals over infinite intervals, which are essential to signal processing in Fourier and Laplace transforms (Kreyszig, 2011).

Probability Theory:

For instance, the probability density function of the normal distribution varies all over the real line, and the convergence of the integral guarantees finite statistical values such as expected value (McCulloch, 2008; Brown & Hwang, 1997).

Acknowledgement

We would like to express our gratitude to our colleagues, mentors, and organizations for their valuable guidance and support towards the completion of this research work on the convergence and divergence of improper integrals and applications.

Conflict of Interest

The authors declare no conflict of interest in the publication of this research.

Conclusion

Convergence and divergence in improper integrals are fully addressed in this research paper with focus on the mathematical techniques used to examine the behaviors. It demonstrates the significance of improper integrals through examples and applications in areas where precise calculation is crucial, like probability theory, engineering, and physics.

Improper integrals are very valuable in theory and in applications but are made complicated by infinite limits or singularities. Whether mathematical models based on various domains of science are true for a particular case rests on their convergence or divergence. The paper illustrates how techniques such as the Cauchy principal value and comparison test help in the solution of such integrals, which are thus central to the solution of real problems.

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