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Relationship Between Differential Equations and Difference Equation

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Abstract

The study of Differential equations and Difference equations play an important and significant role in many sciences. These equations are used as mathematical tool used in solving various problems in modeling, physics, chemistry, biology, anthropology, etc. or even in social studies. Differential equations are used to solve real life problems by approximation of numerical methods. Theory of Differential and Difference equations has been taught at all levels in high schools and at the universities for all students, including students majoring in Mathematics.

This is a micro –study in which the research is designed in an exploitative, qualitative, descriptive and analytic framework to analyze the differential-difference equations. In this study, theoretical concepts, descriptive, analytical and numerical methods about differential-difference equations which are clarified by related examples. This research article seeks to study the relationship between them.

The finite difference method for solving equations leads to difference equation theory, developing a parallel between difference equations and differential equations. The first differences are related to the first derivatives. Difference equations are discrete versions of differential equations, and similarly differential equations are continuous versions of difference equations.

Keywords: Calculus, Differentiation, Differential equation, Difference equation, Finite difference

Introduction

Calculus is one of the most dynamic fields of Mathematics and deals mostly with the rate of changes. There are two major branches of calculus, e.g. Differential Calculus and Integral Calculus. They are opposite to each other. Differential calculus deals with finding the instantaneous rate of change, for example, the speed, velocity, or an acceleration of an object. Integral Calculus used to find linear distance travel, areas under a curve, and surface area, or volume. Differentiation operation is an important in natural sciences, social sciences, engineering and technology. Differential equation used to solve real life problem by approximation of numerical methods.

The equation involving dependent variable, independent variable and differential co-efficient of a dependent variable with respect to independent variable or constant is called ordinary differential equation for example:

$$\frac{dy}{dx} = 2y.$$

The solution of a differential equation does not involve the derivatives of the variables. In most of case, the differential equation related to time which is continuous variables. In these cases, approximate numerical solutions are obtained by various methods.
The new trend in the higher secondary level study in Nepal, the number of practical questions related to the application of these equations has been increasing. In our best knowledge, current curriculum of mathematics is not sufficient and the number of practical exercises book is still very limited.

Differential equations contain the derivatives:

$$\frac{dy}{dx} = \frac{\Delta y}{\Delta x} \quad \text{of a function } y = f(x).$$

i.e. $$\frac{dy}{dx} + p_y = q$$

Difference equation involves differences of terms in a sequence and it can be expressed in terms of shift operator E or forward difference operator Δ.

First order difference equation defined in a sequence:

$$y_{n+1} = [(n, y_n)] \qquad n = 0,1,2,3\text{-----}$$

It is of first order because value of y_{n+1} depends on the value of y_n.

Order and Degree

Differential equations are of two types, e. g, ordinary and partial differential equations. Ordinary differential equations are those in which all derivatives contain one independent variable and also its derivatives. Partial differential equations are those which contain two or more independent variables and also partial derivatives. The order of differential equations is order of the highest derivatives in it. The degree of derivatives is highest degree in it after made free from radicals. (Sahadevan, 2001). The equation:

$$3\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 2y = 0$$ is of second order and first degree.

The order of a difference equation is the highest differences in the equation. The order of difference equation: $y_{n+1} - y_n = 2$ is 1. and order of difference equation:

$$y_{n+2} - y_{n+1} + y_n = 0$$ is 2.

The degree of a difference equation is the degree of the term of the highest order. A difference equation of degree one is called linear. The equation:

$$y_{t+2}^3 - 3y_t = 2$$ is of second order and third degree.

Statement of Problem and Research Question

Most of the principles of this universe can be written in the language of mathematics. Algebra is enable to solve most of static problems, but the most motivating usual phenomenon can be described by equations that relate varying quantity. The whole thing is varying with respect to moment. So it is requiring to explore the relationship between them (Sahadevan, 2001).

This research article seeks to study the relationship between differential and difference equations. The research problem is identified as "Relationship between differential equations and difference equations". The related previous pieces of work are reviewed to answer the research question.

To tackle the problem statement and the knowledge gap described above, the following research question is seeking to be answered. How are differential equations and difference equations related?

Objective of the Study

- The general objective of this research article is to study the differential equations and difference equations. The specific objective is as follows.
- To highlight the relationship between differential equations and difference equations.

Literature Review

Towers (2009) presented two finite variation methods FDM1 and FDM2 for discretizing a delta function to check the consistency. These joint algorithms seem to be consistent in all reasonable conditions, with numerical experiments representing O(h^2) convergence for their new gradient-normalized FDM2 algorithm.

Bojovic (2010) worked with finite difference method to examine the convergence of variation schemes for one-dimensional heat equation.

Sahadevan (2001) reviewed on recent result of non-linear Lie theory of differential-difference equations. They concluded that autonomous non-linear equation of an arbitrary order with one or more independent variables can be linear. They extended the Lie's theorem of differential-difference equations.

Several researchers have contributed in the theory of these equations and their applications. Literatures concerning this theory can be found in any standard textbooks, reference books and monographs of calculus.

Research Methodology

In this study, theoretical concepts, descriptive, analytical and numerical methods about differential-difference equations has been used which are clarified by related examples. Secondary sources has been used in this research.

Research Design

This is micro -study in which the research is designed in an exploitative, qualitative descriptive and analytic framework to analyze the differential-difference equations. The study is based on secondary sources, books, articles, thesis, etc. The literature review related to research question has been thoroughly reviewed to meet the gaps of the study.

