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Research Article

On the Unweighted L^p Boundedness of a Weight-Adapted Dyadic Square Function

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Abstract

We study the boundedness of a weight-adapted dyadic square function S_w on the unweighted Lebesgue spaces $L^p(\mathbb{R})$. The operator is obtained from the classical dyadic square function by inserting the normalized local weight factor

$$\frac{w(x)}{\langle w\rangle_I},$$

where I ranges over dyadic intervals and $\langle w\rangle_I$ denotes the average of w over I. This factor is natural in dyadic weighted harmonic analysis and is closely related to the symbols appearing in t-Haar multipliers. We prove a sharp dichotomy between the ranges $1 < p \leq 2$ and $p > 2$. For $1 < p \leq 2$, the operator S_w is bounded on $L^p(\mathbb{R})$ for every locally integrable weight $w > 0$ almost everywhere. For $p > 2$, we show that S_w is bounded on $L^p(\mathbb{R})$ if and only if

$$w \in \bigcup_{s>p/2} \mathrm{RH}_s^{\mathcal{D}},$$

where $\mathrm{RH}_s^{\mathcal{D}}$ denotes the dyadic reverse Hölder class.

Keywords: Dyadic square function, Haar functions, reverse Hölder classes, dyadic maximal operator, t-Haar multipliers.

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1 Introduction and Preliminaries

Square functions are fundamental objects in harmonic analysis. In the dyadic setting, they measure the size of the Haar coefficients of a function across all dyadic scales and locations. They are closely connected with martingale transforms, Littlewood–Paley theory, dyadic singular integral models, and weighted inequalities [3, 6, 10].

Let $\mathcal{D}$ denote the collection of standard dyadic intervals in $\mathbb{R}$, that is,

$$\mathcal{D} = \{[k2^{-n}, (k+1)2^{-n}) : k, n \in \mathbb{Z}\}.$$

If $I \in \mathcal{D}$, we denote its left and right halves by I_- and I_+, respectively. The L^2-normalized Haar function supported on I is

$$h_I = |I|^{-1/2}(\mathbf{1}_{I_+} - \mathbf{1}_{I_-}).$$

The collection $\{h_I\}_{I\in\mathcal{D}}$ forms the standard dyadic Haar basis for $L^2(\mathbb{R})$.

For a locally integrable function f, the classical dyadic square function is defined by

$$Sf(x) = \left(\sum_{I\in\mathcal{D}} \frac{|\langle f, h_I\rangle|^2}{|I|}\mathbf{1}_I(x)\right)^{1/2}.$$

It is well known that S is bounded on $L^p(\mathbb{R})$ for $1 < p < \infty$ and that S is norm-preserving on $L^2(\mathbb{R})$.

In this paper, we study a weight-adapted version of the dyadic square function. Let w be a locally integrable function with $w > 0$ almost everywhere. For $I \in \mathcal{D}$, write

$$\langle w\rangle_I = \frac{1}{|I|}\int_I w(y)\, dy.$$

We define

$$S_w f(x) = \left(\sum_{I\in\mathcal{D}} \frac{w(x)}{\langle w\rangle_I}\, \frac{|\langle f, h_I\rangle|^2}{|I|}\mathbf{1}_I(x)\right)^{1/2}. \tag{1.1}$$

The sum in (1.1) may initially be interpreted as the monotone limit of finite partial sums. The factor

$$\frac{w(x)}{\langle w\rangle_I}$$

is a normalized local weight. It compares the pointwise size of w at x with the average size of w on the dyadic interval I. This normalization is important: for each fixed $I \in \mathcal{D}$,

$$\frac{1}{|I|}\int_I \frac{w(x)}{\langle w\rangle_I}\, dx = 1.$$

This exact cancellation is the reason why S_w is automatically norm-preserving on $L^2(\mathbb{R})$, regardless of any Muckenhoupt or reverse Hölder condition on w. The same ratio also appears naturally in the theory of t-Haar multipliers. A typical t-Haar multiplier has a dyadic symbol depending on both the spatial variable x and the dyadic interval I, of the form

$$\sigma_I \left(\frac{w(x)}{\langle w\rangle_I}\right)^t,$$

where σ_I is a choice of signs and $t \in \mathbb{R}$. Thus the ratio $w(x)/\langle w\rangle_I$ can be viewed as a dyadic space-frequency symbol associated with the weight w. The square function S_w studied here corresponds to the positive quadratic object obtained by inserting this normalized weight factor into the dyadic square function. This places S_w in the same general circle of ideas as weighted Haar multipliers and weighted dyadic square functions; see, for example, [2, 4, 5].

