Exploring Difficulties of Master's Degree Students in Learning Linear Algebra

Abatar Subedi, PhD
Central Department of Education, Tribhuvan University
abatar.subedi@cded.tu.edu.np
Received Date: October 22, 2024; **Reviewed Date:** Nov. 28, 2024
Accepted Date: Dec. 28, 2024

ABSTRACT

This paper aims to explain the difficulties of students in learning algebraic concepts in linear algebra course at master degree level. For, I conducted a case study of ten master's degree students selected purposively, who are studying linear algebra course in the department of mathematics education, Tribhuvan University, Kirtipur in 2024. I used semi-structured interview guidelines, observation notes and performance test to explore students' difficulties in learning linear algebra. The results shows that students have conceptual as well as procedural difficulties in learning algebraic concepts, proving theorems, solving numerical problems, and constructing examples of abstractions. Likewise, they have poor prerequisites to learn linear algebra which make them difficult to learn new concepts at master level. Thus, it is suggested that the teacher who are teaching linear algebra course at that level need to have develop strong prerequisites to improve their students' performance and to mitigate conceptual as well as procedural difficulties to learn linear algebra.

Keywords: Algebraic concepts, proving theorems, solving problems, constructing examples

Introduction

Linear algebra is a major course of pure mathematics in higher education. It is a branch of mathematics that deals with the abstract mathematical structures closed under addition and scalar multiplication together with the theory of solving system of linear equations, matrices, determinants, vector spaces, and linear transformations. It is the study of vectors and linear functions (Cherney et al., 2013). It represents a process of abstraction where students move from concrete objects and methods in familiar vector spaces like $\mathbb{R}^2$ and $\mathbb{R}^3$ to generalizations at $\mathbb{R}^n$, an abstract level (Rensaa et al., 2020, p.297). Thus, the mathematical concepts included in linear algebra at master level are more theoretical that are abstract in nature like vector space, modules, transformations and their relations.

The major tasks of the students at master level are to memorize facts, conceptualize definitions with examples and counter examples, and understand theoretical/numerical proofs of the theorems by using notations, symbols, and language of linear algebra in general. For the completion of these tasks, students need to connect learned ideas to learn new concepts in the course. These learned ideas, generally named as prerequisites, and their connection are important to understand new concepts, properties and relations; to construct examples, counter examples; and to derive, prove, and establish theorems in linear algebra. In this context, how the master degree students were completing these major tasks while studying and learning linear algebra course was become a concern issue of this paper.

Likewise, the students who do not have sufficient prerequisites for the linear algebra course at master level can have difficulties to learn and conceptualize the abstract algebraic structures and their relations in it. It is understood that learning is continuous process in every aspect of life including, in particular to understand linear algebra concepts at master level. Fry, Ketteridge, & Marshall (2009) have stated that "Learning is not a single thing; it may involve mastering abstract principles, understanding proofs, remembering factual information, acquiring methods, techniques and approaches, recognition, reasoning, debating ideas, or developing behavior appropriate to specific situation; and it is about change (p.8)". So, linear algebra learning can involve lots of mental and physical activities like mastering abstract principles, axioms, propositions; remembering facts, definitions, notation, symbols; and understanding proofs, proofs techniques, language together with the skills and capacity of recognition, reasoning and debating ideas of abstract concepts included in linear algebra course.

Regarding the abstract learning, Lalonde (2013) pointed that students need to have think about examples that satisfy the definition and its useful to understand the concepts in algebra. Similarly, learning algebra at graduate level involves proving algebraic statement that is theorem, lemma, propositions, and corollaries accurately (Judson & Austin, 2015, p.2). In this context, this paper intends to explain how the teacher at master level was experiencing students' difficulties on completing their tasks for learning activities of the concepts included in the linear algebra course.

Objectives

The objective of this paper was to explore the difficulties of master's students in learning algebraic concepts, proving theorems, and constructing examples in linear algebra course.

Literature Review

Algebra is one of the important branches of mathematics, including in the courses of school mathematics to university courses in mathematics. School algebra is a generalization of arithmetic and higher algebra is the generalization of school algebra (Findell, 2001, p.9). Algebra is the study of "symbol manipulation, equations, functions; a way of expressing generality, solving certain classes of problems and model them in real situations; and a formal system involving set theory, logic and operations of entities, and real numbers" (Stancy & Chick, 2004).

It is the study of algebraic structures including groups, rings, modules, fields and vector spaces in general in higher education. Booth (1986) argued that the main purpose of algebra is to learn how to represent general relationships and procedures, such that through these representations, a wide range of problems can be solved, and new relationship can be developed from those known. Algebra can be viewed into two types: the abstract or modern algebra and the classical or linear algebra.

Linear algebra is the branch of mathematics that study of vectors and linear functions (Cherney et.al., 2013); represents a process of abstraction where students move from concrete objects and methods in familiar vector spaces to generalizations (Rensaa et.al., 2020, p.297). In general, linear algebra includes algebraic structures that are basically closed under addition and scalar multiplication for example the vector spaces, modules etc. together with the concepts and relations related to systems of linear equations, matrices, determinants, vector spaces, and linear transformations. A duality exists in linear algebra courses, concrete processes such as calculating a determinant or doing Gaussian row eliminations on one hand and lots of definitions and the use of formalism on the other hand; this duality may conflict with students' expectation from prior mathematics courses (Rensaa et. al., 2020, p. 297).

Proving theorems, lemmas, propositions and corollaries is a major task in addition to understand generalizations of the concepts included in linear algebra. These theorems are the tools that make new and productive applications of mathematics possible (Judson & Austin, 2015, p.2). Without understanding the logic and approaches included in the theorems, students have difficult to conceptualize the abstraction in linear algebra. These approaches of proving theorems are generally named as techniques of proofs. There are some direct techniques of proving theorems such as formal deduction, induction, and axiomatic methods and indirect techniques such as methods of contradiction and contrapositive (Lalonde, 2013). Students need to understand which techniques of proofs is suitable for proving certain statement in linear algebra. Likewise, there is need of connecting learned ideas in proving theorems. Proofs are the convincing argument about the accuracy of the mathematical statement (Judson & Austin, 2015). These arguments are derived from the learned facts. If students are unaware about techniques of proofs as well as ways of connecting learned facts to prove theorems then, it will create difficulty to understand concepts in learning linear algebra.

