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Results on Chatterjea's Contraction in b-Metric Space

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Abstract

In 1922, Polish mathematician S. Banach introduced contraction conditions in metric space and confirmed that every contraction mapping in a complete metric space gives a unique fixed point. In 1968, R. Kannan generalized Banach contraction, and S. K. Chatterjea modified the Kannan contraction in 1972.

This paper aims to show that any contractive condition need not be a weak contraction mapping in b-metric space.

Keywords: b-Metric space, Contraction mappings, Kannan-type and Chatterjea-type contraction.

Introduction

Fixed point theory and metric space play a vital role in studying non-linear analysis and topology. It guarantees the existence and uniqueness of fixed points of certain self-maps of metric spaces and provides a constructive method to find those fixed points. Also, fixed point theory has been used effectively in many other branches, such as physics. Chemistry, economics, computer sciences, Computational mathematics, biology, signal, image, graph theory, game theory, optimization problems, integral equations, differential equation, variational problem, and applied engineering areas. The study of fixed-point theory satisfying contractive-type conditions. During the last four decades, contractive conditions have emerged in many areas. 'Sur Quelques points Du Calcul Fonctionnel' furnishes the common idealization of many mathematical, physical, and other scientific problems in which the notion of "distance" appears. The notion of distance was later known as metric space. In 1906, M. Frechet [16] first introduced the concept of distance. In 1914, Housdroff inaugurated the distance notion by metric space name. Metric spaces have been generalized in many different ways. For example, b-metric space and probabilistic metric space are generalizations of metric space. In 1942, Karl Menger introduced the notion of probabilistic metric space as a generalization of metric space [21-22]. In 1922, Police Mathematician Stefan Banach proved very important results regarding contraction mapping, which is said to be the Banach contraction principle. Banach's contraction principle [6] is one of the pivotal results of the analysis.

In 1968, R. Kannan introduced Kannan-type contractions [19] as an extension of the classical Banach contraction theorem, which deals with mappings that contract distances between points. Kannan-type contractions provide a different criterion based on a combination of distances involving the image and pre-image of points. S.K. Chatterjea introduced Chatterjea-type contraction mapping [7], which is a generalization of the Banach contraction principle. Chatterjea-type contraction guarantees the existence of a unique fixed point under certain conditions. It is a specific type of contraction mapping used fixed point theory.

In 1989, Russian Mathematician I.A. Bakhatin [2] and in 1993, S. Czerwik worked on b-metric space [12-13]. This paper, by using Chatterjea's contraction conditions of an extension of the Banach Contraction Principle, provides different criteria based on a combination of distances involving the image and pre-image of points. Also, any contraction condition is not necessary for weak contraction conditions in b-metric space.

Preliminaries

We started by briefly recalling important definitions related to examples and notions from contractions that we use in the sequel.

Definition 2.1 [23] Let X be an abstract set and d be a distance function from $X \times X \to \mathbb{R}^+$ .Then, an order pair (X, d) is said to be a ***Metric Space*** if d satisfies the following conditions for all $x, y, z \in X$:

(i) $d(x, y) \geq 0$ iff $x = y$ (Identity)

(ii) $d(x, y) = 0$ iff $x \neq y$ (Positivity)

(iii) $d(x, y) = d(y, x)$ for all x, y in X (Symmetry)

(iv) $d(x, \mathrm{z}) \leq d(\mathrm{x}, \mathrm{y}) + d(\mathrm{y}, \mathrm{z})$ for all x, y, z in X (Triangle inequality)

Here, (i) and (ii) guarantee that the distance between any two points of X is always positive and only zero when the points coincide. (iii) assures that the order of measurement of the distance between two points is insignificant. (iv) is a statement of the familiar triangular inequality.

Definition 2.2 [1] Let X be a non-empty abstract set, and $\alpha \geq 1$ be a given real number. Then a distance function d from $X \times X$ into $[0,\infty)$ is said to be a b-metric space provided that for all $p,q,r \in X$ satisfies the following conditions.

B_1. $d(p,q)=0 \ iff \ \ p=q$

B_2. $d(p,q)=d(q,p) \ \ for \ all \ \ p,q \in X$

B_3. $d(p,r) \leq \alpha\left[d(p,q)+d(q,r)\right] \quad for \ all \ \ p,q,r \in X$

A pair (X,d) is said to be a ***b-metric space***. Thus, the definition of b-metric space is an extension of b-metric space. Also, if we assume that $\alpha = 1$ then we get the definition of the usual metric space. This reason shows that our results are more general than some results in metric spaces. Some related examples of b-metric space.

