The Nepali Mathematical Sciences Report
Regd No.: 53/033-034, ISSN: 2392–411X
http://www.journal.cdmathtu.edu.np

Year: 2026, Volume: 43, No.: 1
Pages: 58–75
https://doi.org/10.3126/nmsr.v43i1.98501

Survey Article

Dynamics of Composite Holomorphic Functions

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Received: 21 May, 2026, Accepted: 27 June, 2026, Published Online: 10 August, 2026

Abstract

The dynamics of compositions of transcendental entire functions constitute an important area of modern complex dynamics, particularly in connection with the structure of Fatou and Julia sets and the existence of wandering domains. In this paper, we present a survey of several classical and recent developments concerning the iteration of transcendental entire functions and their compositions. We review the classification of Fatou components, the dynamics of escaping, bounded orbit, and unbounded non-escaping sets, and the role of singular values in determining the global behavior of entire functions. Particular attention is devoted to results on wandering domains, including the pioneering work of Baker, the absence of wandering domains for functions in the Speiser class, and the construction of oscillating wandering domains by quasiconformal folding techniques.

We further discuss the dynamics of composite holomorphic functions, emphasizing the relationships between the Fatou and Julia sets of individual functions and those of their compositions. The theory of Carleman sets and approximation methods for entire functions is reviewed as a fundamental tool in the construction of transcendental entire functions with prescribed dynamical properties. In this context, we summarize several results concerning the existence of wandering, periodic, and preperiodic Fatou components for compositions of transcendental entire functions and formulate a natural extension of these phenomena to arbitrary finite collections of entire functions. The paper highlights the interaction between approximation theory and complex dynamics and outlines several open problems and directions for future research.

Keywords: Wandering domains, Fatou set, Entire functions, Carlemann set.
AMS(MOS) Subject Classification: MSC 2020: 37-02 (primary) 37F10, 30D05 (Secondary)

1 Introduction

Let f be a holomorphic function defined on the extended complex plane $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$, or on a suitable subset thereof. The nth iterate of f is defined recursively by $f^n(z) = f\big(f^{n-1}(z)\big),\ n \in \mathbb{N}$, where

$$f^n = \underbrace{f \circ f \circ \cdots \circ f}_{n \text{ times}}.$$

The zeroth iterate is defined to be the identity map, $f^0(z) = z$. For a point $z \in \mathbb{C}_\infty$, the sequence $\{f^n(z)\}_{n=0}^{\infty}$ is called the *orbit* of z under f. Holomorphic (or complex) dynamics is concerned with the study of the behavior of such orbits under repeated iteration of holomorphic functions. One of the fundamental questions in holomorphic dynamics is whether the family of iterates $\{f^n\}_{n=0}^{\infty}$ forms a normal family in the sense of Montel. More precisely, for a given point z, the behavior of the orbit $\{f^n(z)\}$ is said to be stable if every sequence of iterates contains a subsequence that converges locally uniformly either to a holomorphic function or to ∞ on compact subsets of the domain under consideration. The existence or failure of such normality leads naturally to a decomposition of $\mathbb{C}$ or $\mathbb{C}_\infty$ into two disjoint sets, namely the Fatou set and the Julia set, which are the central objects of study in holomorphic dynamics.

Definition 1.1. Let f be a holomorphic function. The *Fatou set* of f is defined by

$$F(f) = \{z \in \mathbb{C} \text{ or } \mathbb{C}_\infty : \{f^n\}_{n=0}^{\infty} \text{ forms a normal family in a neighborhood of } z\}.$$

The complement of $F(f)$ in $\mathbb{C}$ or $\mathbb{C}_\infty$ is called the *Julia set* of f, and is denoted by $J(f)$.

A maximal connected subset of the Fatou set $F(f)$ is called a *Fatou component*. In general, the Fatou set consists of infinitely many connected components, and the action of f maps Fatou components to Fatou components, thereby inducing a dynamical system on the collection of these components. By definition, the Fatou set $F(f)$ is an open set (possibly empty), whereas the Julia set $J(f)$ is closed. Furthermore, the Julia set is nonempty, perfect, and completely invariant under f; that is, equivalently, as a consequence, the Fatou set $F(f)$ is also completely invariant under f.

For a holomorphic function f, a point z is called *preperiodic* (or *eventually periodic*) if there exist integers $m \geq 0$ and $n > 0$ such that $f^{m+n}(z) = f^m(z)$. When $m = 0$, the point z is called a *periodic point* of period n. If n is the smallest positive integer for which $f^n(z) = z$, then n is called the *minimal period* of z. Let z be a periodic point of minimal period n. The complex number $\lambda = (f^n)'(z)$, where $(f^n)'$ denotes the complex derivative of f^n, is called the *multiplier* of the periodic point z. It is well known that the multiplier is constant along the entire periodic orbit. According to the value of $|\lambda|$, a periodic point is classified as follows:

- *attracting* if $|\lambda| < 1$;
- *super-attracting* if $\lambda = 0$;
- *indifferent* (or *neutral*) if $|\lambda| = 1$;
- *repelling* if $|\lambda| > 1$.