Regarding the difficulty in learning algebraic concepts, Nardi & Iannone (2006) argued that algebraic arguments are highly valued by students but difficult to produce or understand; students have some common difficulties when learning new concepts (Judson & Austin ,2015).

The study of Zuya' (2017) has revealed that students have difficulty in developing conceptual and procedural knowledge while learning in higher algebra.

Likewise, high school students have faced difficulty in developing symbol manipulation skills in algebra and due to which they have confusion in understanding algebraic concepts (Nathan & Koedinger, 1999 – 2003). Also, school students have difficulties in understanding variables and its manipulation, using rules of manipulation to solve equations, use of knowledge of algebraic structure and syntax to form equations, and generalization of rules for repetitive patterns or sequence of shapes (p.iii).

Moreover, Karki (2017) had explored that students at lower secondary level have difficulties in comprehending variables, simplifying algebraic expression, solving equations, and in solving word problems (p.36). These difficulties are due to incorrect generalization of arithmetic rules in algebra, guessing reasoning, and having poor algebraic language. Premesti & Retnawati (2019) have explained that students' difficulties in learning algebra are understanding the problems, understanding the meaning of variables, and operating the algebraic form.

On the basis of review of literature, the following conceptual framework was developed to explore students' difficulties while learning linear algebra at master level.

.

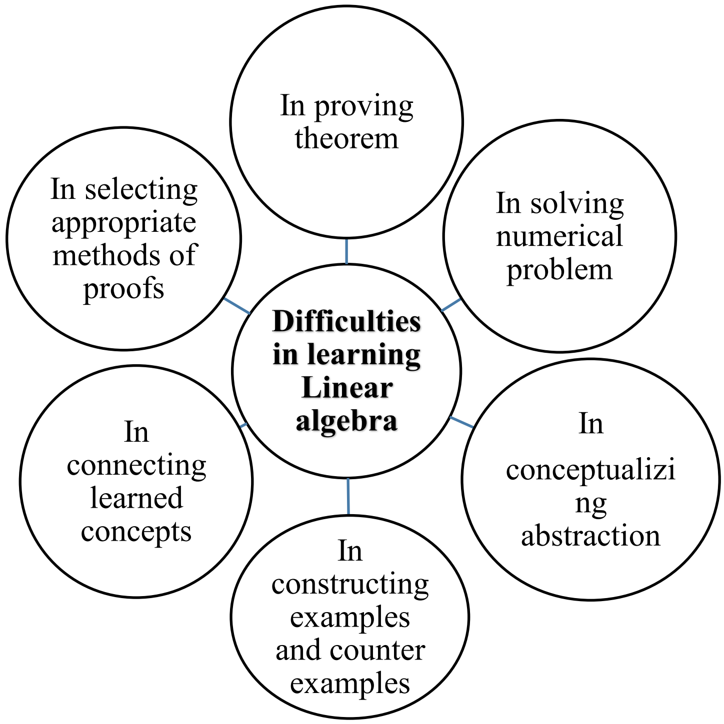


Figure 1: *Areas of Difficulties While Learning Linear Algebra*

Methodology

I adopted constructivist research paradigm which believes on multiple reality, subjective knowledge and qualitative research methodology (Guba & Lincoln, 2005; Creswell, 2009; Fard, 2012). I used case study research design (Gillham, 2000) in which ten case respondents (six boys and four girls) from department of mathematics education, Tribhuvan University, Kirtipur in 2024 were selected purposively including very poor (less than 10 marks out of 30 full marks) and talented (More than 25 marks out of 30 full marks) students in terms of their academic performance and to make inclusive in terms of gender. I used semi-structured interview guidelines to take in-depth face to face interviews, observation notes and performance test to explore difficulties in in learning linear algebra. I applied general inductive approach as explained by Thomas (2006) to analyze data and used triangulation to maintain the validity of the data.

Results

The results were presented and discussed under separate heading as below. Here, B1 to B6 represents the boys responds from first to sixth and G1 to G4 represents the girls' respondents from first to fourth.

Achievement of Students in Two Performance Tests

The following table displays the achievement marks of selected students in two respective Performance Tests in Linear Algebra Course. Each Performance Test was of 30 full marks which consisted five objective questions each of one mark, three short answer questions each of five marks, and one long answer question of ten marks.

These performance tests were taken with the gap of two months. The test items of the first performance test were prepared from the topic of linear maps and matrices, and bilinear forms and standard operators.

Students	Achievement Marks in 1st Test	Achievement Marks in 2nd Test
B1	24	19
B2	21	21
B3	21	18
B4	20	20
B5	18	22
B6	04	07
G1	21	21
G2	09	25
G3	26	27
G4	19	13