Result and Discussions

Differentiation or derivative from first principles is the instantaneous rate of change of y = f(x) with respect to x is:

Slope= m = $\frac{change\ in\ y}{change\ in\ x} = \frac{dy}{dx} = \frac{\lim_{\Delta x \to 0} f(x+\Delta x)-f(x)}{\Delta x}$ is called the derivative

from first principle or derivative from delta method.

Let Y_t is the function of t, then first finite difference Y_t can be written

$$\Delta Yt = (Y_{t+1} - Y_t) \quad (1)$$

$Y_t =$ Value of the variable y in the time period *t*.

Y_{t+1} = Value of the variable y in the immediately following the time t+*1*.
As right hand derivative, left hand derivative exists in process of differentiation, same process exists in finite difference called forward difference and backward difference. So, a difference equation is analogues to a differential equation in many contexts. (Butcher, 2005).

To further highlight the relationship between difference equation and differential equation

Let us begin with homogeneous linear ordinary differential of first order,

$$5\frac{dy(t)}{dt} + 3\,y(t) = 0 \quad (2)$$

and approximate it by a difference equation. Approximation of derivatives by finite difference,

$$\frac{dy(t)}{dt} = \lim_{\Delta t \to 0} \frac{y(t+\Delta t)-y(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{y(t)-y(t-\Delta t)}{\Delta t}$$

$$\frac{dy(t)}{dt} = \lim_{\Delta t \to 0} \frac{y(t+\Delta t)-y(t-\Delta t)}{2\,\Delta t}$$

At any point at which, *y (t)* is differentiable, any of these definitions of a derivative give exactly the same result. When limiting value is taken, a derivative in continuous time can be approximated by a finite difference in discrete time by

$$\frac{y((n+1)\Delta t)-y(n\,\Delta t)}{\Delta t} \quad (3)$$

This is called a forward difference as it uses the present value $y(n\,\Delta t)$ and the forward value of y.

The expression:

$$\frac{y(n\Delta t)-y\,(n-1)\Delta t}{\Delta t} \quad (4)$$

uses the present value $y(n\,\Delta t)$ of y and backward is $y\,(n-1)\Delta t.$

Next we consider

$$\frac{y(n+1)\Delta t - y(n-1)\Delta t}{2\Delta t} \tag{5}$$

Expression (5) uses both of forward value and backward value, so it is a central difference. In the case of limit when $\Delta x \to 0$, these all are same. But in discrete time, Δt is fixed and is not zero and for these, there approximations to a continuous time derivative are different.

The differential equation (1) can be converted to a difference equation by using a forward difference approximation:

$$5\frac{y((n+1)\Delta y) - y(n\Delta t)}{\Delta t} + 3\, y(n\Delta t) = 0 \tag{6}$$

To simplify the notation, suppose $y[n] = y(n\Delta t)$ where the square brackets, [.] discriminate a function of discrete time from a function of continuous time which is indicated by using parentheses, (.). The above equation (6) gets the form:

$$5(y[n+1] - y[n]) + 3\Delta t y[n] = 0 \tag{7}$$

Which is a homogenous difference equation.

In short, the differential of continuous-time function is $y[n] = y(n\Delta t)$.

$$\Delta(x[n]) = x'[n] \tag{8}$$

Formulae for difference and differentiation

If c is constant and y= y (t) is given functions then

Difference	Differentiation
(1) Constant $\Delta C = 0$	$\frac{dC}{dt} = 0$
(2) Constant times, $\Delta(Cy[n]) = Cy'[n]$	$\frac{d}{dt}(Cx(t)) = Cx'(t)$
(3) Power function	
(a) $\Delta[y]^n = n[y]^{n-1}$	$\frac{d(t^n)}{dt} = n\,t^{n-1}$
(b) Sum of two functions	
$\Delta(x[n] + y[n]) = x'[n] + y'[n]$	$\frac{d\,((x(t)+y(t))}{dt} = x'(t) + y'(t)$
(4) Product of two functions	
$\Delta(x[n]y[n]) = x[n]y'[n] + y[n]x'[n]$	$\frac{d}{dt}(x(t)+y(t)) = x(t)y'(t)+x'(t)y(t)$

Both differential equation and difference equation follow the rules of calculus and the nature of solution depends on the choice of Δt.

Conclusion

Difference equations may be defined in a similar way to differential equations. Analogous to differential equations, difference equations can may be either linear or non-linear, homogeneous or non-homogeneous, and of the first or second or higher orders. The first differences are related to first derivatives. Difference equations are a discrete version of differential equations and a differential equation are a continuous version of a difference equations.

References

Butcher, J. C.(2005). *Differential and Difference equations.* Numerical Method for Ordinary Differential Equation,1-44.Doi:10:1002/047086279.ch1.

Chaudhary, A. K.(2018*). Business Mathematics-II*. Khanal Publication, Pvt. Ltd, Kathmandu.

Feudel, F.& Bichler, R.(2020). *Students' understanding of derivatives for concept in the context of Mathematics for Economics*, Journal of Mathematics, 42(8), 273-305.

Panagiota, K. K., Eleni, T., Maria, M.M., & Mary, K. (2013). *Differentiation of teaching and learning mathematics: an action research study in tertiary education,* International Journal of Mathematical Education in Science and Technology.

Rogers, R.(2005). *Putting the differentials back into differential calculus*, Washington, D.C: Mathematics Association of America.

Sahadevan, R. (2001*). Non-linear Differential-Difference and Difference equations: Integral and exact solvability*, 42(2001), 627-637.

Siddiqi, A.H. & Manchanda, P. (2006*).* A First Course in Differential Equation with Application, *Macmillan India Ltd.*

Towers, J.D. (2009). Discretizing delta functions via finite differences and gradient normalization. *Journal of Computational Physics*, 228(10), p.3816-3836.