The purpose of this paper is to determine exactly when S_w is bounded on unweighted Lebesgue spaces $L^p(\mathbb{R})$. The answer exhibits a sharp dichotomy. For $1 < p \leq 2$, no structural condition on w is needed. For $p > 2$, boundedness is equivalent to dyadic reverse Hölder regularity.

We now recall the dyadic reverse Hölder classes.

Definition 1.1. Let $1 < s < \infty$. A weight w belongs to the dyadic reverse Hölder class $\mathrm{RH}_s^{\mathcal{D}}$ if

$$[w]_{\mathrm{RH}_s^{\mathcal{D}}} := \sup_{I\in\mathcal{D}} \frac{\langle w^s\rangle_I^{1/s}}{\langle w\rangle_I} < \infty.$$

Definition 1.2. For a locally integrable function ϕ, the dyadic Hardy–Littlewood maximal operator is defined by

$$M_{\mathcal{D}}\phi(x) := \sup_{\substack{I\in\mathcal{D}\\ x\in I}} \frac{1}{|I|}\int_I |\phi(y)|\,dy.$$

Remark 1.3. All reverse Hölder conditions in this paper are dyadic. Thus, $\mathrm{RH}_s^{\mathcal{D}}$ requires the reverse Hölder inequality only over intervals $I \in \mathcal{D}$, not over all intervals in $\mathbb{R}$.

Our main result is the following characterization.

Theorem 1.4 (Main theorem)**.** *Let $1 < p < \infty$, and let $w > 0$ be locally integrable. Then the following hold.*

1. *If $1 < p \leq 2$, then S_w is bounded on $L^p(\mathbb{R})$ for every such weight w.*
2. *If $p > 2$, then S_w is bounded on $L^p(\mathbb{R})$ if and only if*

$$w \in \bigcup_{s>p/2} \mathrm{RH}_s^{\mathcal{D}}.$$

The proof reveals the mechanism behind the dichotomy. In the range $1 < p \leq 2$, the dyadic cancellation of the Haar functions causes the bad part in the Calderón–Zygmund decomposition to vanish outside the exceptional set. Thus no structural condition on w is needed. For $p > 2$, testing on individual Haar functions forces a dyadic reverse Hölder condition, while the sufficiency argument uses duality and the boundedness of the dyadic maximal operator.

We first record the baseline L^2 identity.

Proposition 1.5. For every locally integrable weight $w > 0$ almost everywhere and every $f \in L^2(\mathbb{R})$,

$$\|S_w f\|_{L^2} = \|f\|_{L^2}.$$

Proof. Squaring (1.1) and integrating over $\mathbb{R}$, Tonelli's theorem applies because the summands are nonnegative. Hence

$$\begin{aligned}\|S_w f\|_{L^2}^2 &= \int_{\mathbb{R}} \sum_{I \in \mathcal{D}} \frac{w(x)}{\langle w \rangle_I} \frac{|\langle f, h_I \rangle|^2}{|I|} \mathbf{1}_I(x)\, dx \\ &= \sum_{I \in \mathcal{D}} \frac{|\langle f, h_I \rangle|^2}{|I| \langle w \rangle_I} \int_I w(x)\, dx.\end{aligned}$$

Since

$$\int_I w(x)\, dx = |I| \langle w \rangle_I,$$

the weight terms cancel exactly. Therefore

$$\|S_w f\|_{L^2}^2 = \sum_{I \in \mathcal{D}} |\langle f, h_I \rangle|^2 = \|f\|_{L^2}^2,$$

by Parseval's identity for the standard dyadic Haar basis of $L^2(\mathbb{R})$. □

2 The Range $1 < p \leq 2$: Unconditional Boundedness

In this section we prove the first part of Theorem 1.4. The key point is that, although the operator S_w contains the weight factor $w(x)/\langle w \rangle_I$, the dyadic cancellation of the Haar functions causes the bad part in the Calderón–Zygmund decomposition to vanish identically outside the exceptional set. This is why no structural condition on w is needed in the range $1 < p \leq 2$.
We shall use the following standard dyadic Calderón–Zygmund decomposition.