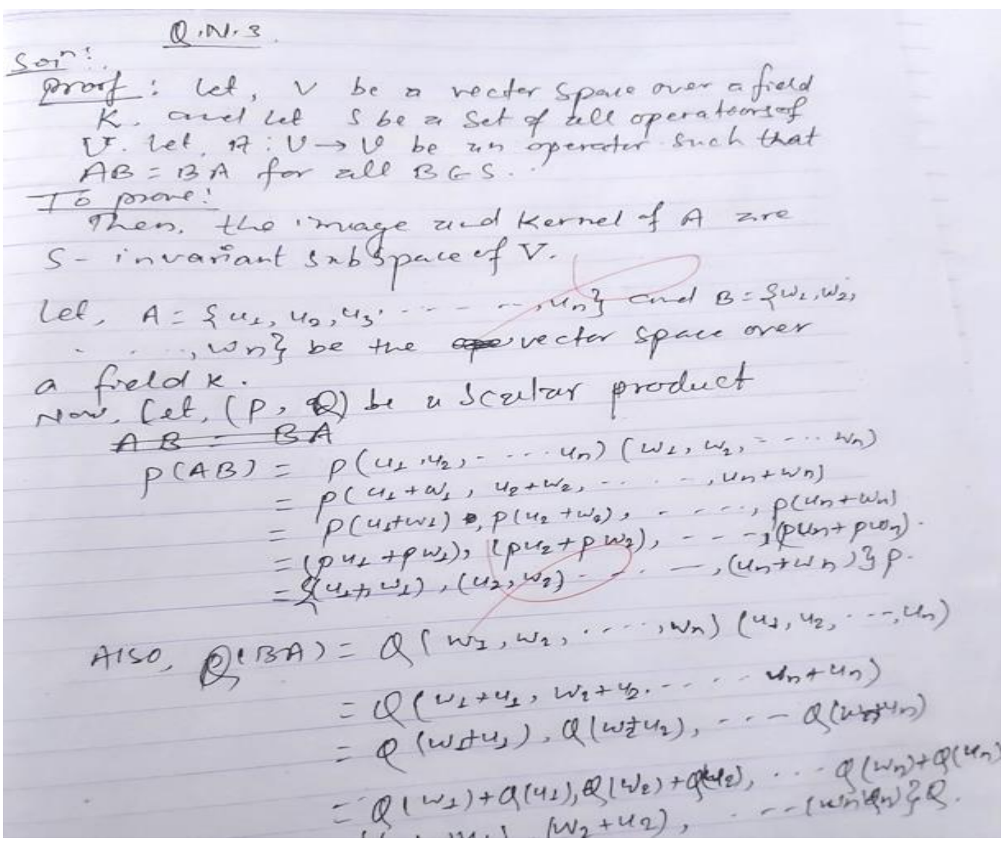
The results in the table show that one boy student has poor result in both performance test. However, the majority of students have more than average performance in both tests. The answer sheets of students have shown that they have conceptual as well as procedural difficulties in solving questions on the performance tests. Regarding the following question of objective type in the first performance test, only one selected student gave correct answer.

Let $V = \mathbb{R}^3$ be the vector space over $\mathbb{R}$. Which of the following association is not scalar product?

(a) $< X, Y > = x_1y_1 + x_2y_2 + x_3y_3$ (b) $< X, Y >= x_1y_1 + 2x_2y_2 + 3x_3y_3$
(c) $< X, Y >= x_1y_1 - 2x_2y_2 - x_3y_3$ (d) $< X, Y >= x_1y_1 + x_2y_2 + x_3y_3 + 3$

When I asked students to get the reason behind of doing such mistake, they did not explain the concept of scalar product on vector space and they were unable to state the procedure of testing whether it is a scalar product or not. These responses had justified that students have faced conceptual as well as procedural difficulties while learning linear algebra.

Regarding the short answer type question: "*Let V be a vector space over the field K, and let S be a set of all operators of V. Let A: V → V be an operator such that AB = BA for all B ∈ S. Then prove the image and kernel of A are S-invariant subspace of V.*" One student gave answer as follows.


Q.N. 3
Soln:
Proof: Let, V be a vector space over a field K. and let S be a set of all operators of V. let, A : V → V be an operator such that AB = BA for all B ∈ S.
To prove:
Then, the image and Kernel of A are S- invariant subspace of V.
Let, A = {u1, u2, u3, ..., un} and B = {w1, w2, ..., wn} be the vector space over a field K.
Now, Let, (P, Q) be a Scalar product
~~AB = BA~~
P(AB) = P(u1, u2, ... un) (w1, w2, ... wn)
= P(u1 + w1, u2 + w2, ..., un + wn)
= P(u1+w1), P(u2 + w2), ..., P(un + wn)
= (pu1 + pw1), (pu2 + pw2), ..., (pun + pwn)
= {(u1+w1), (u2, w2) ... , (un+wn)} P
Also, Q(BA) = Q(w1, w2, ..., wn) (u1, u2, ..., un)
= Q(w1+u1, w2+u2, ... wn+un)
= Q(w1+u1), Q(w2+u2), ... Q(wn+un)
= Q(w1)+Q(u1), Q(w2)+Q(u2), ... Q(wn)+Q(un)
... (w2+u2), ... (wn+un)} Q.

The solution shows that student did not have concepts of image and kernel of linear map as well as *S*-invariant space of vector space. This student also started the solution with wrong step shows that student had learnt poor procedural skills to show image and kernel of the linear map *A* are *S*-invariant subspace of the vector space *V*.

Difficulties in Learning Algebraic Concepts in Linear Algebra

The algebraic concepts mean definition, notation and symbols that are used in linear algebra course at master level. Basically, the algebraic concepts included in the course of linear algebra at master degree of Tribhuvan University are abstractions. For example, the vector spaces, linear transformations, bilinear forms, modules etc.

When I asked selected students to define the concepts of vector space, all the student except one student defined it as:

A vector space is a non-empty set V over the field K and satisfying the following.

For every $u, v, w \in V$ *and* $\alpha, \beta \in V$, *we have* (i) $u + v \in V$ (ii) $(u + v) + w = u + (v + w)$ (iii) $v + 0 = v = 0 + v$ (iv) $v + (- v) = 0 = (- v) + v$ (v) $u + v = v + u$ (vi) $\alpha v \in V$ (vii) $(\alpha + \beta)v = \alpha v + \beta v$ (viii) $\alpha (u + v) = \alpha u + \alpha v$ (ix) $(\alpha\beta)v = \alpha(\beta v)$ and (x) $1.v = v = v.1$.