Example 2.1 [4] *The Space* $l_q \quad (0<q<1)$.

$$l_q = \left\{ x_n \subset R : \sum_{n=1}^{\infty} |x_n|^q < \infty \right\}$$

And function $f : l_q \times l_q \rightarrow R$ *then* $d(x,y) = \sum_{n=1}^{\infty} \left[|x_n - y_m|^q \right]^{\frac{1}{q}}$ *where* $x \rightarrow x_n, y \rightarrow y_n \in l_q$ *is b-metric space solving these we get* $d(x,z) \leq 2^{\frac{1}{q}} \{ d(x,y) + d(y,z) \}$

Example 2. The Space $l_q \quad (0<q<1)$ *of all real function* $x(t), t \in [0,1]$ *such that* $\int_0^1 |x(t)|^q \, dt < \infty$ *is b-metric space. If we choose* $d(x,y) = \int_0^1 \left\{ |x(t) - y(t)|^q \, dt \right\}^{\frac{1}{q}}$ *for each* $x, y \in l_q$.

Definition 2.3 [4] (Cauchy Sequence in b-metric Space). Let (X, d) be a b-metric space. A sequence $\{x_n\}_{n=1}^{\infty}$ in X is called the ***Cauchy Sequence*** if for every $\in > 0$ there exists a positive integer N such that $m,n \geq N \Rightarrow d(x_m, x_n) < \in$.

Thus, just like in the case of metric spaces, we can equivalently say that $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence if $d(x_m, x_n) \rightarrow 0$ as $m,n \rightarrow \infty$.

Definition 2.4 [3] (Convergent Sequence in b-metric Space). Let (X, d) be a b-metric space. A sequence $\{x_n\}_{n=1}^{\infty}$ in X is called converge to some $x \in X$ if for every $\in > 0$ there exists a positive integer N such that $n \geq N \Rightarrow d\left(x_n, x\right) < \in$ this is denoted by writing $\lim_{n\to\infty} x_n = x$.

Since $\left\{\left(d\left(x_n, x\right)\right)\right\}_{n=1}^{\infty}$ is a sequence of positive real numbers, this definition suggests the convergence of this sequence to zero to be a characterization of ***convergence sequence in b-metric space***. This is analogical to a similar characterization in a metric space.

Definition 2.5: [3] (Complete b-metric Space). If a b-metric space (X, d) is such that every Cauchy sequence in the space converges, then we define it to be a ***complete b-metric space***.

Definition 2.6 [11] (Continuity of b-metric Space,). Let (X, d) be a b-metric space. then, d is said to be continuous if for any two convergent sequences $\left\{x_n\right\}_{n=1}^{\infty}$ and $\left\{y_n\right\}_{n=1}^{\infty}$ of points in X. We have $\lim_{n\to\infty} d\left(x_n, y_n\right) = d\left(x, y\right)$ Where $\lim_{x\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$. Then it is called ***Continuity of b-metric space***.

Definition 2.7 [5] Let E be a nonempty set and $f : E \to E$ a self-map. We say that $x \in E$ is a fixed point of f if $f\left(x\right) = x$ and denoted by F f or Fix(f) set of all ***fixed points*** of f. Geometrically, the fixed point of a function $f\left(x\right)$ are the point of intersection of the curve $f\left(x\right) = x$ and the line y = x. In other words, a point that does not change under a certain map is a fixed point.

Example 2.1 *Let a cubic function* $y = f\left(x\right) = x^3 - 7x^2 + 15x - 8$, *be the fixed point at* $x = 1, x = 2, x = 4$ *because* $f\left(1\right) = 1,\ f\left(2\right) = 2,\ f\left(4\right) = 4$.

Example 2.2 *Let a function* $y = f\left(x\right) = x^2 + x + 1$,*be fixed point or not a fixed point.*

Solution:- Here $y = f\left(x\right) = x^2 + x + 1$ *and the line* $x = y = f\left(x\right)$

Thus, $y = x = x^2 + x + 1$

or , $x = x^2 + x + 1$

or $0 = x^2 + 1$

$\therefore x = \pm i$

Hence, it has no fixed point.

Definition 2.8 [9-10] Let E be a non-empty set and $f, g: E \to E$ a self-map. We say that $x \in E$ is a ***common fixed point*** of f and g if $f(x) = x = g(x)$.

