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# Weyl Algebra

Himal Belbase
himalbelbase@gmail.com
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Let k be be a field of characteristic zero and $K[x] = K[x_{1,\ldots,}x_n]$ rhe ring of polynomials in n commuting indeterminates over K where nis some positive integer.

Let $\partial/\partial x_{1,\ldots,}\partial/\partial x_n$be the usal K-linear derivations on K[x].then the K-linear map $\partial/\partial x_i$ map a polynomial into $\partial f/\partial x_i$

We will use the notation $\partial=\partial/\partial x_i$ so that $\partial_i(f)=\partial f/\partial x_{i.}$

**Definitions** The ring of K-linear operators on K[x] which is generated by the derivations $\partial_{1,\ldots,}\partial_n$ and the multiplication operators defined by the polynomials in K[x], is called the ring of K-linear differentialoperators on K[x].

This ring is denoted by $A_n(K)$ and known as Weyl algebra in n variable. For $1\leq i\leq n$,consider the operator $\partial_i x_i$ in the ring $A_n(K)$. Apply it to a polynomial $f\in K[x]$. Using the chain rule, we get,

$\partial_i(x_i f)= \partial_i(x_i)f+x_i\partial_i(f)$.

In other words,

$\partial_i x_i= 1+ x_i\partial_i$

Where 1 is the identity operator. We can re write this formula by using commutators.

If $P,Q\in A_n(K)$ then theircommutator is the operator$[P,Q]= PQ-QP$. Thus, the above formula becomes,

$[\partial i,xi]=1$

In the similar way we can get that

$[\partial i,xi]=\delta ij$,

$[\partial_i,\partial_j]= [x_i,x_j]= 0$

where$1\leq I,j\leq n$. Here, $\delta_{ij}$is the kronecker delta symbol: it equals 1 if i=j and zero otherwise. We will use multi index notation. A multi index is an element of$\mathbf{Z}^{\mathbf{n}}_{\geq 0}$, say $\alpha=(\alpha_1,\ldots,\alpha_n)$. Now by $x^\alpha$we mean the monomial $x_1^{\alpha 1}\ldots x_n^{\alpha n}$and similarly $\partial^\beta$denotes a $\partial$- monomial$\partial_1^{\beta 1}\ldots.\partial_n^{\beta n}$.

Here the length $|\alpha|$of multi-index $\alpha$is ,

$|\alpha|=\alpha_1+\ldots+\alpha_n$,

and the degree of $X^\alpha$ is $|\alpha|$.

**Theorem:**The ring $A_n$ (K) is simple. That is, if J is a two-sided ideal, then J=0 or J= $A_n$ (K).

**Proof**. Let J be a non-zero two sided ideal of $A_n(K)$. Choose $D\neq 0 \in J$. We will use induction on n. If n=0 then $A_0(K) = K$ , which is field and the result is obvious. We can suppose that the sub algebra $K < x_{1,\ldots\ldots,}x_{n-1}, \partial_{1,\ldots\ldots,}\partial_{n-1} > = A_{n-1}(K)$ is simple. Now it is enough to prove that $J\cap A_{n-1}(K) \neq 0$. Then $J\cap A_{n-1}(K) = A_{n-1}(K)$ by induction hypothesis . Since $1\in A_{n-1}(K)$ , $1\in J$ and hence $J = A_n(K)$ follows.

To prove that $J\cap A_{n-1}(K)\neq 0$ we can write

$D=\delta_0+\delta_1\partial_n+\ldots+\delta_s\partial_n^s$

Where $\{ \delta_j \}$ belongs to the sub algebra $A_{n-1}(K)[x_n]$.Here $\delta_s\neq 0$.If $s\geq 1$ we use the Relations$\partial_n^j x_n - x_n\partial_n^j = j\partial_n^{j-1}$

Then,

$Dx_n-x_nD=\delta_1+2\delta_2\partial_n+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots+s\delta_s\partial_n^{s-1}$

Since J is a two-sided ideal , $D_1=Dx_n-x_nD\in J$. If $s\geq 2$ we can continue as above and we get

$D_2=D_1x_n-x_nD_1$

After s steps we see that J contains the non-zero element $D_S=s!\delta_s$.(s! is non-zero because K has characteristic zero). Call this element E and we can write

$E = e_0+e_1x_n+\ldots+e_tx_n^t$

Where $\{ e_j\}$ belong to $A_{n-1}(K)$. if $t \geq 1$ we get

$E_1 =\partial_nE-E\partial_n$

$= e_1+2e_2x_n+\ldots+te_tx_n^{t-1}$

After t steps we get $t!e_t\in J\cap A_{n-1}(K)$ and hence $J\cap A_{n-1}(K)\neq 0$ as required.