This is the correct definition of the vector space. The remaining one student also stated almost all the conditions of non-empty set *V* to become a vector space over the field *K* with little confusions of scalar multiplication. From these responses of selected students, it is said that they have good concepts of vector space. But, when I asked the binary operations used in the concept of vector space, most students did not give correct answer. Three students said *addition* (+) *is only the binary operation because of properties first on the definition of vector space*. Four students said *addition* (+) *and multiplication* (.) *are both binary operations because of properties first, and sixth on the definition of vector space*. The remaining two students only said correct answer that *addition and scalar multiplication are the binary operations because the elements of V are the vectors and the elements of the field K are considered the scalar multiplication. So, in condition six of vector space shows the closure property is the multiplication of scalar* α *to the vector v*.

From these responses, it was experienced that students have conceptual difficulties while learning abstraction; for example, the vector space; even they defined properly. It means that students are just memorizing the conditions of non-empty set V over the field K to become the vector space even for the talented students.

In both performance tests in linear algebra, the majority of selected students did not properly define the concepts of "*linear map associated with a matrix, symmetric bilinear form, quadratic form, index of nullity, index of positivity, eigenvalues, and eigenvectors*". Similarly, I experienced that "*students had given more emphasis on rote memorization rather than understanding the concepts and properties of definitions*" while learning linear algebra concepts. That is why, they can easily forget the concepts of definition in linear algebra at master level. Moreover, the selected students had experienced as:

"*We had faced difficulties in learning new notation, symbols and structures in linear algebra, for example, we are confusion upon the symbol of scalar product and bilinear form (i.e., <,>), notation of function, notation of matrix associated with linear and bilinear form, notation of transition matrix in functional form etc.*"

These responses shows that the students have difficulties in understanding new notation, symbols, and structures in addition to conceptual and procedural difficulties in linear algebra.

Difficulties in Constructing Examples and Counter Examples

Constructing examples and counter examples is an important task while learning concepts in linear algebra at master level. For example, most of the selected students explained and defined the definition of the vector space. But they were unable to give proper example of vector space. As a teacher I asked them one example of vector space. The one selected student replied as

"*I read repeatedly to memorize the definition in linear algebra for example the definition of vector space, linear maps, bilinear forms on vector space, eigenvalues and eigenvectors etc. I am trying to memorize the example given by teacher by repeating the concepts, but I think it is very difficult to construct new examples of related concepts because in bachelor level, the teacher says just focus for memorization of the concepts, really, I have no conceptual understanding of the concepts in linear algebra*".

The above responses of the sampled students show that students have poor conceptual background in understanding the concepts of linear algebra from bachelor level. The other students also replied in same direction of experiences. The students just trying to memorize definition by repeating the concepts given by teacher and they have given less priority to conceptualize it. As a teacher I gave examples and counter examples of such concepts in my master degree classroom in second semester course of M.Ed. Program of TU.

For example, I asked students: is $V = \mathbb{R}^2$ a vector space over the real field $\mathbb{R}$? If yes and how? The students in the classroom replied "yes". But they were unable to prove such V is a vector space. So, I explained all the conditions of non-empty set V to become a vector space over the field K and the nature of elements of $V = \mathbb{R}^2$ such as $v = (x, y)$ where $x, y \in \mathbb{R}$. Then, asked them to take next two elements of $V = \mathbb{R}^2$. The majority of students took it as $v_1 = (x_1, y_1)$ and $v_2 = (x_2, y_2)$. Again, I asked them "to add these vectors and take a scalar α from the field $\mathbb{R}$ and multiply $v = (x, y)$ by such scalar". The students added and multiplied as $v_1 + v_2 = (x_1 + x_2, y_1 + y_2)$ and $\alpha v = (\alpha x, \alpha y)$. I asked them: are $v_1 + v_2$ and αv the elements of V? At that time more students replied yes. I explained the detailed of the concepts of vector space and asked them to test all the properties of vector space. Finally, students were able to show $V = \mathbb{R}^2$ is a vector space over the field $\mathbb{R}$. Likewise, the students were able to show $V = \mathbb{R}^3 \ldots V = \mathbb{R}^n$ are the vector spaces over the field $\mathbb{R}$.

This classroom discussion for constructing the example of vector space shows that students have difficulties of conceptual understanding while constructing examples of concepts in linear algebra. Likewise, students have difficulties to give the examples and counter examples of linear maps, bilinear forms, eigenvalues and eigenvectors and other concepts in linear algebra.

The collected responses of selected students as: "*faced difficulty to give example from the definition of eigenvector and eigenvalues, ... difficulty for exemplify because of poor knowledge in abstract algebra, more difficult to construct counter example than examples such as to construct counter example of eigenvector, difficult to find matrix associated with linear map and linear map associated with matrix*"

The above responses of students are based on their performance test in linear algebra. When I asked students to give counter example of eigenvalues and eigenvector of the linear map $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ given by $f(x, y) = (6x, 7y)$ after performance test. They were able to define eigenvalues and eigenvectors in performance test but unable to give proper example of those concepts. I asked them: is $v = (1, 2)$ an eigenvector of f? The sampled students did not give correct answer. However, some students gave (1, 0) is the eigenvector of f having eigenvalues 6. It means that the students have just memorized (1, 0) is eigenvector from classroom discussion. In fact (1, 2) is not an eigenvector of this linear map f. This is a counter example. That is why the students have difficulties in constructing examples and counter examples in linear algebra because of poor conceptual and procedural understanding of concepts

Difficulties in Proving Theorems and Solving Numerical Problems

Proving theorems/propositions is an essential activity for the students at higher level in mathematics, particularly in linear algebra at that level. Theorems are tools that make new and productive applications of mathematics possible (Judson & Austin, 2015). It means that understanding the concepts in theorems, the students need to have rigorous skills to justify/prove them. There are varieties of proof techniques of theorems in mathematics including direct proof, proof by contradiction, mathematical induction, and proof by contrapositive (the indirect one) (Lalonde, 2013). If students do not have good skills in selecting appropriate technique of proving theorem, then they cannot prove it.