In the indifferent case, the multiplier can be written in the form

$$\lambda = e^{2\pi i\theta}, \qquad 0 \leq \theta < 1.$$

If θ is rational, then z is called a *rationally indifferent* (or *parabolic*) periodic point. If θ is irrational, then z is called an *irrationally indifferent* periodic point. A fundamental result in holomorphic dynamics states that every attracting periodic point belongs to the Fatou set $F(f)$, whereas every repelling and rationally indifferent periodic point belongs to the Julia set $J(f)$. For irrationally indifferent periodic points, both possibilities may occur: such points may belong either to the Fatou set or to the Julia set, depending on the local dynamics. It is also well known that the Julia set can be characterized as the closure of the set of repelling periodic points; that is,

$$J(f) = \overline{\{z : z \text{ is a repelling periodic point of } f\}}.$$

2 Different Fatou Components and Their Dynamics

Let V_0 be a Fatou component of a holomorphic function f. Since the Fatou set $F(f)$ is completely invariant, for each $n \in \mathbb{N}$, the image $f^n(V_0)$ is contained in a Fatou component V_n of $F(f)$. Moreover, the set $V_n \setminus f^n(V_0)$ is either empty or consists of a single point. If the Fatou components V_n, $n \in \mathbb{N}$, are all distinct, then V_0 is called a *wandering domain*. Otherwise, V_0 is said to be *preperiodic*. In particular, if there exists a smallest positive integer n such that $f^n(V_0) = V_0$, then V_0 is called a *periodic Fatou component* of period n. Using quasiconformal deformations of Riemann surfaces, Sullivan [36] solved a long-standing problem that had remained open since the pioneering works of Fatou [17, 18, 19] and Julia [22, 23, 24]. He proved that the Fatou set of a rational function contains no wandering domains. In contrast, transcendental entire functions may possess wandering domains; see, for example, [1, 2, 3, 7, 9, 13, 29, 36]. Examples of transcendental entire functions without wandering domains can be found in [2, 4, 31]. Let V_0 be a periodic Fatou component of period n for a holomorphic function f. Then V_0 belongs to one of the following types:

1. *(Super) attracting domain* A periodic Fatou component V_0 is called a *(super) attracting domain* if it contains a (super) attracting periodic point z_0 of period n such that $f^{np}(z) \longrightarrow z_0 \quad \text{as } p \to \infty$, for every $z \in V_0$.

2. *Parabolic domain (or Leau domain)* A periodic Fatou component V_0 is called a *parabolic domain* (or *Leau domain*) if ∂V_0 contains a parabolic periodic point z_0 of period n such that $f^{np}(z) \longrightarrow z_0 \quad \text{as } p \to \infty$, for every $z \in V_0$. In this case, the local dynamics near z_0 are organized into finitely many invariant regions, called *attracting petals*, within which the iterates converge to the parabolic periodic point z_0. This description is a consequence of the Leau–Fatou Flower Theorem.

3. *Siegel disk* A periodic Fatou component V_0 is called a *Siegel disk* if there exists a conformal map $\varphi : \mathbb{D} \to V_0$, where $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, such that $(\varphi^{-1} \circ f^n \circ \varphi)(z) = e^{2\pi i\alpha} z$, for some irrational number $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Moreover, $\varphi(0) = z_0$, where z_0 is an irrationally indifferent periodic point satisfying $f^n(z_0) = z_0$. Thus, the dynamics on a Siegel disk are analytically conjugate to an irrational rotation of the unit disk.

4. *Herman ring* A periodic Fatou component V_0 is called a *Herman ring* if there exists a conformal homeomorphism $\varphi : A \to V_0$, where $A = \{z \in \mathbb{C} : 1 < |z| < r\}, \quad r > 1$, such that $(\varphi^{-1} \circ f^n \circ \varphi)(z) = e^{2\pi i\alpha} z$, for some irrational number $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. In other words, the dynamics on V_0 are conformally conjugate to an irrational rotation on an annulus. Unlike Siegel disks, Herman rings are doubly connected Fatou components. They occur for certain rational maps but cannot occur for transcendental entire functions.

5. *Baker domain* A periodic Fatou component V_0 is called a *Baker domain* if $f^{np}(z) \longrightarrow \infty$ as $p \to \infty$, for every $z \in V_0$. Baker domains can occur only for transcendental entire or meromorphic functions and never for rational functions. They provide examples of stable dynamics in which all points of the Fatou component escape to infinity under iteration. A classical example is the transcendental entire function $f(z) = z + 1 + e^{-z}$. Although the iterates tend to infinity, the convergence is locally uniform on V_0, and hence the dynamics remain stable within the Fatou set.

