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Novel Approaches Characterizing Separation in Fuzzy Topology with Quasi-Coincidence

Dr.Kamalesh Kumar Lal Karn

Patan Multiple Campus T.U.
Email: *kkamalesh349@gmail.com*
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Abstract

Fuzzy sets were introduced by Zadeh in 1965, and three years later, Chang defined fuzzy topological spaces. Fuzzy topological spaces are families of fuzzy sets that satisfy the three classical axioms of topology. In this paper, we introduce and study some new notions of T_o separation axioms in fuzzy topological spaces using the quasi-coincident relation for fuzzy sets. Every ordinary (crisp) topological space vacuously satisfies the condition of being quasi-T_o. We define the quasi-separation axioms for fuzzy topological spaces, as quasi-T_o, quasi-T_1, Quasi-T_2. We have introduced and studied some new notions of T_o separation axioms in fuzzy topological spaces using the quasi-coincident relation for fuzzy sets. These quasi-separation axioms are weaker than the corresponding classical T_o separation axioms, but they are still useful for characterizing and studying fuzzy topological spaces.

Key Words: Fuzzy topological space, quasi - coincidence, Fuzzy quasi-To, $T_1$and T_2 space.

Introduction

Building on fuzzy sets introduced by (Zadeh, 1965), fuzzy topology emerged with Chang's work in (Chang, 1968). This framework classified fuzzy sets as open or not. (Goguen, 1973) later extended this concept by employing lattice structures instead of a closed interval. (Ying, 1991) linked fuzzy topology to Hohles topology, sparking further research. (Zhang and Xu, 1999) then defined neighborhoods in fuzzifying topological spaces. Separation axioms, a cornerstone of fuzzy topology (Shen, 1993; Yue & Fang, 2006), have traditionally focused on crisp points. This paper explores separation axioms (T_0, T_1, T_2) within the framework of fuzzy topological spaces.

Preliminaries:

Definition 2.1: X is a fuzzy T-space iff for all x, distinct grades λ, μ in [0, 1],
either, $X_\lambda \notin X_\mu$
or, $X_\mu \notin X_\lambda$.

Definition 2.2: X is a fuzzy T_0-space iff distinct fuzzy points e, d exist in X such that $e \neq d$,
either, $e \notin d$,

or, $d \notin e$

Definition 2.3: X is a fuzzy T1-space iff all fuzzy points are closed
The following implications are obvious:

$T_1 \Longrightarrow T_o \Rightarrow \text{quasi} - T_o$

Every ordinary (crisp) topological space vacuously satisfies condition of being quasi- T_o and, hence the quasi- T_o separation is a particularity in a fuzzy topology.

Let (X, T) be a quasi- T_o space,
Let, $x \in X$, and $\Delta = (p_1, p_2)(0 \leq p_1 < p_2 < 1)$;
then there exists $B \in T$ such that $B(x) \in \Delta$

In fact,

Let $\lambda = 1 - p_1$, $\mu = 1 - p_2$
Then, $\lambda > p > 0$

Since (X, T) is a quasi- T_o space, $X_\lambda \notin X_\mu$.
Hence there exists some open Q-neighborhood, i.e. $(B(x) > 1 - \lambda = p_1)$, which is not a quasi-coincident with x_μ, i.e. $(B(x) \leq 1 - \mu = p_2)$.
Hence $B(x) \in \Delta$.

Proving Quasi-T_0 space properties as theorems

The following properties concerning quasi-T_o space can be sharpened in the form of theorem as follows:

Theorem 1 *(Necessity)*:

- $B = \emptyset$ works for p = 0.
- For $0 < p < 1$, take an increasing sequence (Δn) converging to p with open sets (Bn) in each Δn.
- Their union ($B = \bigcup Bn$) is open with B(x) = p.

Theorem 1 *(Sufficiency)***:**

- Given distinct fuzzy points (x, λ) and (x, μ) ($\lambda \neq \mu$), there's B open such that $B(x) = 1 - \mu > 1 - \lambda$.
- B is open and not quasi-coincident with (x, λ), making it closed.

Theorem 2:

- (X, T) is T_0 iff quasi-T_0 and for any distinct x, y, and $\lambda \in [0, 1]$, there exists B open such that $B(x) = \lambda$ and $B(y) > \lambda$ (or $B(x) > \lambda$ and $B(y) = \lambda$).

Proof (Necessity):

Quasi-T_0 spaces are also T_0.
Given distinct points x, y, define $\lambda = 1 - \mu$.
We get distinct fuzzy points (x, λ) and (y, μ).
If (y, μ) isn't closed, there's an open set containing y but not (x, λ), making them uasi-coincident (contradiction).
Similar argument for (x, λ).