The collection of responses of selected students regarding to proving and understanding the theorems in linear algebra are as:

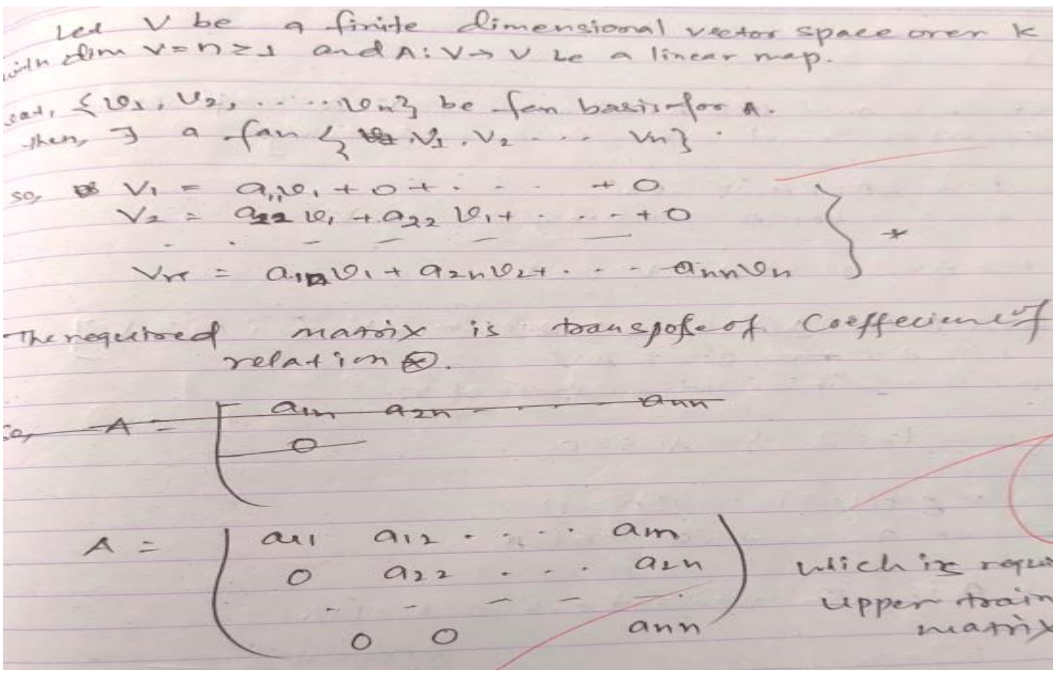
"*We are always trying to memorize algebraic concepts rather than learning concepts, very difficult to complete proof, we understand steps in theorem, but for exam we always rote for memory, difficult to use definition in proving, abstraction is more general so difficult to understand, construct example etc.*"

These responses indicate that students have difficulty in completing the proof of the theorem in examination because that were habituated in rote learning rather than conceptualized the concepts and steps in theorem and so forget in examination period. Another experience they responded that students have felt difficulty in using the concept of definition while using it in proving theorem. Also, because of poor conceptualization on abstraction and in construction of examples in linear algebra have made difficulty for master students in proving theorem on linear algebra.

Other selected students explained that: "*abstraction is relative, when I understand meaningfully then it is very; easy otherwise difficult. For exam we always rote theorems without understanding, we do not know which method is suitable- direct or indirect, induction or deduction or method of contradiction or contrapositive or axiomatic etc.*"

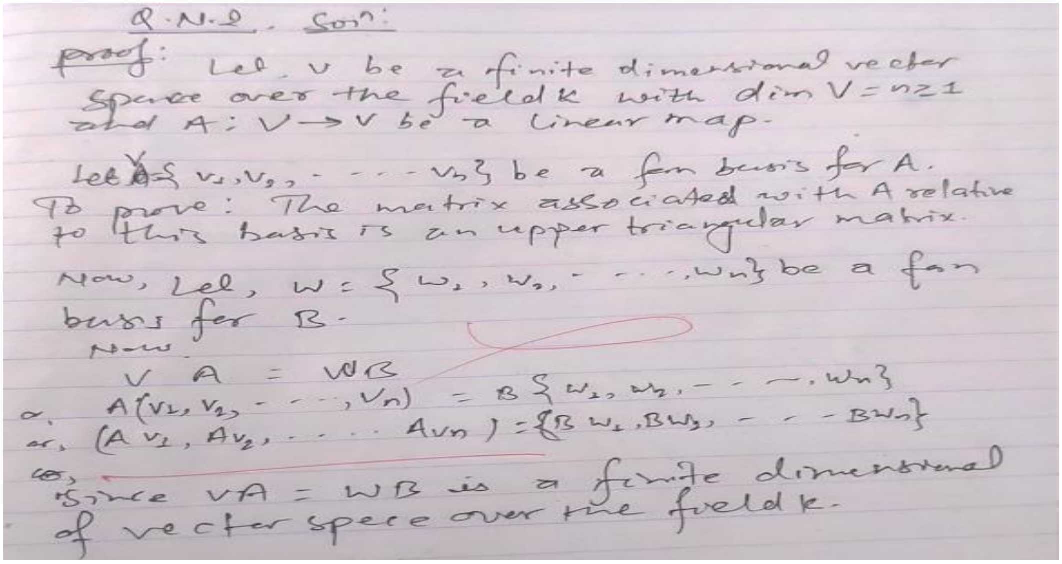
These responses reveal that students had good experiences on "abstraction is relative" in understanding linear algebra. Generally, abstraction are the generalized concepts in mathematics. To understand it, it is necessary to understand the particular case related to it like examples. According to selected students, it is very difficult to construct example and counter example of the abstraction in linear algebra. Because of that they have faced difficult in understanding the abstraction, and hence in the proofs, in selecting appropriate approaches of proof, for algebraic statements in linear algebra.