Example 2.3 *Let* $f, g: E \to E$ *be functions sch that* $f(x) = x^3$ *and* $g(x) = \sin x$, then $x = 0$ *is a common fixed point.*

In 1986, G. Jungck introduced a new class of mappings, known as compatible mappings and proved some common fixed-point theorem in metric spaces.

Definition 2.9 [5] Let E be any set and $f: E \to E$ a self-mapping. For any given $x \in E$. We define $f^n(x)$ inductively by $f^0(x) = x$ and $f^{n+1}(x) = f(f^n(x))$ We recall $f_n(x)$ the nth iterative of x under T.

For any $x_0 \in X$ the sequence $\{x_n\}^{\infty}_{n \geq 0} \subset X$ given by

$x_n = f(x_{n-1}) = f^n(x_0)$, $n = 1, 2.3 \ldots$ is called the sequence of successive approximations with the initial value x_0. It is also known as the ***Picard iteration*** starting at x_0.

Definition 2.10 [8] Let (X, d) be a metric space then a mapping $f: \mathrm{x} \to \mathrm{x}$ is said to be a ***contraction*** on X if and only if there exists a positive number $\alpha < 1$ such that for all x, y in X.

$$d(f(x), f(y)) \leq \alpha d(x, y) \ldots \ldots \ldots (1.1)$$

If (x, d) is a complete metric space then the mapping 1.1 satisfying has a unique fixed point, and also inequality means that the function f brings points closer together in a specific Mathematical sense by at least a factor of α in terms of their distances. The concept of contraction mapping is central to the fixed-point theorem. This theorem is fundamental in many more areas of analysis because it provides a guarantee of the existence and uniqueness of solutions under some conditions. A natural question is whether we can find contractive conditions that will imply the existence of a fixed point in a complete metric space but will not imply continuity. R. Kannan established the following results in which the above question has been answered in the affirmative.

Definition 2.11 [7] Let (x, d) be a metric space then a function $f: \mathrm{X} \to \mathrm{X}$ satisfied the inequality $d(fx, fy) \leq \alpha [d(x, fx) + d(y, fy)] \ldots \ldots \ldots (1.2)$

Where α the triangular constant, $\alpha \in \left[0, \frac{1}{2}\right)$ and $x, y \in X$ then f has a unique fixed point. The function that satisfies 1.2 is called ***Kannan-type mapping***. In the same way, a contractive condition has been notion by Chatterjea as following way.

Definition 2.12 [7] Let (X, d) be a complete metric space satisfied by the inequality

$$d(fx, fy) \leq \alpha\left[d(x, fy) + d(y, fx)\right] \ldots \ \ldots \ \ldots (1.3)$$

Where α triangular constant, $\alpha \in \left[0, \frac{1}{2}\right)$ and $x, y \in X$ then f has a unique fixed point. The function that satisfies 1.3 is called ***Chatterjea type mapping***.

Particularly concerning the concept of distance lies in how these mappings contact the space under their particular conditions. both mapping conditions ensure that the mappings are points closer together in a specific manner, though their approaches are different. Some problems, particularly the problems of the convergent of measurable functions concerning measurable functions concerning measure, lead to a generalization of the notion of metric space. S. Czerwik [19] presented a generalization of famous Banach's fixed points theorem in so-called b-metric Space.

Definition 2.13 [5] Let (X, d) be matric space. A mapping $f: X \to X$ is called ***Weak contraction*** if there exists a constant $\delta \in (0,1)$ and some $L \geq 0$ such that inequalities $d(fx, fy) \leq \delta d(x, y) + Ld(\mathrm{x}, f\ \mathrm{y})$ and $d(fx, fy) \leq \delta d(x, y) + Ld(\mathrm{y}, f\ \mathrm{x})$ for all $x, y \in X$ Satisfied.

Definition 2.14 [5] Let (X, d) be a metric space. A mapping $f: \mathrm{X} \to X$ is called ***weak contraction*** if there exists a constant $\delta \in (0,1)$ and some $L \geq 0$ such that

$$d(fx, fy) \leq \delta d(x, y) + Ld(y, fx) \ldots \ \ldots \ (2)$$

For all $x, y \in X$.

Remak 2.1 [5] Due to the symmetry of the distance. The weak contractive condition (2) implicitly includes the following dual one

$$d(fx, fy) \leq \delta d(x, y) + Ld(\mathrm{x}, fy) \ldots \ \ldots \ (3)$$

For all $x, y \in X$.

Consequently, if we have to check the weak contractiveness of f. Then it is necessary to check both equations (2) and (3).

