Received Date: April 2022 Revised: May 2022 Accepted: June 2022

Fourier Series and Transformation with Integration and Summation

Yogendra Prasad Shah
Assistant Prof. of Mathematics Patan Multiple Campus
Tribhuvan University
Email : yog.9841@gmail.com

Abstract

The main objective of this article is to study the basic of Fourier transformation, Fourier Series, Integration and summation. This article gives the basic idea about Integration and summation: definitions, types, use in different field, limitations and may more. In this article first reader gain general information about integral and summation, rule and regulation to used integration and summation, condition to used integration and summation and application in different field.

Keywords: Fourier transformation, Fourier Series, Integration, Summation, Application.

Introduction

Fourier Series:A periodic function is broken down into a sum of sines and cosines with various frequencies and amplitudes using the Fourier series. A mathematical technique called the Fourier Transform separates a signal into its component frequencies. The time domain representation of the signal refers to the initial signal that changed through time. Given that it is frequency dependent, the Fourier transform is referred to as the frequency domain representation of a signal. The idea of a Fourier series was introduced by Euler and D. Bernoulli in their research on vibrating strings, but Fourier's groundbreaking work on heat conduction and a sequence of sine functions, as shown below, is where the theory of Fourier series really got its start:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \qquad (1)$$

Where b_n is the coefficient and value can be calculated as

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx \qquad (2)$$

However, it took the work of Dirichlet, Riemann, Lebesgue, and others over the course of the following 200 years to properly define which functions were expandable in such trigonometric series. He did not provide a convincing proof of convergence of the infinite series.The Fourier series of the function f is expressed as if the function has a period of 2π.

$$c_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad (3)$$

Where,

$$\sum_{n=1}^{\infty} a_n \cos nx = a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \ldots \quad \text{and} \sum_{n=1}^{\infty} a_n \sin nx = a_1 \sin x + a_2 \sin 2x + a_3 \sin 3x + \ldots \quad (4)$$

In integral form of Fourier coefficient c_0, a_n, b_n are expressed as,

$$c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\,dx,\ a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\,dx,\ b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\,dx \quad (5)$$

According to Euler the complex exponential $e^{i\theta} - \cos\theta + i\sin\theta$.Therefore

$$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) \text{ and } \sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta}) \quad (6)$$

Hence from above equation we get,

$$c_0 + \sum_{n=1}^{\infty} c_n e^{inx} + c_{-n} e^{-inx} \text{ and} c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx}\,dx \quad (7)$$

Hence if series is converges then we have,

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \quad (8)$$

Then, f is written as the superposition of two elementary functions, $c_n e^{inx}$, with a frequency of n/2π and an amplitude of c_n. The meaning of Equation can be interpreted in a variety of ways (8). Mathematical explorations by luminaries like Dirichlet, Riemann, Weierstrass, Lipschitz, Lebesgue, Fejer', Gelfand, and Schwartz have been sparked by inquiries into the types of functions permitted on the left side of (8) and the forms of convergence considered for its right side.

Fourier Transformer:The Fourier transform explains the connection between a signal's representation in the frequency domain and its time domain counterpart. A function is broken down into oscillatory functions using the Fourier transform. The Fourier transform is originally inspired by the study of Fourier series. The contribution of each wave to the sum can be determined using an integral thanks to the characteristics of sines and cosines. A few fundamental characteristics of the Fourier transform are linearity, translation, modulation, scaling, conjugation, duality, and convolution. Due to its close relationship to the Laplace transformation, the Fourier transform is used to solve differential equations.

