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Research Article

Revan Degree-Based Topological Indices of Some Generalized Thorn Cog Graphs

K. M. Saranya[1] and S. Manimekalai[1*]

[1]Department of Mathematics, Dr. N.G.P. Arts and Science College, Coimbatore, Tamil Nadu, India.
manimekalai@drngpasc.ac.in

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Abstract

Topological descriptors based on Revan vertex degrees provide an effective framework for analyzing the structural characteristics of complex graphs. This work investigates Revan degree-based topological indices for generalized thorn cog graphs, including the first and second Revan indices, the Atom-Bond Connectivity Revan index, and the Geometric-Arithmetic Revan index. Closed-form expressions are derived for thorn cog complete, wheel graph, and star graphs, along with their special cases, using an edge-partitioning approach based on Revan degrees. The results offer a systematic method for evaluating these indices and demonstrate their relevance in modeling physicochemical and biological properties of chemical compounds.

Keywords: Revan Index; Thorn Cog Graphs; Degree-based descriptors.
AMS(MOS) Subject Classification: 05C05,05C12,05C35.

1 Introduction

Topological indices are numerical measures that capture the structural characteristics of a graph based on the arrangement and connectivity of its vertices [4, 10]. Over the years, numerous indices have been proposed by researchers worldwide, each designed with different computational approaches and specific applications in mind. These descriptors play a significant role in QSPR and QSAR studies and have been widely applied in areas such as environmental analysis, pharmaceutical research, drug design, and materials science. In addition, ongoing investigations continue to expand their applicability to new scientific domains. These descriptors play a significant role in QSPR and QSAR studies, particularly in

*Corresponding author

predicting physicochemical properties such as boiling point, entropy, stability, and molecular branching of chemical compounds. They are also applied in pharmaceutical chemistry for molecular characterization, drug activity prediction, and the analysis of polycyclic aromatic hydrocarbons and nanostructures. Representing chemical compounds as graphs allows researchers to predict molecular properties without relying heavily on complex experimental procedures, thereby saving both time and cost.

A major breakthrough in this field was achieved by Harold Wiener through the introduction of the Wiener index, which laid the foundation for modern chemical graph theory [13, 7]. This distance-based index has been successfully used to relate molecular structure with physicochemical properties such as boiling and melting points of alkanes. Although many new indices have been developed since then, the Wiener index continues to serve as a standard reference for comparison. The Zagreb index, introduced by Gutman and Trinajstic [3], is a degree-based topological descriptor used to examine the relationship between molecular structure and total π-electron energy. Building on this early contribution, a wide variety of topological indices have been introduced for studying molecular stability, physicochemical behavior, and structure-property relationship of chemical compounds through QSPR modeling. This work focuses on Revan indices derived from the Revan vertex degree, as introduced by V. R. Kulli [6].

In recent years, the study of topological indices associated with thorn and cog-type graph structures has attracted considerable attention due to their applicability in mathematical chemistry. Several researches have focused on analyzing degree-based and distance-based indices for such graphs. Rasool et al. [9] studied M_{ve}-polynomials for cog-related graph structure. Pawar and Soner [8], Parvathi N et al. [12], Muhammad et al. [5]and Azari [1] also worked on different topological indices of thorn and related graph structures. Their work shows the importance of studying topological indices in graphs with pendant vertices. Jagadeesh R et al. [2] and Parvathi N et al. [11] studied thorn cog graphs where each cog vertex is connected to exactly two pendant (thorn) vertices. They analyzed different degree-based indices for this fixed structure. In this paper, we consider a more general case where each cog vertex is connected to l pendant vertices. Using this idea, we study thorn cog graphs based on complete, wheel, and star graphs. We use vertex classification and edge partition methods to find formulas for revan degree-based indices such as the first and second Revan indices, Atom-Bond Connectivity Revan index, and Geometric-Arithmetic Revan index.

2 Preliminaries

Let $G = (V(G), E(G))$ be a connected graph with vertex set $V(G)$ and edge set $E(G)$. The degree of a vertex x is denoted by $d_G(x)$ is the number of vertices adjacent at x. The maximum and minimum degree of a graph G is written as $\Delta(G)$ and $\delta(G)$, respectively. For each vertex $x \in V(G)$, the revan vertex degree is defined by

$$r_G(x) = \Delta(G) + \delta(G) - d_G(x).$$

The edge xy denotes the revan edge connecting the revan vertices x and y. The revan indices are defined as follows:
The first Revan index of G is expressed as

$$R_1(G) = \sum_{xy \in E(G)} \left[r_G(x) + r_G(y) \right].$$

The second Revan index is given by

$$R_2(G) = \sum_{xy \in E(G)} \left[r_G(x) r_G(y) \right].$$

The Atom Bond Connectivity Revan index is defined as

$$ABCR(G) = \sum_{xy \in E(G)} \sqrt{\frac{r_G(x) + r_G(y) - 2}{r_G(x) r_G(y)}}.$$

The Geometric-Arithmetic Revan index is written as

$$GAR(G) = \sum_{xy \in E(G)} \frac{2\sqrt{r_G(x) r_G(y)}}{r_G(x) + r_G(y)}.$$

3 Main Results

In this section, we compute the revan degree-based indices of generalized thorn cog graphs. Specifically, we consider three graphs: complete, wheel, and star graphs. Throughout this section, we use n, a, and l to denote the number of central vertices, cog vertices, and pendant vertices, respectively.

3.1 Thorn Cog Complete Graph

Let K_n be a complete graph with n vertices ($n \geq 3$). To each vertex of K_n, two distinct cog vertices are attached, and each cog vertex is further connected to l pendant vertices. The resulting graph is called the thorn cog complete graph, denoted by $K^{c^*}_{n,a,l}$, where $a \geq 3$. Let the central vertex set be $V = \{v_1, v_2, ..., v_n\}$ for $i = 1, 2, ..., n$, inducing a complete graph K_n. Let $U = \{u_1, u_2, ..., u_a\}$ denote the set of cog vertices, where each u_j is adjacent to two central vertices v_i and v_{i+1} (cyclically). Further, each cog vertex u_j supports a set of l pendant vertices $W_{jk} = \{w_{j1}, w_{j2}, ..., w_{jl}\}$. Hence, the vertex set of G is $V(G) = \{v_i : 1 \leq i \leq n\} \cup \{u_j : 1 \leq j \leq a\} \cup \{w_{jk} : 1 \leq j \leq a, 1 \leq k \leq l\}$, which yields $|V(G)| = n+$