One example of students' proof of the theorem/statement on performance test is given as follows. The theorem was: "*Let A: V → V be a linear map. Let* $\{v_1, v_2, . . . , v_n\}$ *be a fan basis for A. Then prove the matrix associated with A relative to this basis is an upper triangular matrix.*


Let V be a finite dimensional vector space over K
with dim V = n ≥ 1 and A: V → V be a linear map.
Let, {v1, v2, . . . vn} be fan basis for A.
then, ∃ a fan {V1, V2 . . . Vn}.
so, V1 = a11 v1 + 0 + . . . + 0
V2 = a22 v1 + a22 v1 + . . . + 0
Vn = a1n v1 + a2n v2 + . . . ann vn
The required matrix is transpose of coeffecient of relation (*).
so, A = a1n a2n . . . ann 0
A = (a11 a12 . . . am; 0 a22 . . . a2n; 0 0 ann)
which is requ upper train matrix

The above proof of the theorem is incomplete. There are conceptual as well as procedural errors in the proofs. This proof does not explain the characteristics of fan for the linear map A in the vector space V, i.e., $AV_i \in V_i$ for all indices i. In the equation (*), it should be written $AV_1 = \ldots$, $AV_2 = \ldots.$ and so on. These several difficulties are due to poor conceptual understanding of the concepts. The other incomplete proof done by student on the same question is as follows.

Perspectives on Higher Education, Vol., 15(1), January 2025, pp. 183-200, ISSN: 2542-2707, https://doi.org/10.3126/phe.v15i01.81001

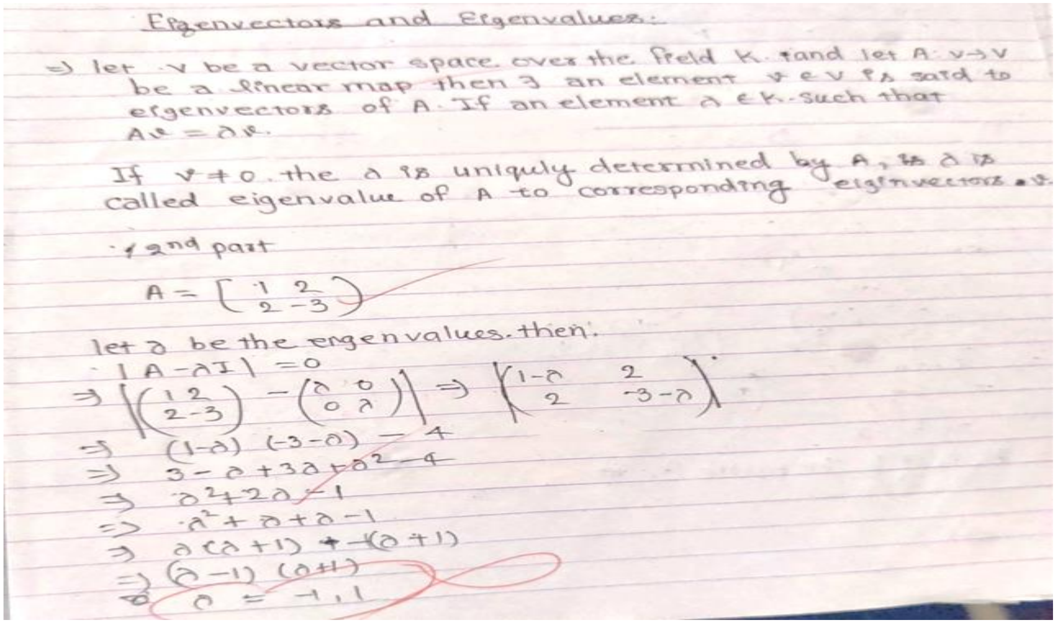

Q.N.2. Soln:
proof: Let V be a finite dimensional vecter space over the field k with dim V = n ≥ 1 and A : V → V be a linear map.
Let V = {v1, v2, - - - vn} be a fan basis for A.
To prove: The matrix associated with A relative to this basis is an upper triangular matrix.
Now, Let, W = {w1, w2, - - -, wn} be a fan basis for B.
Now.
V A = WB
or, A(v1, v2, - - -, vn) = B{w1, w2, - - -, wn}
or, (A v1, Av2, - - - Avn) = {B w1, Bw2, - - - Bwn}
Since VA = WB is a finite dimensional of vecter space over the field k.

This solution of the performance test shows that students did not have conceptual as well as procedural knowledge to prove the matrix associated with the linear map with respect to given fan basis is an upper triangular matrix.

One selected student expressed as: "*It is very difficult to understand language of theorem, and statement, having poor prerequisite, difficult to linkage of learned concepts etc. are difficulties in proving theorem. I am also unknown to real life application*".

These responses display that language is one of the major problems for the students to understand the concepts given in theorem. Likewise, having poor prerequisites, unable to make linkage between learned concepts with new concepts, and could not showing real life application of the algebraic concepts in linear algebra are also the causes of difficulties experienced by master degree students in linear algebra.

Solving numerical problems related to theoretical concepts in linear algebra is another area of abstraction where most selected students did mistake. In the following answer sheet of the student, the definition of eigenvalues and eigenvectors are clearly defined. But student did not follow the correct process to find the eigenvalues and eigenvectors associated to the given matrix.



Eigenvectors and Eigenvalues:

⇒ let V be a vector space over the field K and let A: V→V be a linear map then ∃ an element v ∈ V is said to eigenvectors of A. If an element λ ∈ K such that Av = λv.

If v ≠ 0 the λ is uniquly determined by A, λ is called eigenvalue of A to corresponding eigenvectors v.