In order to define the Fourier series of a periodic function $f(x)$ across the complete real line, x ∈ R, we must take the limit a → ∞ in the intravel −a/2 ≤ x < a/2,

$$f(x) = \lim_{x\to\infty} \sum_{n=-\infty}^{\infty} e^{ik_n x} f_n \text{ where} k_n = n\Delta k, \qquad \Delta k = \frac{2\pi n}{a} \qquad (9)$$

The wave-number quantum Δkdecreases to zero as a → ∞ result. As a result, the set of k_n values becomes a continuum, and the discrete sum is replaced by an integral over the set of k_n values. To accomplish this, we increase the sum by the fraction $\left(\frac{\Delta k}{2\pi}\right) / \left(\frac{\Delta k}{2\pi}\right) = 1$, which equals 1.

$$f(x) = \lim_{x\to\infty} \sum_{n=-\infty}^{\infty} \frac{\Delta k}{2\pi} e^{ik_n x} \frac{2\pi f_n}{\Delta k} \qquad (10)$$

On defining $F(k_n) = \frac{2\pi}{\Delta k} f_n$ then equsiton (10) become

$$f(x) = \lim_{x\to\infty} \sum_{n=-\infty}^{\infty} \frac{\Delta k}{2\pi} e^{ik_n x} F(k_n) \qquad (11)$$

This limiting phrase corresponds to the fundamental outline of an integral become.

$$f(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} F(k) \qquad (12)$$

The Fourier transform and the factor of 2π are essentially arbitrary.

$$F(k_n) = \lim_{a\to-\infty} \frac{2\pi}{\Delta k} f_n = \lim_{a\to-\infty} \frac{2\pi}{2\pi/a} \frac{1}{a} \int_{-a/2}^{a/2} dx\, e^{-ik_n x} = \int_{-\infty}^{\infty} dk\, e^{-ikx} f(x) \quad (13)$$

Hence, we have a pair of equations called the Fourier relations

$$F(k) = \int_{-\infty}^{\infty} dk\, e^{-ikx} f(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} F(k) \qquad (14)$$

The Fourier transform is represented by equation (14), and the inverse Fourier transform is represented by equation (14). According to these relationships, there is a specific counterpart function $F(k)$ defined over $k \in R$for any function $f(x)$ defined over $x \in R$, and vice versa. The inverse Fourier transform does the opposite of what the Fourier transform does, converting $f(x)$to $F(k)$.

Integrals

An integral of function $f(x) > 0$ on closed interval [a, b] is define as area under graph of $f(x)$ over an interval a to b. Mathematically, $I = \int_a^b f(x)dx$ and proceeding the integration one can write an integration I as the sum of areas of rectangular strips and to compute a limit as the number of strips increases as, where N= Number of strips, k= Index associated with the k'th strip & $\Delta x = x_{k+1} - x_k$ is the width of the rectangle [1]. Mathematician Georg Riemann provides a strict definition of the definite integral, stating that if $f(x)$ is a continuous function, it must be defined everywhere in the interval [a, b] and have a limit to the Riemann sums.

Finite and Infinite Summation

Summation uses "discrete" values, while integration usually uses continuous values over an uncountably infinite interval. Until Planck made the radical assumption at the spectrum was built on discrete values, not continuous, and switched the integration to sums, which then perfectly modeled the observations. Summation is adding up of a sequence of numbers. Usually, the summation is given in this form $\sum_{i=1} x_i$ x_i when the terms in the sequence have a pattern. $\sum_{i=1}^{N} a_i$ this form of summation is defined as finite sum or summation while the form $\sum_{i=1}^{\infty} a_i$ is called infinite sum or summation.

Integration typically employs continuous numbers over an uncountably infinite range, whereas summation typically uses "discrete" values. Before Planck made the radical assumption that the spectrum was constructed using discrete values rather than continuous values and converted the integration to sums, the observations were perfectly modeled. A series of numbers is added up in a summation. When the terms in the sequence follow a pattern, the summation is sometimes expressed in this form, $\sum_{i=1} x_i$ x_i. While the other is known as infinite sum or summation, $\sum_{i=1}^{N} a_i$ this form of summation is referred to as finite sum or summation. The index of summation is referred to as the i. The summation operator, a class of baryons in particle physics, macroscopic cross sections in nuclear and particle physics, self-energy in condensed matter physics, the balance of invoice classes and the total amount of debts and demands in economics, the collection of symbols that make up an alphabet in linguistics and computer science, etc. are all examples of things where sigma is used [2].