$a(1+l)$. The edge set $E(G)$ is partitioned into three mutually disjoint classes: $E_1 = \{v_i v_{i+1} : 1 \le i \le n\}$ with $\frac{n(n-1)}{2}$ edges, $E_2 = \{u_j v_i, u_j v_{i+1} : 1 \le j \le a\}$ with $2a$ edges, and $E_3 = \{u_j w_{jk} : 1 \le j \le a, 1 \le k \le l\}$ with al edges. Thus, the total number of edges is $|E(G)| = \frac{n(n-1)}{2} + 2a + al$. From the construction of G, the degrees of vertices are given by $d_G(v_i) = n+1$, $d_G(u_j) = l+2$, and $d_G(w_{jk}) = 1$. Here, the maximum degree of G is $\Delta(G) = max\,\{n+1, l+2\}$ and the minimum degree $\delta(G) = 1$.

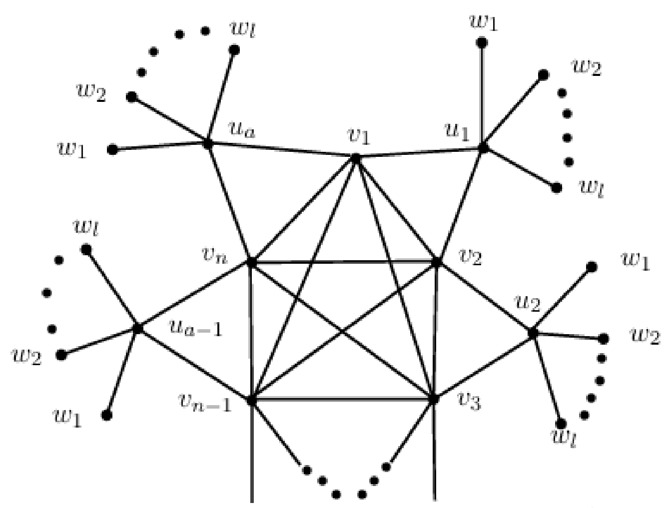

Figure 1: Thorn Cog Complete Graph $(K^{c^*}_{n,a,l})$

Theorem 3.1. *Let $G = K^{c^*}_{n,a,l}$ be the thorn cog complete graph with $n, a \ge 3$, if $\Delta(G) = n+1$ and $\delta(G) = 1$. Then, the Revan degree-based indices of G are obtained as follows:*

$$\begin{aligned}
R_1(G) &= n(n-1) + 2a\,(n-l+1) + al(2n-l+1)\\
R_2(G) &= \frac{n(n-1)}{2} + 2a\,(n-l) + al(n^2+n-nl-l)\\
ABCR(G) &= 2a\sqrt{\frac{n-l-1}{n-l}} + al\sqrt{\frac{2n-l-1}{n^2+n-nl-l}}\\
GAR(G) &= \frac{n(n-1)}{2} + \frac{4a}{n-l+1}\left(\sqrt{n-l}\right) + \frac{2al}{2n-l+1}\left(\sqrt{n^2+n-nl-l}\right)
\end{aligned}$$

Proof. As illustrated in Figure 1, for $n, a \ge 3$, given that $\Delta(G) = n+1$ and $\delta(G) = 1$, the Revan degree of a vertex $u \in V(G)$ is defined as: $r_G(u) = n+2-d_G(u)$. Thus, $r(v_i) = 1$, $r(u_j) = n-l$, $r(w_{jk}) = n+1$.

Substituting the revan degrees into the definitions, we obtain

$$
\begin{aligned}
R_1(G) &= \sum_{v_iv_{i+1}\in E_1}[r(v_i)+r(v_{i+1})] + \sum_{v_iu_j\in E_2}[r(v_i)+r(u_j)] + \sum_{u_jw_{jk}\in E_3}[r(u_j)+r(w_{jk})]\\
R_1(G) &= \frac{n(n-1)}{2}(1+1) + 2a(1+n-l) + al(n-l+n+1)\\
&= n(n-1) + 2a\,(n-l+1) + al(2n-l+1)\\
R_2(G) &= \frac{n(n-1)}{2}(1) + 2a((1).(n-l)) + al((n-l)(n+1))\\
&= \frac{n(n-1)}{2} + 2a\,(n-l) + al(n^2+n-nl-l)\\
ABCR(G) &= \frac{n(n-1)}{2}\sqrt{\frac{1+1-2}{1}} + 2a\sqrt{\frac{1+n-l-2}{n-l}} + al\sqrt{\frac{n-l+n+1-2}{(n-l)(n+1)}}\\
&= 2a\sqrt{\frac{n-l-1}{n-l}} + al\sqrt{\frac{2n-l-1}{n^2+n-nl-l}}\\
GAR(G) &= \frac{n(n-1)}{2}\left(\frac{2\sqrt{1}}{1+1}\right) + 2a\left(\frac{2\sqrt{n-l}}{1+n-l}\right) + al\left(\frac{2\sqrt{(n-l)(n+1)}}{n-l+n+1}\right)\\
&= \frac{n(n-1)}{2} + \frac{4a}{n-l+1}(\sqrt{n-l}) + \frac{2al}{2n-l+1}(\sqrt{n^2+n-nl-l})
\end{aligned}
$$

□

Theorem 3.2. *For the thorn cog complete graph $G = K^{c^*}_{n,a,l}$ with $n, a \geq 3$, if $\Delta(G) = l+2$ and $\delta(G) = 1$. Then, the Revan degree-based indices of G are obtained as follows:*

$$
\begin{aligned}
R_1(G) &= n(n-1)(l-n+2) + 2a\,(l-n+3) + al(l+3)\\
R_2(G) &= \frac{n(n-1)}{2}(l-n+2)^2 + 4al - 2na + 4a + al^2\\
ABCR(G) &= \frac{n(n-1)}{2}\frac{\sqrt{2(l-n+1)}}{l-n+2} + 2a\sqrt{\frac{l-n+1}{l-n+2}} + al\sqrt{\frac{l+1}{l+2}}\\
GAR(G) &= \frac{n(n-1)}{2} + \frac{4a}{l-n+3}\left(\sqrt{l-n+2}\right) + \frac{2al}{l+3}\left(\sqrt{l+2}\right)
\end{aligned}
$$