Lemma 2.1 (Dyadic Calderón–Zygmund decomposition)**.** *Let $f \in L^1(\mathbb{R})$ and let $\lambda > 0$. Then there exists a pairwise disjoint collection of maximal dyadic intervals $\{J_k\} \subset \mathcal{D}$ such that, with*

$$\Omega = \bigcup_k J_k,$$

the following properties hold:

1. $|f(x)| \leq \lambda$ *for almost every* $x \in \mathbb{R} \setminus \Omega$*;*
2. *for each* k*,*
$$\lambda < \frac{1}{|J_k|} \int_{J_k} |f(x)|\, dx \leq 2\lambda;$$
3. $$|\Omega| \leq \frac{1}{\lambda} \|f\|_{L^1};$$

4. *f admits a decomposition*

$$f = g + b, \qquad b = \sum_k b_k,$$

where

$$b_k = (f - \langle f \rangle_{J_k})\, \mathbf{1}_{J_k},$$

so that $\operatorname{supp} b_k \subset J_k$ *and*

$$\int_{J_k} b_k(x)\, dx = 0;$$

5. *the good function g satisfies*

$$\|g\|_{L^2}^2 \leq 2\lambda \|f\|_{L^1}.$$

Proof. Choose $\{J_k\}$ to be the maximal dyadic intervals satisfying

$$\frac{1}{|J_k|} \int_{J_k} |f(x)|\, dx > \lambda.$$

Maximality implies that the intervals J_k are pairwise disjoint. It also implies that $|f(x)| \leq \lambda$ for almost every $x \notin \Omega$, since otherwise x would belong to some dyadic interval on which the average of $|f|$ exceeds λ.

The lower bound in (2) follows from the definition of the intervals. For the upper bound, let $\widehat{J}_k$ denote the dyadic parent of J_k. Since J_k is maximal,

$$\frac{1}{|\widehat{J}_k|} \int_{\widehat{J}_k} |f(x)|\, dx \leq \lambda.$$

Because $|\widehat{J}_k| = 2|J_k|$, we get

$$\frac{1}{|J_k|} \int_{J_k} |f(x)|\, dx \leq \frac{1}{|J_k|} \int_{\widehat{J}_k} |f(x)|\, dx \leq 2\lambda.$$

Moreover,

$$|\Omega| = \sum_k |J_k| < \frac{1}{\lambda} \sum_k \int_{J_k} |f(x)|\, dx \leq \frac{1}{\lambda} \|f\|_{L^1}.$$

Define

$$g = f \mathbf{1}_{\mathbb{R}\setminus\Omega} + \sum_k \langle f \rangle_{J_k} \mathbf{1}_{J_k}$$

and

$$b = f - g = \sum_k (f - \langle f \rangle_{J_k})\, \mathbf{1}_{J_k}.$$

Then $b_k = (f - \langle f \rangle_{J_k})\, \mathbf{1}_{J_k}$ is supported on J_k and has mean zero. Finally, since $|g| \leq \lambda$ on $\mathbb{R} \setminus \Omega$ and

$$|\langle f \rangle_{J_k}| \leq \frac{1}{|J_k|} \int_{J_k} |f(x)|\, dx \leq 2\lambda$$

on each J_k, we have $|g| \leq 2\lambda$ almost everywhere. Also

$$\|g\|_{L^1} \leq \|f\|_{L^1}.$$

Moreover, since $|g| \leq 2\lambda$ almost everywhere, we have

$$\|g\|_{L^2}^2 \leq \|g\|_{L^\infty}\|g\|_{L^1} \leq 2\lambda\|f\|_{L^1}.$$

This completes the proof. □

Theorem 2.2. *For every $1 < p \leq 2$, there exists a constant $C_p > 0$ such that*

$$\|S_w f\|_{L^p} \leq C_p\|f\|_{L^p}$$

for all locally integrable weights $w > 0$ almost everywhere and all $f \in L^p(\mathbb{R})$.