Remak 2.2 [5] it is clear that any contraction mapping is also weak contraction mapping in a (usual) metric space.

3. Methodology

Finding the result in pure mathematics based on existing theorems in literature. So, here to establish the result in this paper, contraction mapping in mapping metric space given by Banach, generalization of contraction mapping by Kannan, and modification of Kannan contraction by Chatterjee paper make the motivation and framework to establish the result. This is the main methods for this article.

Result.

Here, mentions first the useful theorem to reach our main result.

Theorem 4.1 [20] Let (X, d) be a complete b-metric space with a triangular constant $\rho > 1$. Let $f : X \to X$ be a self-mapping for which there exists $\alpha > 0$ such that $\alpha \in (0,1)$ and $\rho\alpha < 1$ which also satisfies

For all $x, y \in X$ $d(fx, fy) \le \alpha d(x, y)$

Then, *f* has a unique fixed point.

Theorem 4.2 [20] Let (X, d) be a complete b-metric space with a triangular constant $\rho > 1$. Let $f : X \to X$ be a function for which there exists $\alpha > 0$ such that $\alpha \in \left(0, \frac{1}{2}\right)$ which is also satisfies For all $x, y \in X$ $d(fx, fy) \le \alpha\left[d(x, fx) + d(y, fy)\right]$ Then, *f* has a unique fixed point.

Proposition 1.[20] Contrary to the known usual metric space, any mapping satisfying the contractive condition $x, y \in X$ $d(fx, fy) \le \alpha\left[d(x, fx) + d(y, fy)\right]$ need not be a weak contraction in a b-metric spaces unless under the term $\rho\alpha \in \left[0, \frac{1}{2}\right)$.

Proof: Here,

Let $T : X \to X$ be a mapping satisfy $x, y \in X$ $d(fx, fy) \le \alpha\left[d(x, fx) + d(y, fy)\right]$.

We have

$$d(fx, fy) \le \alpha\left[d(x, fx) + d(y, fy)\right]$$

$$\le \alpha\left[\rho d(x, y) + \rho d(y, fx) + \rho d(y, fx) + \rho d(fx, fy)\right]$$

$$\leq \alpha\left[\rho d(x,y)+2\rho d(y,fx)+\rho d(fx,fy)\right].$$

$$d(fx,fy)\leq \frac{\alpha\rho}{1-\alpha\rho}d(x,y)+\frac{2\alpha\rho}{1-\alpha\rho}d(y,fx).$$

According to the inequality, f is not a weak contraction unless the terms $\alpha\rho\in\left[0,\frac{1}{2}\right)$.

Theorem 4.3 [20] Let (X, d) be a complete b-metric space with a triangular constant $\rho>1$. Let $f:X\to X$ be a function for which there exists $\alpha>0$ such that $\rho\alpha\in\left(0,\frac{1}{2}\right)$ which is also satisfies

For all $x,y\in X$ $\quad d(fx,fy)\leq\alpha\left[d(x,fy)+d(y,fx\right]$.Then, f has a unique fixed point.

These theorems had one more condition. Which was actually a hint to develop a Cauchy sequence to prove, rather than a condition that was need to develop the proof. For this it was choose x_0 in X and construct a sequence $\sum_{n=0}^{\infty}x_n$ by $x_n=f^n(x_0)$.This sequence is then shown to be a Cauchy's sequence using these conditions in this theorem. This construction has not been overlooking in this paper. The proof of this theorem had a flaw and proof of $\sum_{n=0}^{\infty}x_n$ being a Cauchy sequence has some unstated assumptions. We include them in the following ways

- The flaw is that step marked as. Let (X, d) be a complete b-metric space and define a sequence $\{x_n\}_{n=1}^{\infty}\subset X$ by the recursion $x_n=fx_{n-1}=f^nx_0,\ n=1,2,3,\ldots\ \ldots\ \ldots$ where $x_0\in X$ and $\{x_n\}_{n>0}\subset X$ is called the sequence of successive approximations with the initial value x_0.Which is also called ***Picard iteration*** [30]starting at x_0. Then we obtain

$$\begin{aligned}
d(x_n,x_{n+1}) &= d(fx_{n-1},fx_n)\\
&\leq \alpha\left[d(x_{n-1},f(x_n)+d\left(x_n,f(x_{n-1})\right)\right]\\
&= \alpha\left[d(x_{n-1},x_{n+1})+d(x_n,\mathrm{x}_n)\right]\\
&= \alpha d(x_{n-1},x_{n+1})\\
&\leq \alpha\rho\left[d(x_{n-1},x_n)+d(x_n,x_{n+1})\right]
\end{aligned}$$

$$d\left(x_n, x_{n+1}\right) \le \frac{\alpha\rho}{1-\alpha\rho} d\left(x_{n-1}, x_n\right)$$

Note that $\alpha\rho \in \left[0, \frac{1}{2}\right)$ then $\frac{\alpha\rho}{1-\alpha\rho} \in [0,1)$ .Thus, f is a contraction mapping .Then there exists $x^* \in X$ such that $x_n \to x^*$ and x^* is a unique fixed point of f .