Proof. As illustrated in Figure 1, for $n, a \geq 3$, given that $\Delta(G) = l+2$ and $\delta(G) = 1$, the Revan degree of a vertex $u \in V(G)$ is defined as: $r_G(u) = l+3-d_G(u)$. Thus, $r(v_i) = l-n+2$, $r(u_j) = 1$, $r(w_{jk}) = l+2$.
Substituting the revan degrees into the definitions, we obtain

$$
\begin{aligned}
R_1(G) &= \sum_{v_iv_{i+1}\in E_1}[r(v_i)+r(v_{i+1})] + \sum_{v_iu_j\in E_2}[r(v_i)+r(u_j)] + \sum_{u_jw_{jk}\in E_3}[r(u_j)+r(w_{jk})]\\
R_1(G) &= \frac{n(n-1)}{2}((l-n+2)+(l-n+2)) + 2a(l-n+2+1) + al(1+l+2)\\
&= n(n-1)(l-n+2) + 2a\,(l-n+3) + al(l+3)
\end{aligned}
$$

$$R_2(G) = \frac{n(n-1)}{2}((l-n+2).(l-n+2)) + 2a((1).(l-n+2)) + al((1).(l+2))$$
$$= \frac{n(n-1)}{2}(l-n+2)^2 + 4al - 2na + 4a + al^2$$
$$ABCR(G) = \frac{n(n-1)}{2}\sqrt{\frac{l-n+2+l-n+2-2}{(l-n+2)^2}} + 2a\sqrt{\frac{l-n+2+1-2}{l-n+2}} + al\sqrt{\frac{1+l+2-2}{l+2}}$$
$$= \frac{n(n-1)}{2}\frac{\sqrt{2(l-n+1)}}{l-n+2} + 2a\sqrt{\frac{l-n+1}{l-n+2}} + al\sqrt{\frac{l+1}{l+2}}$$
$$GAR(G) = \frac{n(n-1)}{2}\left(\frac{2\sqrt{(l-n+2).(l-n+2)}}{l-n+2+l-n+2}\right) + 2a\left(\frac{2\sqrt{l-n+2}}{l-n+2+1}\right) + al\left(\frac{2\sqrt{l+2}}{l+2+1}\right)$$
$$= \frac{n(n-1)}{2} + \frac{4a}{l-n+3}(\sqrt{l-n+2}) + \frac{2al}{l+3}(\sqrt{l+2})$$

□

Table 1: Numerical values of Revan-type topological indices for $K^{c^*}_{n,a,l}$ graphs.

$K^{c*}_{n,a,l}$	$\Delta(G)$	$\delta(G)$	$R_1(G)$	$R_2(G)$	$ABCR(G)$	$GAR(G)$
$K^{c^*}_{3,3,2}$	5	1	48	33	5.1962	13.8
$K^{c^*}_{3,3,3}$	5	1	84	69	14.4138	15.3651
$K^{c^*}_{4,4,2}$	6	1	92	102	11.3137	20.7705
$K^{c^*}_{4,4,3}$	6	1	100	74	10.7331	22.9443
$K^{c^*}_{5,5,2}$	7	1	150	220	14.4011	28.0883
$K^{c^*}_{5,5,3}$	7	1	170	210	17.6777	32.4185
$K^{c^*}_{6,6,2}$	8	1	222	399	17.1957	36.1451
$K^{c^*}_{6,6,3}$	8	1	258	429	20.9078	41.8896
$K^{c^*}_{7,7,2}$	9	1	308	651	19.8636	45.0571
$K^{c^*}_{7,7,3}$	9	1	364	749	23.8637	51.9990

3.2 Thorn Cog Wheel Graph

Let W_n be a wheel graph with n vertices ($n \geq 4$). To each vertex of W_n, two cog vertices are attached, and each cog vertex is further connected to l pendant vertices. The resulting graph is called the thorn cog wheel graph, denoted by $G = W^{c^*}_{n,a,l}$, where $n, a \geq 4$. Let v_1 denote the central vertex and $v_2, v_3, v_4, ..., v_n$ be the wheel vertices forming a cycle, where the central is adjacent to all wheel vertices. Let $U = \{u_1, u_2, u_3, ..., u_{a-1}\}$ represent the set of cog vertices, where each cog vertex u_j is adjacent to two consecutive wheel vertices v_i and v_{i+1} (cyclically). Furthermore, each cog vertex u_j supports l pendant vertices $W_{jk} = \{w_{j1}, w_{j2}, ..., w_{jl}\}$. Hence, the vertex set of G is given by $V(G) = \{v_1\} \cup \{v_i : 2 \leq i \leq n\} \cup \{u_j : 1 \leq j \leq a-1\} \cup \{w_{jk} : 1 \leq j \leq a-1, 1 \leq k \leq l\}$, which yields $|V(G)| = n+a-1+(a-1)l$. The edge set $E(G)$ is partitioned into four mutually disjoint classes: $E_1 = \{v_1v_i : 1 \leq i \leq n\}$ with $(n-1)$ edges, $E_2 = \{v_iv_{i+1} : 1 \leq i \leq n\}$ with $(n-1)$ edges, $E_3 = \{v_iu_j : 2 \leq i \leq n, 1 \leq j \leq a-1\}$ with $2(a-1)$ edges and $E_4 = \{u_jw_{jk} : 1 \leq j \leq a-1, 1 \leq k \leq l\}$ with $(a-1)l$ edges. Thus, the total number of edges is $|E(G)| = 2(n-1) + 2(a-1) + (a-1)l$. From the construction of G, the degrees of vertices are given by $d_G(v_1) = n-1$, $d_G(v_i) = 5$ $(2 \leq i \leq n)$, $d_G(u_j) = l+2$, and $d_G(w_{jk}) = 1$. Here, the maximum degree of G is $\Delta(G) = max\,\{n-1, 5, l+2\}$ and the minimum degree $\delta(G) = 1$.