It is worth noting that the Fatou set $F(f)$ of a transcendental entire function f cannot contain any Herman rings. Indeed, Herman rings may occur for rational or transcendental meromorphic functions, but they do not arise in the dynamics of transcendental entire functions. If V_0 is a Siegel disk, then the family of iterates $\{f^n|_{V_0}\}_{n\in\mathbb{N}}$ has non-constant limit functions. In contrast, for all other types of periodic Fatou components, namely attracting domains, parabolic domains, and Baker domains, the limit functions of the iterates are constant. Suppose that V_0 is a periodic Fatou component of period $p \geq 2$, and let $V_j = f^j(V_0),\ j = 0, 1, \ldots, p-1$. Then the components $V_0, V_1, \ldots, V_{p-1}$ form a periodic cycle of Fatou components. Moreover, all components in this cycle are of the same type in the above classification. That is, if V_0 is an attracting domain, parabolic domain, Siegel disk, Herman ring, or Baker domain, then each of the components $V_1, V_2, \ldots, V_{p-1}$ is of the same type. A systematic account of the classification of periodic Fatou components may be found in [6]. This classification originates in the fundamental work of Fatou [17, 18, 19], who completely classified periodic Fatou components more than a century ago.

3 Escaping Set, Bounded Orbit Set and Unbounded Non-escaping Orbit Set

Another important partition of $\mathbb{C}$ (or $\mathbb{C}_\infty$) arises from the behavior of orbits under iteration and is described in terms of the following dynamically significant sets.

Definition 3.1. Let f be a holomorphic function. The set

$$I(f) = \{z \in \mathbb{C} : f^n(z) \to \infty \text{ as } n \to \infty\}$$

is called the *escaping set* of f. The set

$$K(f) = \{z \in \mathbb{C} : \{f^n(z)\}_{n\in\mathbb{N}} \text{ is bounded}\}$$

is called the *bounded orbit set* of f. The set

$$BU(f) = \mathbb{C} \setminus \big(I(f) \cup K(f)\big)$$

is called the *unbounded non-escaping set* of f. Equivalently, $BU(f)$ consists of points whose orbits are unbounded but do not escape to infinity.

Prior to Eremenko's seminal work [12], the escaping set was studied primarily in the context of polynomial dynamics, where it is closely related to the Fatou component containing the point at infinity. Eremenko [12] initiated the systematic study of escaping sets for general transcendental entire functions and established the following fundamental results:

1. $I(f) \neq \emptyset$; (Figure 1)
2. $I(f) \cap J(f) \neq \emptyset$; (Figure 2)
3. the closure $\overline{I(f)}$ has no bounded components (Figure 3).

Furthermore, Eremenko proved that

4 $I(f) = I(f^n)$, $n \geq 2$;

5 $I(f)$ is completely invariant under f.

The sets $K(f)$ and $BU(f)$ were subsequently investigated by Bergweiler [8] and Osborne and Sixsmith [26], respectively. They established analogues of the above properties (1)–(5) for the bounded orbit set $K(f)$ and the unbounded non-escaping set $BU(f)$. Let V_0 be a wandering domain of a holomorphic function f. Then V_0 belongs to one of the following classes of wandering domains.

1. Escaping wandering domain if $V_0 \subset I(f)$.
2. Oscillating wandering domain if $V_0 \subset BU(f)$.
3. Bounded orbit wandering domain if $V_0 \subset K(f)$.

Baker [1] constructed the first example of a transcendental entire function possessing a wandering domain. The function is of the form

$$f(z) = Cz^2 \prod_{n=1}^{\infty} \left(1 + \frac{z}{a_n}\right),$$

where $C \in \mathbb{C}$ and the sequence $1 < a_1 < a_2 < \cdots$ is chosen so that f grows sufficiently rapidly and satisfies $a_{n+1} < f(a_n) < 2a_{n+1},\ n \in \mathbb{N}$. The numbers a_n are selected in such a way that the annuli $A_n = \left\{z \in \mathbb{C} : a_n < |z| < a_{n+1}^{1/2}\right\}$ satisfy $f(A_n) \subset A_{n+1}$. Let V_n denote the Fatou component containing A_n. Then $f(V_n) = V_{n+1}$, and the components V_n are all distinct. Consequently, V_n is a wandering domain. Moreover, $f^m(z) \to \infty$ as $m \to \infty$, locally uniformly on V_n. In fact, each V_n is a multiply connected wandering domain. Hence V_n is an escaping wandering domain; moreover, it belongs to the class of fast escaping wandering domains [1, Theorem 3.1].

The wandering domains in Baker's original example are multiply connected. Since then, a wide variety of examples of simply connected wandering domains have been constructed; see, for example, [2, 11, 14, 15, 21, 36]. Osborne and Sixsmith [26, Theorem 1.1] proved that every Fatou component contained in $BU(f)$ is necessarily an oscillating wandering domain. On the other hand, it remains an open problem whether there exist wandering domains whose orbits are bounded; such domains are commonly referred to as *bounded orbit wandering domains*. The preceding discussion may be summarized as follows. Fatou proved that all limit functions of the family of iterates on a wandering domain are constant [17]. Wandering domains for which the only limit function is ∞ are called *escaping wandering domains*. If ∞ is a limit function but there also exist finite limit functions, then the wandering domain is called *oscillating*. Finally, if all limit functions are finite constants, the wandering domain is said to be *dynamically bounded*. A major open problem in transcendental dynamics is whether dynamically bounded wandering domains exist. Any progress toward resolving this question is likely to require a deeper understanding of the dynamics both within and near wandering domains. This problem provides one of the main motivations for the research presented in this paper.