Numerous mathematicians, including Moritz, Andrew Russ Forsyth, Henri Poincare, Rickey, Fasanelli, Furinghettim, Silva &Arajo, and Liu & 'Fauvel's, regard epistemology and the history of mathematics to be appropriate for setting mathematical objects [3].Since its inception more than 4,000 years ago, mathematics has been a product of human invention. It came into being in response to various social and economic demands made by civilizations including Babylon, Egypt, India, China, Greek, and Rome, to name a few.While today's study focuses mostly on structures and covers a much larger range of topics than it did in Ancient Greece, mathematics was once a discipline concerned spatial and quantitative relationships. Florian Cajori discovered a motivating source of knowledge for instructors in the history of mathematics in 1894. The importance of mathematics history and its use in the teaching and learning process have come to light over the past 20 years [4].

Aristotle (384–322 B.C.), Over the course of history, people have accumulated an inexhaustible list of practical mathematical procedures, problem-solving strategies, instruments for surveying and measuring, logical conundrums, and logical arguments. All aspiring math teachers must "Demonstrate knowledge of the historical development" of number and number systems, Euclidean and non-Euclidean geometry, algebra, calculus, discrete mathematics, statistics and probability, measurement and measurement systems, and knowledge about contributions from various cultures, according to the NCTM/NCATE Program Content Standards [5].

In the fifth century B.C., Hippocrates produced the first quadrature of a figure having a curved boundary. Archimedes quadratured a parabolic segment in the third century B.C. Gregory St. Vincent produced a discovery in 1647 that connected the area under the hyperbola with Napier's logarithm function. Both Newton and Leibniz contributed to the development of calculus and its many applications, but neither provided what we would today refer to as a rigorous definition of a definite integral. Cauchy provided the first accurate definition of a definite integral in the 1820s. Lebesgue fixed the problems with the Riemann integral at the start of the 20th century [6].

The limit we find for the exact area for a function $f(x)$, considered between $x = a$ and $x = b$, is represented by $I = \int_a^b f(x)dx$ and is known as the definite integral, or simply the integral, of $f(x)$ with respect to x from $x = a$ to $x = b$.

Mathematicians including Pierre de Fermat, James Gregory, Isaac Barrow, Isaac Newton, and Gottfried Wilhelm Leibniz, but primarily Newton and Leibniz, invented the differential and integral calculus, antiderivatives, and the calculus fundamental theorem in the 17th century [7]. A history of mathematics that focuses on limits, functions, derivatives, integrals, and infinite series includes the history of calculus. ShenKuo, a Chinese genius, created packing equations for integration in the eleventh century. Numerous publications with certain calculus concepts were produced by Indian mathematicians. Aryabhata created the formula for the sum of the cubes in the year 500 AD in order to calculate the volume of a cube, which was a crucial milestone in the evolution of integral calculus.

For orthonormal families of complex valued functions, $\{\varphi_n\}$, Fourier Series are sums of the $\{\varphi_n\}$ that can approximate periodic, complex valued functions with arbitrary precision. Fourier series have a significant role in both theoretical and applied mathematics. Because uniform convergence of the Fourier series is assured for continuous, bounded functions, Fourier series are particularly alluring. Finally, it is demonstrated that when initial boundary value problems are given, Fourier series are related to the solution of linear partial differential equations. Numerous mathematical conclusions, like the general Euler-McLaurin summation formula, depend heavily on periodic Bernoulli functions. Delvos demonstrated that periodic B-splines can be interpolated by translation using Locher's approach for uniform meshes.An infinite sum of sines and cosines is used to represent the expansion of a periodic function f(x) into a Fourier series. The orthogonality relationships between the sine and cosine functions are used in Fourier series [8].

History and Report of FS and FT

Series 1729 of trigonometry Euler, Leonhard Interpolation, the issue of determining function values in an arbitrary point x if its values for x = n, where n is an integer, are known, was formulated and study on it started. 1747. A method for interpolation developed in 1729 was utilized in the trigonometric series of a function derived from the motion of the planets. Daniel Bernoulli is responsible for the first series decomposition of a signal in 1753. 1754 The reciprocal value of the mutual distance between two planets is a series in cosine functions. Le Rondd'Alambert Jean. 1757 Alexis Claude Clairaut Cosine series of a function discovered through research on solar disturbances [9].