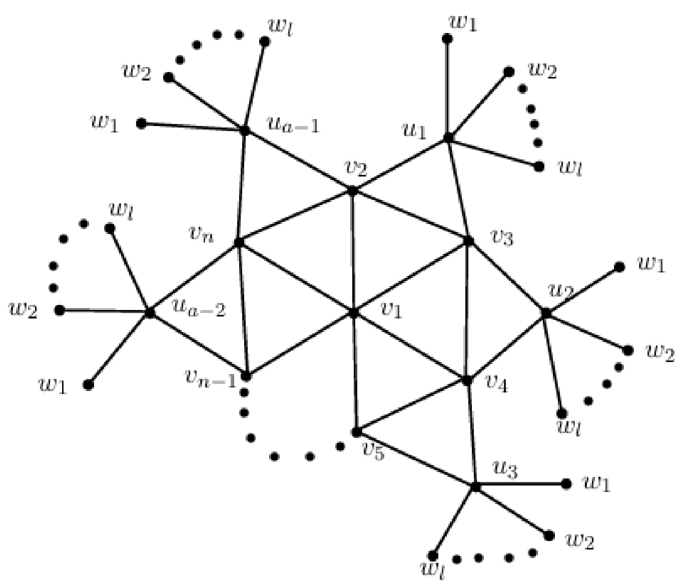

Figure 2: Thorn Cog Wheel Graph $(W^{c^*}_{n,a,l})$

Theorem 3.3. *For the thorn cog wheel graph $G = W^{c^*}_{n,a,l}$ with $n, a \geq 4$, if $\Delta(G) = n-1$ and $\delta(G) = 1$. Then, the Revan-based indices are given by:*

$$R_1(G) = (n-1)(3n-14) + (a-1)(4n-5l-l^2+2nl-14)$$

$$R_2(G) = n^2-6n+5+(n-1)(n-5)^2+2(a-1)(n^2-nl-7n+5l+10)+ l(a-1)(n^2-3n-nl+l+2)$$

$$ABCR(G) = (n-1)\left(\sqrt{\frac{n-6}{n-5}}\right) + (n-1)\left(\frac{\sqrt{2n-12}}{n-5}\right) + 2(a-1)\left(\sqrt{\frac{2n-l-9}{n^2-nl-7n+5l+10}}\right) + l(a-1)\left(\sqrt{\frac{2n-l-5}{n^2-nl+l-3n+2}}\right)$$

$$GAR(G) = \frac{2(n-1)}{n-4}\left(\sqrt{n-5}\right) + n-1+4(a-1)\left(\frac{\sqrt{n^2-nl-7n+5l+10}}{2n-l-7}\right) + 2l(a-1)\left(\frac{\sqrt{n^2-nl+l-3n+2}}{2n-l-3}\right)$$

Proof. As illustrated in Figure 2, for $n, a \geq 4$, given that $\Delta(G) = n-1$ and $\delta(G) = 1$, the Revan degree of a vertex $u \in V(G)$ is defined as: $r_G(u) = n - d_G(u)$. Thus, $r(v_1) = 1$, $r(v_i) = n-5$, $r(u_j) = n-l-2$, $r(w_{jk}) = n-1$.
Substituting the revan degrees into the definitions, we obtain

$$\begin{aligned}
R_1(G) &= \sum_{v_1v_i \in E_1}[r(v_1)+r(v_i)] + \sum_{v_iv_{i+1} \in E_2}[r(v_i)+r(v_{i+1})] + \sum_{v_iu_j \in E_3}[r(v_i)+r(u_j)] + \\
&\quad \sum_{u_jw_{jk} \in E_4}[r(u_j)+r(w_{jk})] \\
&= (n-1)(1+n-5) + (n-1)(n-5+n-5) + 2(a-1)(n-5+n-l-2) + \\
&\quad l(a-1)(n-l-2+n-1) \\
&= 3n^2 - 21n + 4na - 5al - al^2 + 2nal - 14a + 5l + l^2 + 28 \\
R_2(G) &= (n-1)\left(1.(n-5)\right) + (n-1)\left((n-5).(n-5)\right) + 2(a-1)\left((n-5).(n-l-2)\right) + \\
&\quad l(a-1)\left((n-l-2).(n-1)\right) \\
&= n^2 - 6n + 5 + (n-1)(n-5)^2 + 2(a-1)(n^2 - nl - 7n + 5l + 10) + \\
&\quad l(a-1)(n^2 - 3n - nl + l + 2) \\
ABCR(G) &= (n-1)\sqrt{\frac{1+n-5-2}{1.(n-5)}} + (n-1)\sqrt{\frac{n-5+n-5-2}{(n-5).(n-5)}} + 2(a-1)\sqrt{\frac{n-5+n-l-2-2}{(n-5).(n-l-2)}} + \\
&\quad l(a-1)\sqrt{\frac{n-l-2+n-1-2}{(n-l-2).(n-1)}} \\
&= (n-1)\left(\sqrt{\frac{n-6}{n-5}}\right) + (n-1)\left(\frac{\sqrt{2n-12}}{n-5}\right) + 2(a-1)\left(\sqrt{\frac{2n-l-9}{n^2-nl-7n+5l+10}}\right) + \\
&\quad l(a-1)\left(\sqrt{\frac{2n-l-5}{n^2-nl+l-3n+2}}\right) \\
GAR(G) &= (n-1)\frac{2\sqrt{1.(n-5)}}{1+n-5} + (n-1)\frac{2\sqrt{(n-5).(n-5)}}{n-5+n-5} + 2(a-1)\frac{2\sqrt{(n-5).(n-l-2)}}{n-5+n-l-2} + \\
&\quad l(a-1)\frac{2\sqrt{(n-l-2).(n-1)}}{n-l-2+n-1} \\
&= \frac{2(n-1)}{n-4}\left(\sqrt{n-5}\right) + n - 1 + 4(a-1)\left(\frac{\sqrt{n^2-nl-7n+5l+10}}{2n-l-7}\right) + \\
&\quad 2l(a-1)\left(\frac{\sqrt{n^2-nl+l-3n+2}}{2n-l-3}\right)
\end{aligned}$$

□

Theorem 3.4. *For the thorn cog wheel graph $G = W^{c^*}_{n,a,l}$ with $n, a \geq 4$, if $\Delta(G) = 5$ and*

$\delta(G) = 1$. *Then, the Revan-based indices are given by:*

$$R_1(G) = (n-1)(10-n) + (a-1)(10+7l-l^2)$$
$$R_2(G) = (n-1)(8-n) + (a-1)(8+18l-5l^2)$$
$$ABCR(G) = (n-1)\left(\sqrt{\frac{6-n}{7-n}}\right) + 2(a-1)\left(\sqrt{\frac{3-l}{4-l}}\right) + l(a-1)\left(\sqrt{\frac{7-l}{20-5l}}\right)$$
$$GAR(G) = \frac{2(n-1)}{8-n}\left(\sqrt{7-n}\right) + n - 1 + \frac{4(a-1)}{5-l}\left(\sqrt{4-l}\right) + \frac{2l(a-1)}{9-l}\left(\sqrt{20-5l}\right)$$