2nd part

$A = \begin{bmatrix} 1 & 2 \\ 2 & -3 \end{bmatrix}$

let λ be the eigenvalues. then:

$|A - \lambda I| = 0$

$\Rightarrow \left|\begin{pmatrix} 1 & 2 \\ 2 & -3 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right| \Rightarrow \begin{vmatrix} 1-\lambda & 2 \\ 2 & -3-\lambda \end{vmatrix}$

$\Rightarrow (1-\lambda)(-3-\lambda) - 4$

$\Rightarrow 3 - \lambda + 3\lambda + \lambda^2 - 4$

$\Rightarrow \lambda^2 + 2\lambda - 1$

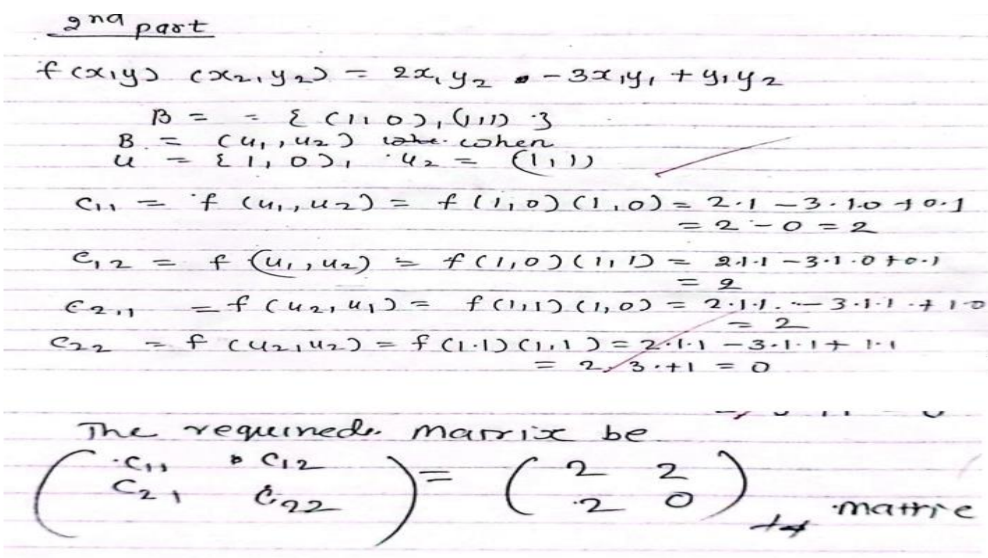
$\Rightarrow \lambda^2 + \lambda + \lambda - 1$

$\Rightarrow \lambda(\lambda+1) + 1(\lambda+1)$

$\Rightarrow (\lambda-1)(\lambda+1)$

$\lambda = -1, 1$

The solution of above answer sheet shows that student had procedural as well as calculation error on solving that problem. Likewise, students have faced difficulty to find the associated matrix of the given bilinear form. One of the examples answer sheet for the question in performance test as: *Define symmetric bilinear form and the quadratic form. If f is a bilinear form on* $\mathbb{R}^2$ *defined by f* $((x_1, y_1)(x_2, y_2)) = 2x_1x_2 - 3x_1y_1 + y_1y_2$*, then find the matrix of f with respect to the ordered basis B* = {(1,0), (1, 1)}.



2nd part

$f(x_1 y_1)(x_2, y_2) = 2x_1 y_2 - 3x_1 y_1 + y_1 y_2$

$B = \{(1,0), (1,1)\}$

$B = (u_1, u_2)$ when

$u_1 = \{1, 0\}, \; u_2 = (1,1)$

$c_{11} = f(u_1, u_2) = f(1,0)(1,0) = 2\cdot1 - 3\cdot1\cdot0 + 0\cdot1 = 2 - 0 = 2$

$c_{12} = f(u_1, u_2) = f(1,0)(1,1) = 2\cdot1\cdot1 - 3\cdot1\cdot0 + 0\cdot1 = 2$

$c_{21} = f(u_2, u_1) = f(1,1)(1,0) = 2\cdot1\cdot1 - 3\cdot1\cdot1 + 1\cdot0 = 2$

$c_{22} = f(u_2, u_2) = f(1\cdot1)(1,1) = 2\cdot1\cdot1 - 3\cdot1\cdot1 + 1\cdot1 = 2 - 3 + 1 = 0$

The required matrix be

$\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 0 \end{pmatrix}$ matrix

This is the correct solution of the numerical part of this question, only solved by two students out of ten selected students. But remaining eight students did not give complete answer, even some of them did not start for the solution. It means that students were habituated to rote learning. This was the problem different from practices in the classroom. So, the students had both conceptual and procedural difficulties to solve the numerical problems related to the theoretical concepts in linear algebra at master level in mathematics education.

Difficulties in Connecting Learned Concepts

It is natural that the already learned concepts are prerequisite for learning new concepts in linear algebra. For example, if the students are not familiar with vector spaces and linear maps, then they could have difficulties in learning bilinear forms and other higher concepts in linear algebra. The concepts of binary operations and the abelian group are necessary prerequisite to define vector spaces and field which are the central objects handled in the courses of higher linear algebra.

Regarding to connection of learned concepts in linear algebra, one selected student had experienced as:

"*I have felt difficulty to understand mathematical relation, difficult to link already learn concepts to learn new concepts, I am unable to define linear map properly, I have difficulty to link concepts in real life situation*".

These responses show that the students at master level are facing difficulties in learning linear algebra because they cannot connect already learned concept while learning new concepts. This view also indicates that the teacher needs to connect classroom teaching with the real-life situation while teaching linear algebra.

Likewise, while teaching theorems in linear algebra, for example to "*show the isomorphism between the vector space of all bilinear forms on the vector space V over the field K (i.e., Bil* $(V \times V) \rightarrow K$*) and the space of* $n \times n$ *matrices C (i.e., Mn(K)) given by* $g \longmapsto g_C$*, where* $g_C: K^n \times K^n \rightarrow K$ *is a bilinear form given by* $g_C(X, Y) = {}^tXCY$*, for every* $X, Y \in K^n$", the students need to have link several learned facts. These concepts are: isomorphism, well defined function, one to one and onto function, bilinear forms, vector spaces of matrices and vector spaces of bilinear maps, linear maps (vector space homomorphism), bases and dimensions etc. Without connection of these concepts in totality, the master students cannot prove that theorem.