Out of all the transformation techniques, the Fourier transform is the most straightforward. It takes less time and is utilized in wireless networks, mechanical systems, and industries. Fast, accurate, and highly noise-resistant methods are needed for the mitigation of power quality disturbances, particularly in power distribution systems. The developments of oversampling, computerized sifting, and clamor shaping are typically welcomed in the Fourier Transform (FT) area for stifling the quantization commotion [10].

In Solid Bodies, which was read to the Paris Institute on December 21, 1807. The Fourier series was criticized by Laplace and Lagrange because "his analysis... left something to be desired on the score of generality and even rigour." The theory of series was created by Fourier, who then used it to solve boundary-value issues in the fields of biology, medicine, photography, computerized tomography for X-rays, and partial differential equations. Transforming Fourier Discrete Fourier Transform (DFT), integral transform, maps one discrete vector to another discrete vector, maps one function to another,Image processing using Fast Fourier Transform (FFT), 2DFFT, Discrete Fourier Transform in two dimensions, spatial frequency in images, and other concepts [11].

Early links to music and the fundamental physics of sound can be found in the history of mathematics. The ratios and intervals inherent in music and contemporary tuning systems occur naturally and contain elements of mathematics. The relationship between mathematics and music dates back at least to the Greek philosopher Pythagoras in the sixth century B.C. The first of these is French theologian, philosopher, mathematician, and music theorist

Marin Mersenne (1588–1648). Calculus started to be utilized as a tool in the 18th century when people started talking about vibrating strings.The Taylor Series was discovered by Brook Taylor, who identified a sine curve as the solution to a differential equation describing the vibrations of a string based on initial conditions. Additionally, Taylor's problem with the vibrating string led D'Alembert to a differential equation,

$$\frac{\partial^2 y}{\partial x^2} = \alpha \frac{\partial^2 y}{\partial t^2} \qquad (15)$$

where the x-axis represents the direction of the string, the y-axis is the displacement at time t, and the origin of the coordinates is at the end of the string [12].Our capacity to use Fourier methods on digital computers has significantly increased with the introduction of the rapid Fourier transform method. Following a discussion of the algorithm and its programming, a theorem connecting its operands, the finite sample sequences, to the continuous functions they frequently are meant to approximate is presented. Runge and Konig and Stumpff published the Fast Fourier Transform Algorithm many years ago, and it mainly explained how to exploit the sine-cosine functions' symmetries to minimize the amount of processing by factors of 4, 8, or even more [13].

Research Methodology

Pseudo-Fourier Transform

A measurable function $f: R \to [0,1]$ has a pseudo-Fourier cosine transform based on the semiri([0,1], ⊗, ⊙),

$$F_c^{\oplus}[f(x)](\omega) = g^{-1}\left(\frac{1}{2^{1/2}}\right) \odot \int_{[-\pi,\pi]}^{\oplus} g^{-1}(\cos(\omega x)) \odot E_p(x)\,dx \quad (16)$$

for each actual numberω (if the right side exists).

A measurable function $f: R \to [0,1]$ has a pseudo-Fourier sine transform based on the semiring

([0,1], ⊗, ⊙),

$$F_s^{\oplus}[f(x)](\omega) = g^{-1}\left(\frac{1}{2^{1/2}}\right) \odot \int_{[-\pi,\pi]}^{\oplus} g^{-1}(\sin(\omega x)) \odot O_p(x)\,dx \quad (17)$$

for each actual number ω (if the right side exists).

If it is possible to reverse the transition and go from pseudo-Fourier pictures back to functions. The straightforward calculation yields,

$$f(\omega) = f(gof)(\omega) = \frac{1}{2^{1/2}} \int_{-\infty}^{\infty} gof(t) e^{-i\omega t} dt \qquad (18)$$

where the conventional Fourier transform, f is used. Further study on this issue should focus on developing and examining discrete pseudo-Fourier transforms based on the background that has been provided as well as potential applications of the transforms that have been obtained, particularly in the field of signal and image processing but also in the study of nonlinear partial differential equations [14].