Proof. As illustrated in Figure 2, for $n, a \geq 4$, given that $\Delta(G) = 5$ and $\delta(G) = 1$, the Revan degree of a vertex $u \in V(G)$ is defined as: $r_G(u) = 6 - d_G(u)$. Thus, $r(v_1) = 7-n$, $r(v_i) = 1$, $r(u_j) = 4-l$, $r(w_{jk}) = 5$.
Substituting the revan degrees into the definitions, we obtain

$$\begin{aligned}
R_1(G) &= \sum_{v_1v_i\in E_1}[r(v_1)+r(v_i)] + \sum_{v_iv_{i+1}\in E_2}[r(v_i)+r(v_{i+1})] + \sum_{v_iu_j\in E_3}[r(v_i)+r(u_j)] + \\
&\quad \sum_{u_jw_{jk}\in E_4}[r(u_j)+r(w_{jk})] \\
&= (n-1)(7-n+1) + (n-1)(1+1) + 2(a-1)(1+4-l) + l(a-1)(4-l+5) \\
&= (n-1)(10-n) + (a-1)(10+7l-l^2) \\
R_2(G) &= (n-1)\left((7-n).1\right) + (n-1)\left(1.1\right) + 2(a-1)\left(1.(4-l)\right) + l(a-1)\left((4-l).5\right) \\
&= (n-1)(8-n) + (a-1)(8+18l-5l^2) \\
ABCR(G) &= (n-1)\sqrt{\frac{7-n+1-2}{1.(7-n)}} + (n-1)\sqrt{\frac{1+1-2}{1.1}} + 2(a-1)\sqrt{\frac{1+4-l-2}{1.(4-l)}} + \\
&\quad l(a-1)\sqrt{\frac{4-l+5-2}{(4-l).5}} \\
&= (n-1)\left(\sqrt{\frac{6-n}{7-n}}\right) + 2(a-1)\left(\sqrt{\frac{3-l}{4-l}}\right) + l(a-1)\left(\sqrt{\frac{7-l}{20-5l}}\right) \\
GAR(G) &= (n-1)\frac{2\sqrt{1.(7-n)}}{7-n+1} + (n-1)\frac{2\sqrt{1.1}}{1+1} + 2(a-1)\frac{2\sqrt{1.(4-l)}}{1+4-l} + l(a-1)\frac{2\sqrt{(4-l).5}}{4-l+5} \\
&= \frac{2(n-1)}{8-n}\left(\sqrt{7-n}\right) + n - 1 + \frac{4(a-1)}{5-l}\left(\sqrt{4-l}\right) + \frac{2l(a-1)}{9-l}\left(\sqrt{20-5l}\right)
\end{aligned}$$

□

Theorem 3.5. *For the thorn cog wheel graph* $G = W^{c^*}_{n,a,l}$ *with* $n, a \geq 4$, *if* $\Delta(G) = l+2$ *and* $\delta(G) = 1$. *Then, the revan-based indices are given by*

$$R_1(G) = (n-1)(2l-n+2) + (n-1)(2l-4) + 2(a-1)(l-1) + l(a-1)(l+3)$$
$$R_2(G) = (n-1)(l^2+2l-nl+2n-8) + (n-1)(l^2-4l+4) + 2(a-1)(l-2) + l(a-1)(l+2)$$

$$ABCR(G) = (n-1)\left(\sqrt{\frac{2l-n}{(l-n+4).(l-2)}}\right) + (n-1)\left(\frac{\sqrt{2l-6}}{l-2}\right) + 2(a-1)\left(\sqrt{\frac{l-3}{l-2}}\right) + \\ l(a-1)\left(\sqrt{\frac{l+1}{l+2}}\right)$$

$$GAR(G) = \frac{2(n-1)}{2l-n+2}\left(\sqrt{l^2+2l-nl+2n-8}\right) + n - 1 + 4(a-1)\left(\frac{\sqrt{l-2}}{l-1}\right) + \\ 2l(a-1)\left(\frac{\sqrt{l+2}}{l+3}\right)$$

Proof. As illustrated in Figure 2, for $n, a \geq 4$, given that $\Delta(G) = l+2$ and $\delta(G) = 1$, the Revan degree of a vertex $u \in V(G)$ is defined as: $r_G(u) = l+3-d_G(u)$. Thus, $r(v_1) = l-n+4$, $r(v_i) = l-2$, $r(u_j) = 1$, $r(w_{jk}) = l+2$.
Substituting the revan degrees into the definitions, we obtain

$$\begin{aligned}
R_1(G) &= \sum_{v_1v_i \in E_1}[r(v_1)+r(v_i)] + \sum_{v_iv_{i+1} \in E_2}[r(v_i)+r(v_{i+1})] + \sum_{v_iu_j \in E_3}[r(v_i)+r(u_j)] + \\
&\quad \sum_{u_jw_{jk} \in E_4}[r(u_j)+r(w_{jk})] \\
&= (n-1)(l-n+4+l-2) + (n-1)(l-2+l-2) + 2(a-1)(l-2+1) + \\
&\quad l(a-1)(1+l+2) \\
&= (n-1)(2l-n+2) + (n-1)(2l-4) + 2(a-1)(l-1) + l(a-1)(l+3) \\
R_2(G) &= (n-1)((l-n+4).(l-2)) + (n-1)((l-2).(l-2)) + 2(a-1)((l-2).(1)) + \\
&\quad l(a-1)((1).(l+2)) \\
&= (n-1)(l^2+2l-nl+2n-8) + (n-1)(l^2-4l+4) + 2(a-1)(l-2) + \\
&\quad l(a-1)(l+2) \\
ABCR(G) &= (n-1)\left(\sqrt{\frac{(l-n+4+l-2-2}{(l-n+4).(l-2)}}\right) + (n-1)\left(\sqrt{\frac{l-2+l-2-2}{(l-2).(l-2)}}\right) + \\
&\quad 2(a-1)\left(\sqrt{\frac{l-2+1-2}{l-2}}\right) + l(a-1)\left(\sqrt{\frac{l+2+1-2}{l+2}}\right) \\
&= (n-1)\left(\sqrt{\frac{2l-n}{(l-n+4).(l-2)}}\right) + (n-1)\left(\frac{\sqrt{2l-6}}{l-2}\right) + 2(a-1)\left(\sqrt{\frac{l-3}{l-2}}\right) + \\
&\quad l(a-1)\left(\sqrt{\frac{l+1}{l+2}}\right) \\
GAR(G) &= (n-1)\frac{2\sqrt{(l-n+4).(l-2)}}{l-n+4+l-2} + (n-1)\frac{2\sqrt{(l-2).(l-2)}}{l-2+l-2} + 2(a-1)\frac{2\sqrt{(l-2).(1)}}{l-2+1} + \\
&\quad l(a-1)\frac{2\sqrt{(1).(l+2)}}{1+l+2}
\end{aligned}$$