Then $d\left(x^*, f\left(x^*\right)\right) \le \alpha d\left(x^*, x_{n+1}\right) + \alpha\rho d\left(x^*, x_{n+1}\right) + \alpha\rho d\left(x_n, f\left(x^*\right)\right)$

Let $n \to \infty$ then it becomes that

$d\left(x^*, f\left(x^*\right)\right) \le \alpha\rho d\left(x^*, f\left(x^*\right)\right)$.

Using the continuity of the b-metric space it has been proved. But this theorem doesn't state that the condition and it had been illustrated by S. Cobzas [22] that a b-metric is not necessarily continuous.

- The proof of $\{x_n\}_{n=1}^{\infty}$ being a Cauchy sequence is said to be followed by using "a similar method as used in the proof of Theorem 1 and Theorem 2"[22]. Theorem 2 suggests the method similar to that of Theorem 1. So, we basically, the authors want us to use the procedure as used in Theorem 1. But while doing so, we obtained

$$d\left(x_{m,} x_n\right) \le bk^m \left(1 + bk + (bk)^2 + (bk)^3 + \ldots\ldots\ldots (bk)^{n-m-1}\right) d\left(x_0, x_1\right)$$

The authors have assumed that $bk < 1$, which led to the conclusion that the geometric series on the right was convergent and, therefore, the sequence was a Cauchy Sequence. Here, $k = \frac{b\alpha}{1-b\alpha}$. But, if we have $b = 20$, $\alpha = \frac{1}{80}$? . In such a case, we have

$$kb = \frac{b^2\alpha}{1-b\alpha} = \frac{\left(400 \times \frac{1}{80}\right)}{\left(1 - \frac{1\times 20}{80}\right)} = \frac{20}{3} > 1$$

In this case, the convergent of the said geometric sequence will not follow at all. The authors have not considered or mentioned such possibilities, which makes the proof incomplete.

Here, we wish to alter the conditions prescribed by Theorem 4.3 so that the new conditions would generalize Chatterjea's Fixed Point Theorem to a b-metric space and have no such questionable assumptions and flaws.

Theorem 4.4 [14-15] Let (X, d) be a complete b-metric space with a continuous b-metric and triangular constant $\rho > 1$. Let $f : X \to X$ be a function for which there exists $\alpha > 0$ such that $0 < \frac{\rho^2 \alpha}{1 - \alpha\rho} < 1$ and For all $x, y \in X$ $d(fx, fy) \le \alpha[d(x, fy) + d(y, fx)]$

Then, *f* has a unique fixed point.

Example 4.1 The following example satisfied Chatterjea-type contraction mapping. and the function $f(x) = \frac{x}{3}$ satisfied Chatterjea-type Contraction.

Solution: - Here, We have

$$d(f(x), f(y)) \le \lambda[d(x, f(y)) + d(y, f(x))] \ldots\ldots\ldots\ldots\ldots\ldots\ldots .. (1)$$

Given that $f(x) = \frac{x}{3}$ and $f(y) = \frac{y}{3}$

Now, $d(f(x), f(y)) = \left|\frac{x}{3} - \frac{y}{3}\right|$

$$= \frac{1}{3}|x - y|$$

Thus, $d(x, f(y)) = |x - f(y)| = \left|x - \frac{y}{3}\right|$

$$= \left|x - y + \frac{2y}{3}\right|$$

$$\le |x - y| + \frac{2}{3}|y|$$

$$d(y, f(x)) = |y - f(x)| = \left|y - \frac{x}{3}\right|$$

$$= \left|y - x + \frac{2x}{3}\right|$$

$$\le |y - x| + \frac{2}{3}|x|$$

$$[d(x, f(y)) + d(y, f(x))] = |x - f(y)| + |y - f(x)|$$

$$= |x-y| + \frac{2}{3}|y| + |y-x| + \frac{2}{3}|x|$$

$$= 2|x-y| + \frac{2}{3}(|x|+|y|)$$

From equation (1) $\frac{1}{3}|x-y| \le \lambda\left[2|x-y| + \frac{2}{3}(|x|+|y|)\right]$