Moreover, one student responded that: "*at bachelor level, our teacher only focused to memorize theorems, they are exam oriented. Because of that I have no actual understanding of notation, poor fundamental concepts, confusing in representation matrix of linear maps, bilinear maps, difficulties in conceptualize, memorize, and application for proving theorem. In our campus, there are rarely classes, I just practiced myself without understanding it*".

These responses reveal that there is lack of subject teacher in mathematics education. Students had just practiced and drilled some questions in the text books with the intension of passing in exam. Hence, the selected students at master level have poor prerequisites with poor conceptual understanding. That is why, they have faced difficulties to link the learned concepts to learn the new concepts in linear algebra.

Conclusion

Learning is the continuous process on which master students have their own techniques for learning linear algebra. The results show that the performance of the majority of students at master level have more than average performance. However, they have conceptual as well as procedural difficulties in learning algebraic concepts, proving theorems, solving numerical problems, and constructing examples of abstractions. Likewise, they have poor prerequisites to learn linear algebra which make them difficult to learn new concepts at master level. Thus, it is suggested that the teacher who are teaching linear algebra course at that level need to have develop strong prerequisites on students to improve their performance and to mitigate conceptual as well as procedural difficulties to learn linear algebra.

References

Booth, L.R. (1986). Difficulties in Algebra. *Australian Mathematics Teacher*, *42*(3), 2 - 4. Retrieved from http://www.rachaelwelder.com/files/vitae/Welder_Prereq_Know_Algebra.pdf

Cherney, D, Denton, T. Thomas, R. & Waldron, A. (2013). Linear Algebra. Retrieved from: https://www.math.ucdavis.edu/~linear/linear-guest.pdf

Chow, T. C. F. (2011). Students' Difficulties, Conceptions, and Attitudes towards Learning Algebra: An Intervention Study to Improve Teaching and Learning. A PhD Dissertation, Curtin University. Retrieved from: https://core.ac.uk/download/pdf/195632071.pdf

Creswell, J.W. (2009). *Research design: Qualitative, quantitative, and mixed methods approach* (3rd Ed.). California: Sage Publication, Inc.

Fard, H.D. (2012). Research paradigm in publish administration. *International Journal of Humanities, 19* (4), 55 -108.

Findell, B. (2001). *Learning and understanding abstract algebra* (Doctoral Dissertation). Retrieved from file:///C:/Users/acer/Downloads/Learning%20and%20understanding%20in%20abstract%20algebra%20(3).pdf

Fry, H., Ketteridge, S., & Marshall, S. (2009). *A handbook for teaching and learning in higher education enhancing academic practice* (3rd Ed.). New York: Routledge.

Gillham, B. (2000). *Case study research methods*. London and New York: Continum.

Guba, E.G., & Lincoln, Y.S. (2005). Paradigmatic controversies, contradiction, and emerging Confluences. In N.K Denzin & Y.S. Lincoln (Eds.), *The SAGE handbook of qualitative research* (3rd ed., pp.191 – 215). Thousand Oaks, CA: Sage.

Judson, T.W., & Austin, S.F.(2015). *Abstract algebra: theory and application*. Retrieved from http://mathbook.pugetsound.edu

Karki, K. (2017). *Students' Difficulties in Learning Algebra at Lower Secondary Level*. An Unpublished Master Thesis in the Department of Education, Central Department of Education, Tribhuvan University, Kirtipur, Nepal.

Lalonde, S. M. (2013). *Notes in abstract algebra*. Retrieved from https://math.dartmouth.edu/archive/m31x13/public_html/Notes%20on%20Abstract%20Algebra%202013.pdf

Nardi, E., & Iannone, P. (2006). *How to prove it: A brief guide for teaching proof to Year 1 mathematics undergraduates.* UK: UEA Print Service.

Nathan, M. J., & Koedinger, K. (1999-2003). Teachers' judgments about algebra problem difficulty. *Technical Report*, Institute of Cognitive Science, University of Colorado, Boulder.

Premesti, T. I., & Retnawati, H. (2019). Difficulties in learning algebra: An analysis of students' errors. *Journal of Physics: Conf. Series* 1320, doi:10.1088/1742-6596/1320/1/012061. Retrieved from: https://www.researchgate.net/publication/337644793_Difficulties_in_learning_algebra_An_analysis_of_students'_errors#fullTextFileContent

Rensaa, R. J, Mogstad, N. M., & Monaghan, J. (2020). Perspectives and reflections on teaching linear algebra. *Teaching Mathematics and its Applications: An International Journal of the IMA*, 39 pp. 296-309, doi:10.1093/teamat/hraa002. Retrieved from: https://www.researchgate.net/publication/344942840_Perspectives_and_reflections_on_teaching_linear_algebra#fullTextFileContent

Stacey, K., & Chick, H. (2004). Solving the problem with algebra. *In:* K. Stacey, H. Chick & M. Kendal (eds.), *The future of the teaching and learning algebra, the 12th ICMI study (pp. 1-20)*. Kluwer Academic Publishers.

Thomas, D. R. (2006). A General inductive approach for analyzing qualitative evaluation data. *American Journal of Evaluation, 27*(2), 237 -246. DOI: 10.1177/1098214005283748

Witzel, S. B. (2005). Using CRA to teach algebra to students with math difficulties in inclusive setting. *Learning Disabilities: A Contemporary Journal, 3*(2), 49 - 60. Retrieved from http://www.catea.gatech.edu/scitrain/kb/Full Text_Articles/ Witzel_Using.pdf

Zuya, H.E. (2017). Prospective teacher' conceptual and procedural knowledge in Mathematics: The case of Algebra. *American Journal of Educational Research*, *5* (3), 310 - 315. Retrieved from http://pubs.sciepub.com/education/5/3/12/index.htm.