$$GAR(G) = \frac{2(n-1)}{2l-n+2}\left(\sqrt{l^2+2l-nl+2n-8}\right) + n - 1 + 4(a-1)\left(\frac{\sqrt{l-2}}{l-1}\right) + 2l(a-1)\left(\frac{\sqrt{l+2}}{l+3}\right)$$

□

Table 2: Numerical values of Revan-type topological indices for $W^{c^*}_{n,a,l}$ graphs.

$W^{c*}_{n,a,l}$	$\Delta(G)$	$\delta(G)$	$R_1(G)$	$R_2(G)$	$ABCR(G)$	$GAR(G)$
$W^{c^*}_{4,4,2}$	5	1	78	84	10.9348	16.676
$W^{c^*}_{4,4,3}$	5	1	84	63	10.4993	18.3063
$W^{c^*}_{5,5,2}$	5	1	100	108	14.1421	22.5418
$W^{c^*}_{5,5,3}$	5	1	108	80	13.5616	24.7155
$W^{c^*}_{6,6,2}$	5	1	120	130	14.1421	28.4632
$W^{c^*}_{6,6,3}$	5	1	130	95	13.4164	31.1803
$W^{c^*}_{7,7,2}$	6	1	210	324	24.4539	34.7281
$W^{c^*}_{7,7,3}$	6	1	234	300	29.6985	39.2453
$W^{c^*}_{8,8,2}$	7	1	322	644	27.3564	40.3879
$W^{c^*}_{8,8,3}$	7	1	364	651	32.677	46.309

3.3 Thorn Cog Star Graph

Let S_n be a star graph with n vertices ($n \geq 4$). To each vertex of S_n, two cog vertices are attached, and each cog vertices is further connected to l pendant vertices. The resulting graph is called the thorn cog star graph, denoted by $G = S^{c^*}_{n,a,l}$, where $n, a \geq 4$. Let v_1 be the central vertex and $v_2, v_3, v_4, \ldots.., v_n$ denote the star vertices. Let $U = \{u_1, u_2, u_3, \ldots.., u_{a-1}\}$ be the set of cog vertices, where each star vertex is adjacent to exactly two cog vertices, and each cog vertex is incident with two star vertices. Furthermore, every cog vertex u_j supports l pendant (thorn) vertices $W_{jk} = \{w_{j1}, w_{j2}, \ldots, w_{jl}\}$. Hence, the vertex set of G is given by $V(G) = \{v_1\} \cup \{v_i : 2 \leq i \leq n\} \cup \{u_j : 1 \leq j \leq a-1\} \cup \{w_{jk} : 1 \leq j \leq a-1, 1 \leq k \leq l\}$, which yields $|V(G)| = n + a - 1 + (a-1)l$. The edge set $E(G)$ is partitioned into three mutually disjoint classes: edges joining the central vertex to all star vertices, edges connecting each star vertex to two cog vertices, and pendant edges incident with cog vertices. Then,$E_1 = \{v_1v_i : 1 \leq i \leq n\}$ with $(n-1)$ edges, $E_2 = \{v_iu_j : 2 \leq i \leq n, 1 \leq j \leq a-1\}$ with $2(a-1)$ edges and $E_3 = \{u_jw_{jk} : 1 \leq j \leq a-1, 1 \leq k \leq l\}$ with $(a-1)l$ edges. Thus, the total number of edges is $|E(G)| = 1(n-1)+2(a-1)+(a-1)l$. From the construction of G, the degrees of vertices are given by $d_G(v_1) = n-1$, $d_G(v_i) = 3$ $(2 \leq i \leq n)$, $d_G(u_j) = l+2$, and $d_G(w_{jk}) = 1$. Here, the maximum degree of G is $\Delta(G) = max\,\{n-1, l+2\}$ and the minimum degree $\delta(G) = 1$.

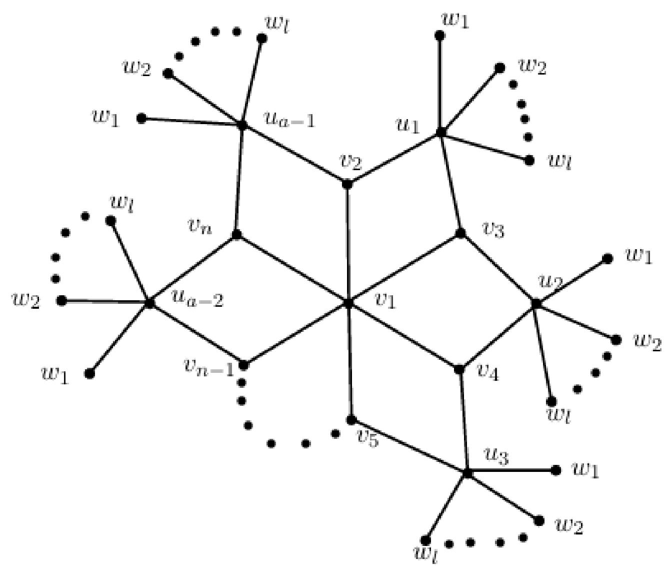

Figure 3: Thorn Cog Star Graph ($S^{c^*}_{n,a,l}$)