Proof. The endpoint $p = 2$ follows from Proposition 1.5. We prove that S_w is of weak type $(1, 1)$, uniformly in the weight w. Since S_w is sublinear, the Marcinkiewicz interpolation theorem will then imply the desired L^p boundedness for $1 < p < 2$.
Fix $f \in L^1(\mathbb{R})$ and $\lambda > 0$. Apply Lemma 2.1 to obtain

$$f = g + b, \qquad b = \sum_k b_k,$$

with exceptional set

$$\Omega = \bigcup_k J_k.$$

By sublinearity,

$$\{x \in \mathbb{R} : S_w f(x) > \lambda\} \subset \{x \in \mathbb{R} : S_w g(x) > \lambda/2\} \cup \{x \in \mathbb{R} : S_w b(x) > \lambda/2\}.$$

We first estimate the good part. By Chebyshev's inequality and Proposition 1.5,

$$|\{x \in \mathbb{R} : S_w g(x) > \lambda/2\}| \leq \frac{4}{\lambda^2}\|S_w g\|_{L^2}^2 = \frac{4}{\lambda^2}\|g\|_{L^2}^2.$$

Using Lemma 2.1, we obtain

$$|\{x \in \mathbb{R} : S_w g(x) > \lambda/2\}| \leq \frac{8}{\lambda}\|f\|_{L^1}.$$

We now estimate the bad part. We claim that

$$S_w b(x) = 0 \qquad \text{for every } x \in \mathbb{R} \setminus \Omega.$$

Fix $x \in \mathbb{R} \setminus \Omega$ and let $I \in \mathcal{D}$ be such that $x \in I$. We shall show that

$$\langle b, h_I \rangle = 0.$$

Since I and J_k are dyadic intervals, if they intersect, then one is contained in the other. The alternative $I \subseteq J_k$ cannot occur, because $x \in I$ while $x \notin J_k$. Therefore, for each k, either $I \cap J_k = \varnothing$ or $J_k \subsetneq I$.

If $I \cap J_k = \varnothing$, then $\langle b_k, h_I \rangle = 0$ because b_k is supported on J_k. If $J_k \subsetneq I$, then J_k lies inside one of the two dyadic children of I. Since h_I is constant on J_k, say $h_I \equiv c_{I,k}$ on J_k, and b_k has mean zero, we have

$$\langle b_k, h_I \rangle = \int_{J_k} b_k(x) h_I(x)\, dx = c_{I,k} \int_{J_k} b_k(x)\, dx = 0.$$

Thus $\langle b_k, h_I \rangle = 0$ for every k, and consequently

$$\langle b, h_I \rangle = 0.$$

Since this holds for every dyadic interval I containing x, every term in the square function defining $S_w b(x)$ is zero. Therefore $S_w b(x) = 0$ for all $x \in \mathbb{R} \setminus \Omega$.

It follows that

$$|\{x \in \mathbb{R} : S_w b(x) > \lambda/2\}| \le |\Omega| \le \frac{1}{\lambda} \|f\|_{L^1}.$$

Combining the estimates for the good and bad parts gives

$$|\{x \in \mathbb{R} : S_w f(x) > \lambda\}| \le \frac{9}{\lambda} \|f\|_{L^1}.$$

Thus S_w is of weak type $(1,1)$, with a constant independent of w.