A specific identity known as the Parseval's identity is applicable if a function $f(x)$ converges uniformly in $(c, c + 2l)$. The identical can be expressed as,

$$\frac{1}{2l}\int_{c}^{c+2l}[f(x)]^2dx = \frac{a_0^2}{2} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2) \quad (19)$$

The Parseval's Identity for the function $f(x)$ in the range $(c, c + 2l)$ is represented in this example as (19). Both the sine and the cosine terms must be present for a Fourier series to be considered complete. In the range $(-l, l)$, the generalized version of the Fourier Series is written as,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right) \qquad (20)$$

Now, if the function $f(x)$ is even, the coefficient bn will be zero inside the range $(-l, l)$, hence the expression for (18) is,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L}\right) \qquad (21)$$

As a result, expression (21) no longer contains a sine term, and this is known as the Half Range Cosine Series. Similar to how the coefficients a0 and a would be zero within the limits $(-l, l)$if the function $f(x)$ were an odd function, the series would be shown as,

$$f(x) = \sum_{n=1}^{\infty}\left(b_n\sin\frac{n\pi x}{L}\right) \qquad (22)$$

As a result, the series (22) is now known as the Half Range of Sine Series because it no longer contains any cosine terms in the statement.For the frequency analysis of discrete temporal data, the Discrete Fourier Transform (DFT) is easier to use. The Fourier Transform in intravel $0 \leq \omega \leq 2$ of a finite time sequence of length L is given by,

$$F(s) = \sum_{n=0}^{L-1} f(n)e^{iwt} \quad (23)$$

$x(n) = 0$outside the range $0 \leq n \leq L-1$is indicated by the higher and lower indices in the summation. When we sample F(s) at evenly spaced frequency of $2\pi k/$, k=0, 1… N-1, where $N \geq L$,

$$F(k) = \sum_{n=0}^{L-1} f(n) e^{-\frac{2\pi i n}{N}} \qquad (24)$$

Equation (24) is referred to as the Discrete Fourier Transform of $f(n)$ because it allows us to convert a finite sequence into a set of frequency samples of length N. The frequency samples are obtained by evaluating the Fourier Transform into a set of equally spaced frequencies, and the sequence $f(n)$ from the frequency samples is given by,

$$f(n) = \frac{1}{N} \sum_{k=0}^{L-1} F(s) e^{\frac{2\pi i n}{N}} \quad (25)$$

The Inverse Discrete Fourier Transform (IDFT) is also known as (25). [15].

Properties of the Definite Integral:

1. $\int_a^b f(x)dx = 0$, the integral is zero if both its upper and lower limits are equal.
2. If a ≤ b ≤ c, $I = \int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx.$ According to this, the integral of a function over two intervals that are joined together equals the sum of the integrals over each individual interval.
3. $\int_a^b cf(x)dx = c\int_a^b f(x)dx.$ for any constant c.
4. $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx.$
5. If f(x) ≤ g(x) in [a, b] then $\int_a^b f(x)dx \le \int_a^b g(x)dx.$
6. $\int_a^b cdx = c(b-a)$ If c> 0 and (b − a)>0.
7. Additionally, based on the relationship stated above, if M is any upper bound and m is any lower bound for f(x) in the range [a, b], so that m ≤ f(x) ≤ M, then

$$m(b-a) \le \int_a^b f(x)dx \le M(b-a)$$

8. In case of exchanging limits. $\int_a^b f(x)dx = -\int_b^a f(x)dx.$

Leibniz created the sign $\int$ to represent the integral. From the Latin word for sum, it is an extended S. The integral of &x1/2 serves as the limit when Sum approaches integral, S approaches $\int f(x)dx = \int x^{1/2}dx$ the rectangles of base x. The "dx" signifies that Δx is getting closer to zero. The heights v(x) of the curve are equal to the heights vj of the rectangles.