Theorem 3.6. *Let $G = S^{c^*}_{n,a,l}$ be a thorn cog star graph with $n, a \geq 4$, if $\Delta(G) = n-1$ and $\delta(G) = 1$. Then, the revan-based indices of G are given by*

$$R_1(G) = n^2 - 3n + 2 + 2(a-1)(2n-l-5) + l(a-1)(2n-l-3)$$

$$R_2(G) = n^2 - 4n + 3 + 2(a-1)(n^2 - 5n - nl + 3l + 6) + l(a-1)(n^2 - 3n - nl + l + 2)$$

$$ABCR(G) = (n-1)\left(\sqrt{\frac{n-4}{n-3}}\right) + 2(a-1)\left(\sqrt{\frac{2n-l-7}{n^2-nl-5n+3l+6}}\right) + l(a-1)\left(\sqrt{\frac{2n-l-5}{n^2-nl+l-3n+2}}\right)$$

$$GAR(G) = \frac{2(n-1)}{n-2}\left(\sqrt{n-3}\right) + 4(a-1)\left(\frac{\sqrt{n^2-nl-5n+3l+6}}{2n-l-5}\right) + 2l(a-1)\left(\frac{\sqrt{n^2-nl+l-3n+2}}{2n-l-3}\right)$$

Proof. As illustrated in Figure 3, for $n, a \geq 4$, given that $\Delta(G) = n-1$ and $\delta(G) = 1$, the Revan degree of a vertex $u \in V(G)$ is defined as: $r_G(u) = n - d_G(u)$. Thus, $r(v_1) = 1$, $r(v_i) = n-3$, $r(u_j) = n-l-2$, $r(w_{jk}) = n-1$.
Substituting the revan degrees into the definitions, we obtain

$$\begin{aligned}
R_1(G) &= \sum_{v_1v_i \in E_1} [r(v_1) + r(v_i)] + \sum_{v_iu_j \in E_2} [r(v_i) + r(u_j)] + \sum_{u_jw_{jk} \in E_3} [r(u_j) + r(w_{jk})] \\
&= (n-1)(1+n-3) + 2(a-1)(n-3+n-l-2) + l(a-1)(n-l-2+n-1) \\
&= n^2 - 3n + 2 + 2(a-1)(2n-l-5) + l(a-1)(2n-l-3)
\end{aligned}$$

$$
\begin{aligned}
R_2(G) &= (n-1)\left(1.(n-3)\right) + 2(a-1)\left((n-3).(n-l-2)\right) + l(a-1)\left((n-l-2).(n-1)\right)\\
&= n^2 - 4n + 3 + 2(a-1)(n^2 - 5n - nl + 3l + 6) + l(a-1)(n^2 - 3n - nl + l + 2)\\
ABCR(G) &= (n-1)\sqrt{\frac{1+n-3-2}{1.(n-3)}} + 2(a-1)\sqrt{\frac{n-3+n-l-2-2}{(n-3).(n-l-2)}} +\\
&\quad l(a-1)\sqrt{\frac{n-l-2+n-1-2}{(n-l-2).(n-1)}}\\
&= (n-1)\left(\sqrt{\frac{n-4}{n-3}}\right) + 2(a-1)\left(\sqrt{\frac{2n-l-7}{n^2-nl-5n+3l+6}}\right) +\\
&\quad l(a-1)\left(\sqrt{\frac{2n-l-5}{n^2-nl+l-3n+2}}\right)\\
GAR(G) &= (n-1)\frac{2\sqrt{1.(n-3)}}{1+n-3} + 2(a-1)\frac{2\sqrt{(n-3).(n-l-2)}}{n-3+n-l-2} +\\
&\quad l(a-1)\frac{2\sqrt{(n-l-2).(n-1)}}{n-l-2+n-1}\\
&= \frac{2(n-1)}{n-2}\left(\sqrt{n-3}\right) + 4(a-1)\left(\frac{\sqrt{n^2-nl-5n+3l+6}}{2n-l-5}\right) +\\
&\quad 2l(a-1)\left(\frac{\sqrt{n^2-nl+l-3n+2}}{2n-l-3}\right)
\end{aligned}
$$

□

Theorem 3.7. *For the thorn cog star graph $G = S^{c^*}_{n,a,l}$ with $n, a \geq 4$, if $\Delta(G) = l+2$ and $\delta(G) = 1$. Then, the revan-based indices are given by*

$$
\begin{aligned}
R_1(G) &= (n-1)(2l-n+4) + 2(a-1)(l+1) + l(a-1)(l+3)\\
R_2(G) &= (n-1)(l^2-nl+4l) + 2l(a-1) + l(a-1)(l+2)\\
ABCR(G) &= (n-1)\left(\sqrt{\frac{2l-n+2}{l^2-nl+4l}}\right) + 2(a-1)\left(\sqrt{\frac{l-1}{l}}\right) + l(a-1)\left(\sqrt{\frac{l+1}{l+2}}\right)\\
GAR(G) &= \frac{2(n-1)}{2l-n+4}\left(\sqrt{l^2-nl+4l}\right) + 4(a-1)\left(\frac{\sqrt{l}}{l+1}\right) + 2l(a-1)\left(\frac{\sqrt{l+2}}{l+3}\right)
\end{aligned}
$$

Proof. As illustrated in Figure 3, for $n, a \geq 4$, given that $\Delta(G) = l+2$ and $\delta(G) = 1$, the Revan degree of a vertex $u \in V(G)$ is defined as: $r_G(u) = l+3-d_G(u)$. Thus, $r(v_1) = l-n+4$, $r(v_i) = l$, $r(u_j) = 1$, $r(w_{jk}) = l+2$.
Substituting the revan degrees into the definitions, we obtain

$$
\begin{aligned}
R_1(G) &= \sum_{v_1v_i \in E_1}[r(v_1)+r(v_i)] + \sum_{v_iu_j \in E_2}[r(v_i)+r(u_j)] + \sum_{u_jw_{jk} \in E_3}[r(u_j)+r(w_{jk})]\\
&= (n-1)\left(l-n+4+l\right) + 2(a-1)\left(l+1\right) + l(a-1)\left(1+l+2\right)\\
&= (n-1)(2l-n+4) + 2(a-1)(l+1) + l(a-1)(l+3)
\end{aligned}
$$

$$
\begin{aligned}
R_2(G) &= (n-1)\left((l-n+4).(l)\right) + 2(a-1)\left((l).(1)\right) + l(a-1)\left((1).(l+2)\right) \\
&= (n-1)(l^2-nl+4l) + 2l(a-1) + l(a-1)(l+2) \\
ABCR(G) &= (n-1)\left(\sqrt{\frac{(l-n+4+l-2}{(l-n+4).(l)}}\right) + 2(a-1)\left(\sqrt{\frac{l+1-2}{l}}\right) + \\
&\quad l(a-1)\left(\sqrt{\frac{l+2+1-2}{l+2}}\right) \\
&= (n-1)\left(\sqrt{\frac{2l-n+2}{l^2-nl+4l}}\right) + 2(a-1)\left(\sqrt{\frac{l-1}{l}}\right) + l(a-1)\left(\sqrt{\frac{l+1}{l+2}}\right) \\
GAR(G) &= (n-1)\frac{2\sqrt{(l-n+4).(l)}}{l-n+4+l} + 2(a-1)\frac{2\sqrt{(l).(1)}}{l+1} + l(a-1)\frac{2\sqrt{(1).(l+2)}}{1+l+2} \\
&= \frac{2(n-1)}{2l-n+4}\left(\sqrt{l^2-nl+4l}\right) + 4(a-1)\left(\frac{\sqrt{l}}{l+1}\right) + 2l(a-1)\left(\frac{\sqrt{l+2}}{l+3}\right)
\end{aligned}
$$