Proof (Sufficiency):

We only need to separate (x, λ) and (y, μ) (λ ≠ μ).

Define ρ = 1 - λ and σ = 1 - μ.

By assumption, there's B ∈ T such that B(x) = ρ and B(y) > σ.

B is open and not quasi-coincident with (x, λ).

Similar argument for (y, μ).

Theorem 3 *(Necessity)***:**

If all fuzzy points are closed, B = X works for λ = 1.

Otherwise, B = (x, λ) is open (given point is closed).

Theorem 3 *(Sufficiency)***:**

Let x be any fuzzy point. There exists B such that B(x) = λ and B(y) = 1 for all y ≠ x.

This implies the complement of (x, λ) is closed.

Definition 4:

T_2 spaces: for distinct points with different supports, there exist open sets not quasi-coincident with each other.

Proposition:

- Accumulation points of (x, λ) are of the form (x, μ) with μ > λ,
    - When μ ≤ λ, any open set can be quasi-coincident with (x, λ) at most once, so x isn't an accumulation point.
    - When μ > λ, T_2 guarantees open sets separating (x, λ) and (x, μ).

Example:

X = {y, z}, T with base {(y, 1), (z, 1 - ε): ε > 0}.

This is T_2, but not quasi-T_0 (no open set with value 0.5 at y).

Theorem 4:

If T_2 and quasi-T_0, then T_1.

Let y be a fuzzy point. The only possible accumulation point is (y, μ) with μ > λ.

By quasi-T_0 and previous theorem, there's B open and not quasi-coincident with (y, μ).

Hence, (y, μ) isn't an accumulation point, and y has none.

Result and Discussion

Since the derived set of every fuzzy point in a T_1 space is obviously ϕ, we obtain the result that the derived set of every fuzzy set on a T_1 space is closed.

The obtained results significantly contribute to the theoretical framework of fuzzy topological spaces, unraveling intricate relationships and shedding light on key properties. Theorems 1 and 2 provide a foundation for understanding the convergence of fuzzy quasi-To and To spaces, establishing conditions that illuminate the nuanced interplay between these essential topological concepts. These findings have implications not only for the theoretical underpinnings of fuzzy spaces but also for applications in various fields where a nuanced understanding of spatial relationships is crucial.

The exploration of T1 spaces in Theorem 3 adds depth to the discussion, linking specific open sets to the T1 property and offering insights into point separation within fuzzy spaces. This connection between topological properties and open sets provides a valuable tool for characterizing spaces based on their structural features. Furthermore, the analysis of fuzzy T2 (Hausdorff) spaces in Definition 4 and the subsequent Proposition contributes to our understanding of separation properties in fuzzy settings. Theorem 4 then establishes a unifying link between T2 and quasi-To spaces, showcasing the intricate harmony between these properties. Collectively, these results contribute to a refined understanding of fuzzy topological spaces, enriching both theoretical foundations and potential practical applications.

Conclusion

In conclusion, the presented theorems and proofs establish fundamental properties of fuzzy quasi-T_o spaces, T_o spaces, and T_1 spaces. The first theorem characterizes fuzzy quasi-T_o spaces and provides a proof for both necessity and sufficiency. The second theorem links T_o spaces to quasi-T_o spaces, outlining the conditions under which a space is both quasi-T_o and T_o. The third theorem introduces T_1 spaces and demonstrates their equivalence to certain open sets. Lastly, Theorem 4 establishes that a space simultaneously satisfying the T_2 and quasi-T_o properties is also a T_1 space. These results contribute to the understanding of fuzzy topological spaces, shedding light on their properties and relationships.

References

Chang, C. (1968). Fuzzy Topological Space. *Journal of mathematical analysis and publication, 24*, 182-190.

Goguen, J. A. (1973). The fuzzy Tychonoff theorem. *Journal of mathematical analysis and application*, *43*, 734-742.

Shen, J. (1993). Separation axiom in fuzzifying topology. *Fuzzy sets and Systems, 57* (01), 113-123.

Ying, M. S. (1991). A new approach for fuzzy topology. *Journal of mathematical analysis and publications*, *39* (03), 303-321.

Yue, Y., & Fang, J. (2006). On Separation axioms I-Fuzzy topological spaces. *Fuzzy Sets and Systems*, *157* (06), 780-793.

Zadeh, L. A. (1965). Fuzzy Sets. *Information and Computation* (08), 338-353.

Zhang, D., & Xu, L. (1999). Categories isomorphic to FNS. *Fuzzy Sets and System, 104* (03), 373-380.