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Exploring Certain Generalization of Arithmetical Functions

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Abstract

This article introduces a few generalized arithmetical functions and focuses on a few specific instances of these arithmetic functions. This study of certain generalizations of arithmetical functions supplements our understanding of number theory and its applications. By spreading the domain, range, or definition of traditional arithmetical functions, mathematicians unexposed new insights into the structure of integers and their relationships with other mathematical objects.

Moreover, the applications of these generalizations span across various fields of mathematics, highlighting their significance and relevance in modern mathematical research. Continued exploration of these generalizations promises further discoveries and advancements in the field of number theory and beyond.

Keywords: Arithmetical function, prime number, Additive lattice, Maximum and minimum exponents

MSC2020: 11A25, 11A51, 11N37, 11N56

1. Introduction

Arithmetical functions play a fundamental role in number theory, providing crucial insights into the properties of integers and their relationships [1]. These functions, such as the divisor function, Euler's totient function, and the Möbius function, are extensively studied due to their significance in various mathematical inquiries. In this article, we study certain generalizations of arithmetical functions, exploring their properties and applications.

Arithmetical functions, which are essential to number theory, are usually defined on the set of positive integers. Arithmetic function generalizations sometimes include expanding these ideas to other domains or changing the definitions to cover a larger variety of mathematical structures.

Functions defined on the set of positive integers are referred to as arithmetical functions, or number-theoretic functions. The study of the characteristics and connections between numbers, particularly integers, is known as number theory, and it depends heavily on these functions [1].

2. Literature review

Cass [2] generalize Euler's totient function, &phis; and sigma, the sum-of-divisors function. Two of the most well-known arithmetic functions, sigma and &phis; have been the subject of much research. After the generalizations are defined, he examines some of the characteristics of these generalized functions, including growth rates, multiplicativity, and fixed points, as well as their products, convolutions, and compositions. He discovers that the dual nature of sigma and &phis which has been observed—continues to grip true for the generalized versions when conjugate pairs are used. Atanassov [3] defined the two natural numbers n are different from prime numbers.

Smarandache [4] described that there are certain arithmetic functions which are similar to the Smarandache function. McCarthy [1] explained that many of the properties of arithmetical functions hold true, particularly the inversion properties and arithmetical identities. This will provide an overview of basic context and examine a number of cases, and derive some general conclusions that can be used in the unique circumstances that each example presents. Dunkan [5] explained that order of magnitude of the average of the exponents in the canonical factorization of an integer is discussed. In particular, it is shown that this average has normal order one and a result which implies that the average order is one is also derived.

3. Generalized Arithmetical Functions

One of the key generalizations involves extending the domain or range of traditional arithmetical functions. For instance, while the classical divisor function σ(n) sums up the divisors of an integer n, a generalized divisor function may consider different types of divisors, such as prime divisors, square-free divisors, or divisors satisfying certain congruence conditions [5].

Similarly, Euler's totient function φ(n) counts the number of positive integers less than n that are co-prime. Generalizations of this function may involve considering coprimality conditions with respect to specific sets of numbers or algebraic structures beyond the integers.

Another direction of generalization lies in modifying the definition of arithmetical functions themselves. For example, instead of considering functions defined over the integers, one may extend them to other algebraic structures such as polynomial rings, finite fields, or algebraic number fields. This extension opens up new avenues for exploration and applications in various branches of mathematics. A real or complex valued function on the positive integer is called an Arithmetic function, e.g.
The identity function defined by

$$I(n) = \begin{cases} 1, & if\ n = 1 \\ 0 & otherwise, \end{cases}$$

The unit function u(n) $\equiv 1, n \geq 1$ $N^{(S)}$ (n) = n^S, n≥ $1,\ S \in \mathbb{C}$ is an arithmetic function [1].
J. Sandor [6] introduced a function which is defined as

$$F_f^A\ (\eta) = \min\{\kappa \in A: \eta/f(\kappa)\}, \tag{1}$$

where A⊂ N, and f : N→ N is an arithmetic function.

Now, J. Sandor [7], for f (k) = $\phi(k), E,$ Euler totient function A = N has introduced, which is defined as

$$E(n) = \min\{k\varepsilon N : n/\phi(k)\}. \quad (2)$$

J. Sandor [8], have calculated the particular case of equation (1) for f(k) = $\emptyset^*(\kappa)$, unitary totient function and called as unitary totient minimum function defined as

$$E^*(n) = \min\{k \geq 1 : n/\emptyset^*(k)\}. \quad (3)$$

Recently an arithmetic function related to Euler minimum function have been introduced in J. Sandor and Egri. [9] defined as

$$H_\emptyset\,(n) = \min\{k \geq 1 \colon \emptyset(n)/\emptyset(k)\}, \quad (4)$$

and more generally,

$$H_g\,(n) = \min\{k \geq 1 : g\,(n)/g\,(k)\}, \quad (5)$$

For a given arithmetic g : N → N. The arithmetic function given in (5) for $g(n) = \emptyset^*\,(n),\ and\ g(n) = R\,(n),$ product of divisors of n, where unitary totient function = $\emptyset^*(n)$ is defined as