Proper and Improper Integration

An integration $\int_a^b f(x)dx$ is called improper integral if , $a = \infty$ or $b = \infty$ i.e.one or both integration limits are infinite, and At one or more points $a \le x \le b$, $f(x)$ is unbounded. Singularities of f are such a point (x).

Additionally, the improper integral of third types is the integral that meets both conditions (a) and (b). Additionally, integrals of unbounded functions can be represented as at point b, at point a, at points a and b, and at point $c \in (a, b)$, as well as integrals on unbounded intervals as <a, ∞), (-∞ , b>, (-∞ , ∞).

Defined and Indefinite Integrals

Indefinite integrals are those that have no upper nor lower bounds. As defined integral is defined as the integral that has both an upper and a lower limit. This is in the form $\int f(x)dx$ of and have result in the form of variables. This takes the form of $\int_a^b f(x)dx$ and yields a number as a consequence.

Conclusion

A periodic function is broken down into a sum of sines and cosines with various frequencies and amplitudes using the Fourier series. Mathematical studies by luminaries like Dirichlet, Riemann, Weierstrass, Lipschitz, Lebesgue, Fejer, Gelfand, and Schwartz. The Fourier transform explains the connection between a signal's representation in the frequency domain and its time domain counterpart. A function is broken down into oscillatory functions using the Fourier transform.The area under the function's graph across the specified interval $a \le x \le b$ can be used to determine the definite integral of a function $f(x) > 0$ on a closed interval [a, b]. Integration typically employs continuous numbers over an uncountably infinite range, whereas summation typically uses discrete values. A series of numbers is added up in a summation.

References

Asher, K. (2013).Fourier series And Fourier Transform.IOSR Journal of Mathematics 4(6):73-76.

Barnes, M. (1999).Introduction to Integration Part 2: The Definite Integral. Mathematics Learning Centre. University of Sydney, 7-13.

Cooley, J.W., Lewis, P.A.W. and Welch, P.D. (1969).The fast Fourier transform and its applications. IEEE Transactions on Education 12(1):27-31.

Dankova, M., Papuga, I.S. and Bede, B.(2006). Pseudo-Fourier Transform: Research report No. 109. University of Ostrava: Institute for Research and Applications of Fuzzy Modeling.

Dejić, M. and Mihajlović,A.M. (2014). History of Mathematics and Teaching Mathematics. Teaching Innovations 27(3):15–20.

Ezhil,S.S. (2017). Real Time Application of Fourier Transforms. Indonesian Journal of Electrical Engineering and Computer Science8(2):574-577.

Glynn,E.F.(2007).Fourier Analysis and Image Processing. Stowers Institute for Medical Research, 3-47.

Hammond,J.K. (2011). Mathematics of Music.UW-L Journal of Undergraduate Research XIV:1-11.

Keshet, L.E. (2014).Integral Calculus with Applications to the Life Sciences. Course Notes for Mathematics 103, 49-52.

Mathews,D., Brown,P., Evans,M. and Hunt,D. (2013). Integration – A guide for teachers (Years 11–12). Education Services Australia, 14-33.

Panasuk,R.M. and Horton,L.B. (2013). Integrating History of Mathematics into the Classroom: Was Aristotle Wrong?. Journal of Curriculum and Teaching 2(2):37-39.

Ryoo,C.S., Kwon,H.I., Yoon, J. and Jang,Y.S. (2014). Fourier Series of the Periodic Bernoulli and Euler Functions. Hindawi Publishing Corporation: Abstract and Applied Analysis, 1-4.

Sen,S. (2017). Integrating History of Mathematics into The Mathematics Teaching: A Study on Concept of Variable in Elementary Algebra in Indian Perspective. International Journal of Multidisciplinary Education and Research **2**(1):16-17.

Shenitzer, A. and Steprans, J. (1994). The Evolution of Integration. The American Mathematical Monthly, 101(1):66-70.

Stanković, R.S., Astola,J.T. and Karpovsky,M.G. (2013). Remarks son History of Abstract Harmonic Analysis. Tampere International Center for signal processing, 3-30.