4 Brief Review of the Dynamics of Composite Holomorphic Functions

The study of the dynamics of compositions of holomorphic functions was initiated by Fatou [17]. In particular, he proved that

$$F(f) = F(g)$$

whenever f and g are permutable rational functions; see [5, Theorem 4.2.9]. A key ingredient in the proof is the fact that

$$g(J(f)) \subset J(f)$$

for any pair of permutable rational functions, and more generally for permutable entire functions that are neither linear nor Möbius transformations. Furthermore, every point of $J(f)$ is a limit point of repelling periodic points of f, a property that plays a crucial role in the analysis. For transcendental entire functions, however, the corresponding question remains open in general. Nevertheless, several partial results are known. For example, the following theorems may be found in [2, Lemmas 4.4 and 4.5].

Theorem 4.1. *Let f and g be permutable transcendental entire functions such that ∞ is not a limit function of any subsequence of $\{f^n\}$ and $\{g^n\}$ on any component of $F(f)$ and $F(g)$, respectively. Then $F(f) = F(g)$.*

Theorem 4.2. *Let f and g be permutable entire functions satisfying $f = g + c$ for some constant $c \in \mathbb{C}$. Then $F(f) = F(g)$.*

These results mark the beginning of the systematic study of the dynamics of compositions of holomorphic functions. They suggest that when two holomorphic functions f and g are permutable, the dynamical behavior of f, g, $f \circ g$, and $g \circ f$ is closely related. In many cases, these functions share the same Fatou and Julia sets. On the other hand, when f and g are not permutable, it is generally difficult to determine whether their dynamical behaviors are similar or substantially different. One of the primary objectives of the present work is to investigate the dynamics of compositions of holomorphic functions in greater detail. Baker and Singh [4, Theorem 1] proved the following result.

Theorem 4.3. *Let f be a non-constant entire function, and let $g(z) = p + qe^{\frac{2\pi i z}{r}}$, where p, q, and r are nonzero constants. If $g \circ f$ has no wandering domains, then $f \circ g$ also has no wandering domains.*

In particular, if f is a non-constant polynomial, then [2, Theorem 6.3] implies that the function $p(z) = f(e^z)$ has no wandering domains. Consequently, by Theorem 4.3, the function $e^{f(z)}$ also has no wandering domains. Baker and Singh [4, Theorem 3] established the following existence result.

Theorem 4.4. *Let f be a transcendental entire function having at least one fixed point. Then there exists an entire function g such that the composition $f \circ g$ possesses a wandering domain.*

Further results concerning the dynamics of compositions of holomorphic functions rely on several concepts related to the growth of entire functions.

Definition 4.1 (Order, lower order, and type of an entire function). Let f be an entire function. The *order* $\rho(f)$ and the *lower order* $\lambda(f)$ of f are defined respectively by

$$\rho(f) = \limsup_{r\to\infty} \frac{\log\log M(r,f)}{\log r}$$

and

$$\lambda(f) = \liminf_{r\to\infty} \frac{\log\log M(r,f)}{\log r},$$

where $M(r,f) = \max_{|z|=r}|f(z)|$ denotes the maximum modulus of f. The *type* of an entire function f of finite order ρ is defined by

$$\sigma(f) = \limsup_{r\to\infty} \frac{\log M(r,f)}{r^\rho}.$$

The *minimum modulus* of f is defined by $m(r,f) = \min_{|z|=r}|f(z)|$. The growth of f is said to be of

- *minimal type* if $\sigma(f) = 0$;
- *mean type* if $0 < \sigma(f) < \infty$;
- *maximal type* if $\sigma(f) = \infty$.

The order of an entire function is an important measure of its growth, comparing the growth rate of its maximum modulus with that of the exponential function. For example, the functions $e^z,\ \sin z,\ \cos z$ all have order 1, whereas the function $\exp(z^d)$ has order d. The dynamics of a holomorphic function are, to a large extent, determined by its singular values. These arise from points at which the inverse function fails to be locally well defined. There are two basic types of singular values: critical values and asymptotic values.