$$\emptyset^*(n) = \{\, (p_1^{a_1} - 1)(p_2^{a_2} - 1)\ldots(p_r^{a_r} - 1) \quad , \quad n = \prod_{i=1}^{r} p_i^{\alpha i} \quad (6)$$
$$n = 1$$

and product of divisor function R (n) is defined as

$$R(n) = d_1 . d_2 \ldots\ldots d_r, \quad (7)$$

where $d_1, d_2, \ldots\ldots, d_r$ are divisors of n. Also,

$$R(n) = (n)^{\frac{d(n)}{2}}, \quad (8)$$

where d(n) denotes number of divisors of n. In analogy with equation (5), we can define

$$H_{\emptyset^*}\,(n) = \min\{k \geq 1 :\ \emptyset^*(n)/\,\emptyset^*\,(k)\}, \quad (9)$$

and $$H_R\,(n) = \min\{k \geq 1 : R(n)/R\,(k)\}. \quad (10)$$

The proof of important results related with equations (9) and (10).

4.Important Results

Theorem 1. (a) $H_{\emptyset^*}\,(p^\alpha) = \begin{cases} 1 & ,p = 2 \text{ with } \alpha = 1 \\ p^\alpha & ,p = 2 \text{ with } \alpha > 1 \\ p^\alpha & ,p \geq 3 \text{ with } \alpha \geq 1 \end{cases}$,

(b) $H_{\emptyset^*}\,(2p^\alpha) = \begin{cases} 2^{\alpha+1}, & p = 2 \text{ with } \alpha > 0 \\ p^\alpha, & p \geq 3 \text{ with } \alpha \geq 1 \end{cases}$

(c) If n is odd then $H_{\emptyset^*}$ (2n) = $H_{\emptyset^*}$ (n).

Proof.

(a) Using equation (9), It is clear that $H_{\emptyset^*}\,(n) \leq n\ as\ \emptyset^*\,(n)/\emptyset^*\,(n)$.

Let $\emptyset^*\,(p^\alpha)\,/\,\emptyset^*\,(k)$ then using equation (6), we have

$$(p^\alpha - 1) = \emptyset^*\,(p^\alpha) \leq \emptyset^*(k) \leq k - 1\ \ for\ \ k \geq 2$$

$$\Rightarrow \quad \emptyset^\alpha \leq k\,.$$

$$\Rightarrow \quad H_{\emptyset^*}\,(p^\alpha) \geq p^\alpha.$$

for the value of $p^\alpha = 2$ then $H_{\emptyset^*}\,(p^\alpha) = 1$.

For the value of p = 2 , with $\alpha > 1\ and\ p \geq 3, with\ \alpha \geq 1,\ H_{\emptyset^*}\,(p^\alpha) = p^\alpha$,

If n is odd then

$\emptyset^*$ (2n) = $\emptyset^*$ (n),

So, (c) follows. This proves (b).

Theorem 2.

(a) $\sqrt{n} \le H_{\emptyset^*}(n) \le n$ for $n \ge 6$,

(b) $H_{\emptyset^*}(n) = H_{\emptyset^*}(m)$ if $\emptyset^*(n) = \emptyset^*(m)$.

Proof.

(a) Using equation (9), $H_{\emptyset^*}(n) \le n$ as $\emptyset^*(n) / \emptyset^*(n)$, and $H_{\emptyset^*}(n) \ge \emptyset^*(n), n \ge 1$

If $\emptyset^*(n) / \emptyset^*(k)$, then

$\emptyset^*(n) \le \emptyset^*(k) \le k - 1$ for $k \ge 2$.

Since, $\emptyset(n) > \sqrt{n}$ for $n \ge 6$ and $\emptyset(n) \le \emptyset^*(n)$,

$\therefore$ $\emptyset^*(n) > \sqrt{n}$ for $n \ge 6$.

Now (a) is proved.

(b) $H_{\emptyset^*}(n) = \min\{k \ge 1 : \emptyset^*(n) / \emptyset^*(k)\}$

$= \min\{k \ge 1 : \emptyset^*(m) / \emptyset^*(k)\}$

$= H_{\emptyset^*}(m)$,

If $\emptyset^*$ (n) $\emptyset^*$ = (m).

Theorem 3. If $H_{\emptyset^*}(m) / H_{\emptyset^*}(n)$, then $[\emptyset^*(m), \emptyset^*(n)] / \emptyset(H_{\emptyset^*}(n))$, where [6,7,8] denotes L.C.M.

proof.

Let $x = H_{\emptyset^*}(m)$ and $y = H_{\emptyset^*}(n)$. Thus from equation (6), $\emptyset^*(m) / \emptyset^*(x)$ and $\emptyset^*(n) / \emptyset^*(y)$.

Now, it is given that $\frac{x}{y}$ so

$\frac{\emptyset^*(x)}{\emptyset^*(y)}$

$\Rightarrow$ $\emptyset^*(m) / \emptyset^*(x) / \emptyset^*(y)$, gives

$\emptyset^*(m)/\emptyset^*(y)$.

Hence, $[\emptyset^*(m), \emptyset^*(n)] / \emptyset^*(y)$.