When we take $\lambda = \frac{1}{6}$ then $\frac{1}{3}|x-y| \le \frac{1}{6}\left[2|x-y| + \frac{2}{3}(|x|+|y|)\right]$

$$= \frac{1}{3}\left[|x-y| + \frac{1}{3}(|x|+|y|)\right]$$

Since $\frac{1}{3}|x-y|$ is always less than or equal to $= \frac{1}{3}\left[|x-y| + \frac{1}{3}(|x|+|y|)\right]$

Hence $f(\mathrm{x}) = \frac{x}{3}$ satisfied Chatterjea types of contraction mapping.

Theorem 4.5 A Metric Space (X, d) that satisfies the Chatterjea-type fixed point in b-metric space needs not to be a week contraction in b-metric spaces unless under the $\lambda s \in \left[0, \frac{1}{2}\right)$**.**

Proof: - Let $f: X \to X$ be a mapping satisfying $d(f(x), f(y)) \le \lambda\left[d(x, f(y)) + d(y, f(x))\right]$

We have $d(f(x), f(y)) \le \lambda\left[d(x, f(\mathrm{y})) + d(y, f(x))\right]$

$$\le \lambda\left[sd(x,\mathrm{y}) + sd(y, f(y))\right] + \lambda d(y, f(x))$$

$$= \lambda sd(x,\mathrm{y}) + \lambda sd(y, f(y)) + \lambda d(y, f(x))$$

$$= \lambda sd(x,\mathrm{y}) + \lambda d(y, f(x)) + \lambda sd(y, f(y))$$

$$\le \lambda sd(x,\mathrm{y}) + \lambda d(y, f(x)) + \lambda s\left[sd(y, f(x)) + sd(f(x), f(y))\right]$$

$$\le \lambda sd(x,\mathrm{y}) + \lambda d(y, f(x)) + \lambda s^2 d(y, f(x)) + \lambda s^2 d(f(x), f(y))$$

$$d(f(x), f(y)) \le \frac{\lambda s}{1-\lambda s^2} d(x,\mathrm{y}) + \frac{(\lambda + \lambda s^2)}{1-\lambda s^2} d(y, f(x)) \text{..................................(i)}$$

Again, $d\left(f(x),f(y)\right)\le\lambda\left[d\left(x,f(y)\right)+d\left(y,f(x)\right)\right]$

$$\le\lambda\left[d\left(x,f(\mathrm{y})\right)+sd\left(y,\mathrm{x}\right)+sd(x,f(x))\right]$$

$$=\lambda d\left(x,f(\mathrm{y})\right)+s\lambda d\left(y,\mathrm{x}\right)+s\lambda d(x,f(x))$$

$$\le\lambda d\left(x,f(\mathrm{y})\right)+s\lambda d\left(y,\mathrm{x}\right)+s\lambda\left[sd(x,f(y))+\mathrm{sd}(f(\mathrm{y}),f(\mathrm{x}))\right]$$

$$=\lambda d\left(x,f(\mathrm{y})\right)+s\lambda d\left(y,\mathrm{x}\right)+s^2\lambda d\left(x,f(y)\right)+s^2\lambda d\left(f(y),f(x)\right)$$

$$=\left(\lambda+s^2\lambda\right)\left(x,f(\mathrm{y})\right)+s\lambda d\left(y,\mathrm{x}\right)+s^2\lambda d\left(f(y),f(x)\right)$$

$$d\left(f(x),f(y)\right)\le\frac{s\lambda}{1-s^2\lambda}d\left(y,\mathrm{x}\right)+\frac{\left(\lambda+s^2\lambda\right)}{1-s^2\lambda}\left(x,f(\mathrm{y})\right)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}$$

Equations of (i) and (ii) show that it is a weak contraction in the interval $\left[0,\frac{1}{2}\right)$**.**So by the Chatterjea type fixed point theorem in b-metric space f is not weak contraction unless the term $\lambda s\in\left[0,\frac{1}{2}\right)$**.**Thus it satisfied the interval $[0,1)$ also**.**

Conclusion

Here, we discuss b-metric space, different contraction mapping like Banach, Kannan, and Chatterjea which help the researchers and readers for its comparative study. Mainly, show how contractive condition need not be a weak contraction in b- metric space, and given examples too.

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