Definition 4.2 (Critical values, asymptotic values, and singular values). Let f be a holomorphic function. The set of *critical values* of f is defined by

$$CV(f) = \left\{w \in \mathbb{C} : w = f(z) \text{ for some } z \in \mathbb{C} \text{ with } f'(z) = 0\right\}.$$

The set of *asymptotic values* of f is defined by

$$AV(f) = \{\, w \in \mathbb{C} : \exists\, \gamma : [0,\infty) \to \mathbb{C},\ \gamma(t) \to \infty,\ f(\gamma(t)) \to w \,\}.$$

It is straightforward to observe that the exponential function $f(z) = e^z$ has no critical points, since $f'(z) = e^z \neq 0$ for all $z \in \mathbb{C}$. However, it possesses the asymptotic value 0, because $f(z) \to 0$ as $\Re(z) \to -\infty$. Similarly, for the sine function $f(z) = \sin z$, the critical points are determined by $f'(z) = \cos z = 0$, which implies $z = \frac{\pi}{2} + n\pi,\ n \in \mathbb{Z}$. Evaluating f at these points gives $f\left(\frac{\pi}{2} + n\pi\right) = (-1)^n$, and hence the critical values are $CV(f) = \{-1, 1\}$. The zeros of $\sin z$ occur at $z = n\pi,\ n \in \mathbb{Z}$, where $f(n\pi) = \sin(n\pi) = 0$. It is noteworthy that among entire functions, only transcendental entire functions can possess finite asymptotic values. In particular, polynomials cannot have finite asymptotic values. Furthermore, if f is a rational function or a transcendental entire function, then the immediate basin of attraction of every attracting or parabolic cycle contains at least one singular value of f.

There exist holomorphic functions whose finite asymptotic values are also critical values. For example, the function $f(z) = z^2 e^{-z^2}$ has $AV(f) = \{0\}$ and $CV(f) = \left\{0, \frac{1}{e}\right\}$.

Definition 4.3 (**Finite Type and Bounded Type Functions**)**.** Let f be a holomorphic function. If the set of singular values $SV(f)$ is finite, then f is said to be of *finite type*. If the set of singular values $SV(f)$ is bounded, then f is said to be of *bounded type*.

Definition 4.4 (**Speiser Class and Eremenko–Lyubich Class**)**.** Let f be a holomorphic function. The collections

$$\mathscr{S} = \{\, f : f \text{ is of finite type} \,\}$$

and

$$\mathscr{B} = \{\, f : f \text{ is of bounded type} \,\}$$

are known respectively as the *Speiser class* and the *Eremenko-Lyubich class*.

It is evident that $\mathscr{S} \subset \mathscr{B}$ and $\mathscr{B} \setminus \mathscr{S} \neq \emptyset$. For example, $\frac{\sin z}{z} \in \mathscr{B} \setminus \mathscr{S}$. Indeed, this function has the asymptotic value 0, while its set of critical values is an infinite set of real numbers, each having modulus at most 1. Among the functions in the class $\mathscr{S}$, the most important examples are the exponential family $f_\lambda(z) = \lambda e^z,\ \lambda \neq 0$, and the cosine family $f(z) = \cos(\alpha z + \beta),\ \alpha \neq 0$. A fundamental result in transcendental dynamics states that transcendental entire functions belonging to the class $\mathscr{S}$ do not possess wandering domains.

In fact, every Fatou component of a function in the class $\mathscr{S}$ is either periodic or preperiodic. The absence of wandering domains for polynomials (and, more generally, for rational functions) [36], as well as for transcendental entire functions in the class $\mathscr{S}$ [14, 20], is regarded as one of the major breakthroughs in complex dynamics.

Definition 4.5 (**Post-singular Point and Post-singular Set**)**.** Let f be a holomorphic function. A *post-singular point* of f is a point lying on the forward orbit of a singular value of f. More precisely, if $s \in SV(f)$, then $f^n(s),\ n \geq 0$, is a post-singular point of f. The collection of all post-singular points is called the *post-singular set* of f and is denoted by

$$P(f) = \bigcup_{n \geq 0} f^n\big(SV(f)\big).$$

It is worth noting that if f is a transcendental entire function, then $P(f)$ contains at least two points. Moreover, $P(f^k) = P(f)$ for every $k \in \mathbb{N}$.

Definition 4.6 (**Post-singularly Bounded and Post-singularly Finite Functions**)**.** A holomorphic function f is called *post-singularly bounded* if its post-singular set $P(f)$ is bounded. Likewise, f is called *post-singularly finite* if $P(f)$ is finite.

Definition 4.7 (Hyperbolic Holomorphic Function). A holomorphic function $f \in \mathscr{B}$ is said to be *hyperbolic* if every singular value of f belongs to the basin of attraction of an attracting periodic cycle of f.

For example, the function $f(z) = \frac{\pi}{2} \sin z$ is hyperbolic. Indeed, $SV(f) = \{\pm\frac{\pi}{2}\}$, and both singular values are super-attracting fixed points. Therefore, by definition, f is hyperbolic.

Baker and Singh [4, Lemmas 1 and 2] established the following results.

Theorem 4.5. *Let g and h be entire functions, each having only finitely many asymptotic values. Then the composition $g \circ h$ also has only finitely many asymptotic values.*

Theorem 4.6. *Let g and h be entire functions of finite order. Then the composition $g \circ h$ has at most finitely many asymptotic values.*

Baker and Singh [4, Theorem 2] also proved the following theorem.