Theorem 4. If $\frac{H_{\emptyset^*}(m)}{H_{\emptyset^*}(n)}$, then

$\emptyset^*(m)/(\emptyset^*(H_{\emptyset^*}(m)), \emptyset^*(H_{\emptyset^*}(n)$.

Proof.

Let $H_{\emptyset^*}(m) = x$ and $H_{\emptyset^*}(n) = y$, then it gives $\frac{x}{y}$

$\Rightarrow$ $\frac{\emptyset^*(x)}{\emptyset^*(y)}$, but $\emptyset^*(m) / \emptyset^*(x) / \emptyset^*(y)$,

So, $\frac{\emptyset^*(m)}{\emptyset^*(y)}$.

Thus $\emptyset^*(m) / (\emptyset^*(x), \emptyset^*(y))$.

Theorem 5. If H $\emptyset^*$ (m) / H $\emptyset^*$ (n), then

$\emptyset^*(m) \le |\emptyset^*(H_{\emptyset^*}(n)) - \emptyset^*(H_{\emptyset^*}(m))|$.

Proof.

since, (a, b,) $\leq\ |b-a|\ for\ a\ \neq b,$ so, by theorem 4.

$$\emptyset^*\ (m) \leq \left(\emptyset^*(\ x), \emptyset^*\ (y)\right) \leq |\emptyset^*\ \ (H_{\emptyset^*}\ (n))\ \ -\ \ \emptyset^*\ (H_{\emptyset^*}\ (\ m))\ |.$$

Theorem 6. (a) H_R (n) = n $\quad \forall n \geq 1,$

(b) H_R (2n)= 2. H_R (n) $\quad \forall n \geq 1,$

(c) $H_R(n^2)$ = n. H_R (n) $\quad \forall n \geq 1,$

(d) H_R (n) = H_R (m) $\quad$ if R(n) = R(m).

Proof.

Now on using equation (10) then H_R (n)$\leq n$ as R(n)/R(n). On the other hand, since K/R(K)

$\therefore$ R(n) $\leq K\ as\ n\ \leq R(n),$ thus n$\leq k.$

$\Rightarrow H_R$ (n) $\geq n.$

So, H_R (n) = n $\forall n\ \geq\ 1$ .Since, H_R (2n) = 2n (using (a)) so H_R (2n) = 2. H_R (n). So, (b) is proved. Using (a) we get, $\Rightarrow H_R\ (\ n^2) = n^2$ = n. H_R (n). So, (c) is proved.

Now (d) is followed by theorem 10 as

H_R (n) = min {k$\geq 1 : R\ (n)/$ R(k)} = min {k$\geq 1 : R\ (n)/$ R(k)

= H_R (m).

So, (d) is proved.

Theorem 7. If R(m) / R(n), then (R(m), R(n)) =H_R (R(m), where denotes g. c. d. of R(n).

Proof:

since R(m) / R (n) therefore, (R (m), R(n)) = R(m), using theorem 6

(a) we get H_R (R(m)) = R (m).

So, (R(m), R (n)) = H_R (R(m)).

Theorem 8. If R (m) / R (n), then [R(m), R(n)] = H_R (R(n)), where denotes L.C.M. of R(m) and R(n).

Proof.

Now, R(m)/R(n) therefore, [R(m), R(n)] = R(n), using theorem 6, then

H_R (R(n)) = [R(m), R (n)].

So, theorem is proved.

Theorem 9. If H_R (m) / H_R (n), then [R(m), R) n)] / R (H_R (n)).

Proof.

Let x = H_R (m) and y = H_R (n), using equation (10), R(m) / R (x) and R (n) /R (y). On the other hand, if x / y then R(x) / R(y). So, using theorem 2.6 (a) R(m) / R (n) / R(y) $\Rightarrow$ R(m) / R(y). But R (n) too. So, [R(m), R(n)] / R (y).

So, [R(m), R(n)] / R (H_R (R(n)).

Theorem 10. If R (m) / R (n), then H_R (R (m)) $\leq\ |R\ (n) -\ R(m)|$.

Proof.

Using theorem 7, if R (m) / R (n) then, (R (m), R (n)) = H_R (R (m)). Since, (a, b) $\leq\ |b - a|$, so, $\left(R\ (m), R(n)\right) =\ H_R\ \left(\mathrm{R(m)}\right)\ \leq\ |\mathrm{R\ (n) - R(m)}|$.

5. Applications and Implications

Generalized arithmetical functions find applications in diverse areas of mathematics, including algebra, combinatorics, and cryptography. In algebra, they are used to study properties of number fields, group theory, and algebraic geometry. Combinatorically, they provide tools for counting and enumerating objects with specific properties, such as counting lattice points or partitions subject to certain conditions. Moreover, in cryptography, generalized arithmetical functions play a crucial role in designing and analyzing cryptographic algorithms, particularly those based on number theory such as RSA and elliptic curve cryptography.

Conclusion

This study adds to our knowledge of number theory and its applications by examining several generalizations of arithmetical functions. Additionally, this article discusses the characteristics, uses, and ramifications of arithmetical functions.

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