□

Table 3: Numerical values of Revan-type topological indices for $S^{c^*}_{n,a,l}$ graphs.

$S^{c*}_{n,a,l}$	$\Delta(G)$	$\delta(G)$	$R_1(G)$	$R_2(G)$	$ABCR(G)$	$GAR(G)$
$S^{c^*}_{4,4,2}$	4	1	60	48	11.5601	13.4569
$S^{c^*}_{4,4,3}$	5	1	96	90	14.9488	14.9044
$S^{c^*}_{5,5,2}$	4	1	76	56	15.4135	17.7137
$S^{c^*}_{5,5,3}$	5	1	124	108	20.0935	19.7917
$S^{c^*}_{6,6,2}$	5	1	140	175	18.2246	23.1632
$S^{c^*}_{6,6,3}$	5	1	150	120	25.6639	24.1707
$S^{c^*}_{7,7,2}$	6	1	222	384	20.4254	27.9906
$S^{c^*}_{7,7,3}$	6	1	246	336	26.4094	31.7022
$S^{c^*}_{8,8,2}$	7	1	322	707	22.4808	32.6001
$S^{c^*}_{8,8,3}$	7	1	364	686	28.0768	38.0198

4 Conclusion

This study evaluates the first and second Revan indices, along with the atom-bond connectivity index and the geometric-arithmetic Revan index, for generalized forms of thorn cog complete, wheel, and star graphs, including their special cases. These graph-based descriptors are useful in QSPR investigations for predicting physicochemical properties, molecular branching behavior, and structural characteristics of chemical compounds. In particular, the obtained results may support the analysis of polycyclic aromatic hydrocarbon, nanostructures, and related molecular graph models used in mathematical chemistry.

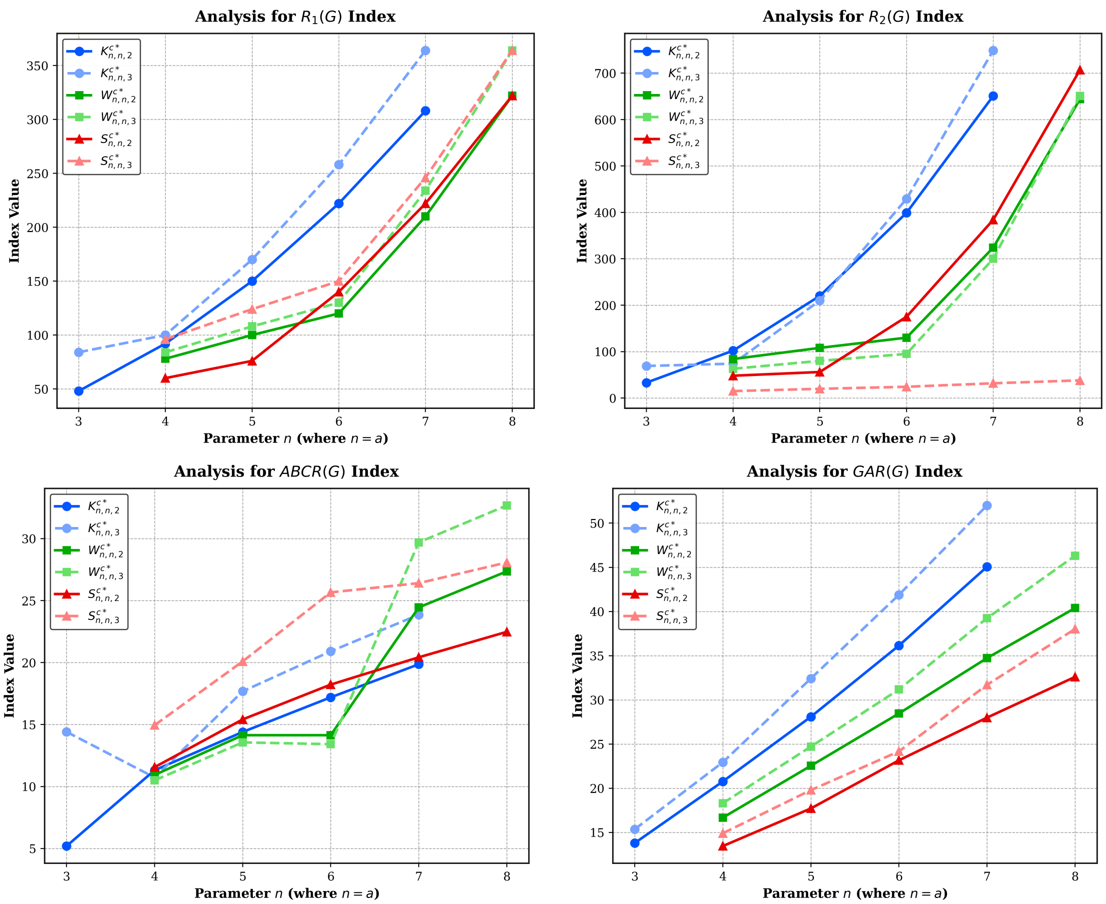

Figure 4: Comprehensive graphical trend analysis of Revan-type topological indices ($R_1(G)$, $R_2(G)$, $ABCR(G)$, and $GAR(G)$) for Thorn Cog-complete ($K^{c*}_{n,a,l}$), Thorn Cog wheel ($W^{c*}_{n,a,l}$), and Thorn Cog-star ($S^{c*}_{n,a,l}$) graphs under synchronized parameter variations ($n = a$) for lengths $l \in \{2, 3\}$.

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