Theorem 4.7. *The function $e^{e^z} - e^z$ has no wandering domains.*

If we let $f(z) = e^z$ and $g(z) = e^z - z$. then the function g has finitely many asymptotic values and exactly one critical value, namely e. Consequently, g belongs to the class $\mathscr{S}$ and therefore has no wandering domains. It then follows from Theorem 4.3 that the composition $g(f(z)) = e^{e^z} - e^z$ also has no wandering domains.

The preceding results concern the dynamics of the compositions $f \circ g$ and $g \circ f$ when one of the functions has a special form. The following theorems of Bergweiler and Wang [9, Theorems 1–4] describe the dynamics of these compositions without imposing such restrictions.

Theorem 4.8. *Let f and g be non-linear entire functions. Then $z \in J(f \circ g)$ if and only if $g(z) \in J(g \circ f)$.*

Theorem 4.9. *Let f and g be non-linear entire functions. Suppose that U_0 is a component of $F(f \circ g)$ and that V_0 is the component of $F(g \circ f)$ containing $g(U_0)$. Then the following statements hold:*

1. *U_0 is wandering if and only if V_0 is wandering.*
2. *If U_0 is periodic, then V_0 is also periodic. Moreover, if U_0 belongs to a particular type in the classification of periodic Fatou components, then V_0 belongs to the same type.*

It should be noted that the difference between the Fatou component V_0 and the image $g(U_0)$ appearing in Theorem 4.9 consists of at most one point. Moreover, Theorem 4.9 shows that the dynamical behavior of the compositions $f \circ g$ and $g \circ f$ is closely related. Consequently, understanding the dynamics of one composition often provides valuable information about the dynamics of the other.

The following theorems establish the absence of wandering domains for compositions belonging to certain classes of entire functions.

Theorem 4.10. *Let $\mathscr{F} = \{e^{iz} \pm z, i(e^z \pm z), \sin z \pm z, \cos z \pm z\}$ and $\mathscr{G} = \{g_1 \circ g_2 \circ g_3 \circ \cdots \circ g_n : g_i = \sin z \text{ or } \cos z,\ i = 1, 2, 3 \cdots, n,\ n \in \mathbb{N}\}$. Then $f \circ g$ has no wandering domains for any $f \in \mathscr{F}$ and $g \in \mathscr{G}$.*

Theorem 4.11. *Let f be a real valued function such that $|f(x)| \leq |x|$ for all $x \in [-1, 1]$. If $SV(f) \subset \mathbb{R}$, then $f \circ \sin z$ has no wandering domains.*

We now consider several results concerning two permutable transcendental entire functions and their Julia sets due to Singh and Wang [30, Theorems 1-4].

Theorem 4.12. *Let f and g be two transcendental holomorphic functions such that $f \circ g = g \circ f$. Then*

1. *$J(f \circ g)$ is completely invariant.*
2. *$J(f) \cup J(g) \subset J(f \circ g)$ and $J(f^n \circ g^m) = J(f \circ g)$ for any non-zero $m, n \in \mathbb{N}$.*
3. *$J(f) = J(f \circ g)$ if f is of bounded type (or f do not have wandering domains).*
4. *Either $J(f) = J(g)$ or $J(f) \subset J(f \circ g) - O^-_{f\circ g}(w_0)$ for every $w_0 \in J(g) - J(f)$ which not an exeptional Fatou value of $f \circ g$, where $O^-_{f\circ g}(w_0) = \cup_{n \geq 0}(f \circ g)^{-n}(w_0)$.*

Analogous statements for Fatou sets follow immediately from Theorem 4.12. In particular $F(f \circ g)$ is completely invariant under f and g, $J(f \circ g) \subset F(f) \cap F(g)$, $F(f) = F(f \circ g)$ if f is of bounded type (or f do not have wandering domains), and $F(f^n \circ g^m) = F(f \circ g)$ for any non-zero $m, n \in \mathbb{N}$. In addition, If $g = af + b$, where a and b are constant and $a \neq 0$, and $g = f + p$ for periodic f with period p. then for both cases $F(f) = F(f \circ g)$ hold.

5 Carleman Set and Dynamics of Composite Holomorphic Maps

Theorem 4.9 provides one of the fundamental results illustrating the similarity between the dynamical behavior of the compositions $f \circ g$ and $g \circ f$. However, the dynamics of individual transcendental entire functions may differ significantly from the dynamics of their compositions. To investigate such phenomena, we require a powerful approximation tool from the theory of entire functions. For this purpose, we employ the notion of a *Carleman set*. Carleman sets play an important role in approximation theory, as they permit the approximation of holomorphic functions by entire functions. This approximation technique will be used extensively in the subsequent study of the dynamics of transcendental entire functions and their compositions.

Definition 5.1 (Carleman Set)**.** Let $Q \subset \mathbb{C}$ be a closed set. Define

$$C(Q) = \{f : Q \to \mathbb{C} \mid f \text{ is continuous on } Q \text{ and analytic on } Q^\circ\}.$$

We say that Q is a Carleman set for $\mathbb{C}$ if for $f \in C(Q)$, $\exists$ continuous function ϵ from Q to the set $\mathbb{R}^+$ of positive real numbers, there is an entire map g such that

$$|f(z) - g(z)| < \epsilon(z)$$

$\forall\, z \in Q$.

A useful criterion for determining whether a subset Q of the complex plane is a Carleman set was established by Nersesjan. This characterization is particularly valuable in applications, as it provides a practical method for verifying the Carleman property of a given set. The theorem is stated as follows.

Theorem 5.1. *Let $Q \subset \mathbb{C}$. Then Q is a Carleman set for $\mathbb{C}$ if and only if*

1. *$\mathbb{C}_\infty \setminus Q$ is connected;*
2. *$\mathbb{C}_\infty \setminus Q$ is locally connected at ∞;*
3. *for any compact subset K, $\exists$ a neighborhood V of ∞ in $\mathbb{C}_\infty$ such that no component of Q° intersects both K and V.*

It is worth noting that the space $\mathbb{C}_\infty \setminus Q$ is connected if and only if every connected component Z of the open set $\mathbb{C} \setminus Q$ is unbounded. This observation, together with Theorem 5.1, provides a useful criterion for determining whether a given set is a Carleman set for $\mathbb{C}$. The sets presented in the following examples are Carleman sets for $\mathbb{C}$.

Example 5.1. The set $E = \{z \in \mathbb{C} : |z| = 1, \Re z > 0\} \cup \{z = x : x > 1\} \cup \bigcup_{n=3}^{\infty} \{z = re^{i\theta} : r > 1, \theta = \pi/n\}$ is a Carleman set by Theorem 5.1.

Example 5.2. The set $E = G_0 \cup \bigcup_{k=1}^{\infty} (G_k \cup B_K \cup L_k \cup M_k)$, where

$$G_0 = \{z \in \mathbb{C} : |z - 2| \leq 1\};$$

$$\begin{aligned} G_k &= \{z \in \mathbb{C} : |z - (4k+2)| \leq 1\} \cup \{z \in \mathbb{C} : \Re(z) = 4k+2,\ \Im(z) \geq 1\} \\ &\quad \cup \{z \in \mathbb{C} : \Re(z) = 4k+2,\ \Im(z) \leq -1\}, \quad (k = 1, 2, 3, 4, \ldots); \end{aligned}$$

$$\begin{aligned} B_k &= \{z \in \mathbb{C} : |z + (4k+2)| \leq 1\} \cup \{z \in \mathbb{C} : \Re(z) = -(4k+2),\ \Im(z) \geq 1\} \cup \\ &\quad \{z \in \mathbb{C} : \Re(z) = -(4k+2),\ \Im(z) \leq -1\}, \quad (k = 1, 2, 3, 4, \ldots); \end{aligned}$$

$$L_k = \{z \in \mathbb{C} : \Re(z) = 4k\}, \quad (k = 1, 2, 3, 4, \ldots);$$

and

$$M_k = \{z \in \mathbb{C} : \Re(z) = -4k\}, \quad (k = 1, 2, 3, 4, \ldots)$$

is a Carleman set by Theorem 5.1.

The Carleman set constructed in Example 5.2 was utilized by Singh [28, Theorems 1–4] to prove the following assertions.

Theorem 5.2. *There exist two transcendental entire functions f and g and a domain U such that U lies in the wandering component of the $F(f),$ and $F(g)$, but lies in the periodic component of $F(g \circ f)$.*

Theorem 5.3. *There exist two transcendental entire functions f and g and a domain U such that U lies in the wandering component of the $F(f),\ F(g),\ F(f \circ g)$ and $F(g \circ f)$.*

Theorem 5.4. *There exist two transcendental entire functions f and g and a domain U such that U lies in the periodic component of the $F(f)$ and $F(g)$ but lies in the wandering components of $F(f \circ g)$ and $F(g \circ f)$.*

Theorem 5.5. *There exist two transcendental entire functions f and g and a domain U such that U lies in the periodic component of the $F(f), F(g),$ and $F(g \circ f)$ but lies in the wandering component of $F(f \circ g)$.*

Singh [27, Theorems 3.2.1–3.2.6] studied the occurrence of various Fatou components of transcendental entire functions within angular regions by employing approximation techniques for entire functions, particularly those arising from Carleman sets. The results stated in Theorems 5.2, 5.3, 5.4, and 5.5 were subsequently extended by Tomar [37], who proved the following result.

Theorem 5.6. *There exist two transcendental entire functions f and g, and infinitely many domains in the angular region which lies in the wandering (pre-periodic) component of the $F(f)$, $F(g), F(f \circ g)$ and $F(g \circ f)$.*

The results discussed above were subsequently extended by Kumar *et al.* [25, Theorems 2.1–2.15]. Among their contributions, the following two theorems are particularly relevant to our study.

Theorem 5.7. *There exist two different transcendental entire functions f and g, and infinitely many domains which lie in different wandering component of the $F(f),\ F(g),$
$F(f \circ g)$ and $F(g \circ f)$.*

Theorem 5.8. *There exist two different transcendental entire functions f and g, and infinitely many domains which lie in different pre-periodic component of the $F(f),\ F(g),$
$F(f \circ g)$ and $F(g \circ f)$.*

A natural question arising from the preceding results is whether the similarity between the dynamics of compositions and those of the individual functions can occur for more than two transcendental entire functions. This question formed part of our Ph.D. research. We constructed three transcendental entire functions with the property that each individual function, together with every composition generated by distinct functions from this collection, has infinitely many domains contained in wandering (respectively, preperiodic) components of the Fatou set. Thus, Theorems 5.2 and 5.3 were generalized to the following assertions.

Theorem 5.9 ([33, Lemma: 7.2.1]). *There exist three different transcendental entire functions f, g and h, and infinitely many domains which lie in different wandering components of $F(f), F(g)$, $F(h), F(f \circ g), F(g \circ f), F(f \circ h), F(g \circ h), F(h \circ f), F(h \circ g), F(f \circ g \circ h), F(f \circ h \circ g), F(g \circ f \circ h), F(g \circ h \circ f), F(h \circ f \circ g)$ and $F(h \circ g \circ f)$.*

Theorem 5.10 ([33, Lemmas: 7.2.2 & 7.2.3]). *There exist three different transcendental entire functions f, g and h, and infinitely many domains which lie in different pre-periodic (or periodic) components of $F(f), F(g)$, $F(h), F(f \circ g), F(g \circ f), F(f \circ h), F(g \circ h), F(h \circ f), F(h \circ g), F(f \circ g \circ h), F(f \circ h \circ g), F(g \circ f \circ h), F(g \circ h \circ f), F(h \circ f \circ g)$ and $F(h \circ g \circ f)$.*

The following result may be regarded as a generalization of Theorems 5.9 and 5.10.

Proposition 5.1. For any $n \in \mathbb{N}$, there exist n different transcendental entire functions and infinitely many domains which lie in different wandering (or pre-periodic or periodic) components of individual functions and their different compositions.

We were unable to establish this result during our Ph.D. research, and consequently it remained an open problem. In the present work, we make an attempt to resolve this problem and prove the asserted statement.

It is worth noting that approximation theory in complex analysis is concerned with the extent to which analytic or continuous functions defined on a given set can be approximated by analytic functions defined on a larger domain. One of the fundamental results in this area is Runge's theorem (1885), which states that every function analytic on a neighborhood of a compact set K with connected complement can be uniformly approximated on K by polynomials.

This theory was subsequently extended by Mergelyan and Arakelyan, whose approximation theorems permit approximation on certain unbounded sets, provided that these sets satisfy suitable regularity conditions at infinity. Such approximation results have proved to be powerful tools for constructing analytic functions with prescribed properties through carefully designed approximations. For a comprehensive overview of developments in this area, we refer the reader to the recent survey [21].

In the setting of complex dynamical systems, the central question is whether certain prescribed model dynamics can be realized by entire functions. Owing to the iterative nature of the problem, one is often required to work with infinitely many sets arising as iterated images of a given set. Consequently, the approximation errors must decrease appropriately as the moduli of the points tend to infinity.

This requirement exposes a limitation of Runge's theorem, which allows approximation only on a single region, and also highlights certain restrictions of Arakelyan's theorem in dynamical applications. To overcome these difficulties, Eremenko and Lyubich developed an extension of Runge's approximation theorem that provides additional control over the approximating functions, making it particularly suitable for applications in complex dynamics.

Theorem 5.11 ([13, Main Lemma p. 460]). *Let (G_k) be a sequence of compact subsets of $\mathbb{C}$ with the following properties:*

1. $\mathbb{C} \setminus G_k$ *is connected, for every k;*
2. $G_k \cap G_m = \emptyset$*, for $k \neq m$;*
3. $\min\{|z| : z \in G_k\} \to \infty$ *as $k \to \infty$.*

Let $z_k \in G_k$, $\epsilon_k > 0$ and ψ holomorphic on $G = \cup_{k=1}^{\infty} G_k$. Then, there exists an entire function f satisfying $|f(z) - \psi(z)| < \epsilon_k$, for $z \in G_k$; $f(z_k) = \psi(z_k)$ and $f'(z_k) = \psi'(z_k)\ \forall\ k \in \mathbb{N}$.

Let us note that this theorem guarantees the existence of an entire function with the desired dynamics, it provides no information on the function outside the region where the model map was defined. In particular, it neither describes how the function behaves globally, nor provides any information about the number or location of its singular values, crucial in determining the global dynamics of the map.

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