Interpolating this weak $(1,1)$ estimate with the strong L^2 estimate from Proposition 1.5 gives

$$\|S_w f\|_{L^p} \le C_p \|f\|_{L^p}$$

for every $1 < p < 2$. Together with the $p = 2$ case, this proves the theorem. □

3 The Range $p > 2$: Dyadic Reverse Hölder Regularity

We now turn to the range $p > 2$. In contrast with the case $1 < p \le 2$, the boundedness of S_w on $L^p(\mathbb{R})$ imposes a genuine regularity condition on the weight. The reason is that, for $p > 2$, the square function estimate is naturally dualized at the level of $(S_w f)^2$. This introduces averages of the form

$$\frac{1}{|I| \langle w \rangle_I} \int_I w(x) g(x)\, dx,$$

and controlling these averages requires reverse Hölder information on w.

We first prove that such a reverse Hölder condition is necessary by testing the operator on a single Haar function. Since the Haar functions are orthonormal, all coefficients vanish except one, and the boundedness of S_w directly forces a reverse Hölder inequality for w.

Lemma 3.1 (Necessity). *Let $p > 2$. Suppose there exists a constant $C > 0$ such that*

$$\|S_w f\|_{L^p} \le C\|f\|_{L^p}$$

for all $f \in L^p(\mathbb{R})$. Then $w \in \mathrm{RH}^{\mathcal{D}}_{p/2}$. More precisely, for every dyadic interval $I \in \mathcal{D}$,

$$\left(\frac{1}{|I|}\int_I w(x)^{p/2}\,dx\right)^{2/p} \le C^2\langle w\rangle_I.$$

Proof. Fix a dyadic interval $I \in \mathcal{D}$ and test the operator on the Haar function $f = h_I$. Since the Haar system is orthonormal,

$$\langle h_I, h_J\rangle = \begin{cases} 1, & J = I, \\ 0, & J \ne I, \end{cases} \qquad J \in \mathcal{D}.$$

Therefore, in the definition of $S_w h_I$, only the term corresponding to the interval I survives. Hence

$$S_w h_I(x)^2 = \frac{w(x)}{\langle w\rangle_I}\frac{1}{|I|}\mathbf{1}_I(x).$$

Taking the L^p norm gives

$$\begin{aligned}\|S_w h_I\|_{L^p}^p &= \int_I \left(\frac{w(x)}{\langle w\rangle_I |I|}\right)^{p/2} dx \\ &= \frac{1}{\langle w\rangle_I^{p/2}|I|^{p/2}}\int_I w(x)^{p/2}\,dx.\end{aligned}$$

On the other hand,

$$|h_I(x)| = |I|^{-1/2}\mathbf{1}_I(x),$$

and so

$$\|h_I\|_{L^p}^p = \int_I |I|^{-p/2}\,dx = |I|^{1-p/2}.$$

The assumed boundedness of S_w on $L^p(\mathbb{R})$ gives

$$\|S_w h_I\|_{L^p}^p \le C^p\|h_I\|_{L^p}^p.$$

Substituting the preceding computations yields

$$\frac{1}{\langle w\rangle_I^{p/2}|I|^{p/2}}\int_I w(x)^{p/2}\,dx \le C^p|I|^{1-p/2}.$$

Multiplying by $\langle w\rangle_I^{p/2}|I|^{p/2}$ and dividing by $|I|$, we obtain

$$\frac{1}{|I|}\int_I w(x)^{p/2}\,dx \le C^p\langle w\rangle_I^{p/2}.$$

Taking the power $2/p$ gives

$$\left(\frac{1}{|I|}\int_I w(x)^{p/2}\,dx\right)^{2/p} \le C^2\langle w\rangle_I.$$

Since this holds for every dyadic interval I, we have $w \in \mathrm{RH}^{\mathcal{D}}_{p/2}$. □

Lemma 3.1 gives the endpoint reverse Hölder condition $w \in \mathrm{RH}^{\mathcal{D}}_{p/2}$. In the sufficiency argument below, however, the boundedness of the dyadic maximal operator requires a strict inequality in the exponent. We therefore use the self-improvement property of reverse Hölder classes.

Lemma 3.2 (Dyadic Gehring self-improvement). *Let $q > 1$. If $w \in \mathrm{RH}^{\mathcal{D}}_q$, then there exists $\varepsilon > 0$, depending only on q and $[w]_{\mathrm{RH}^{\mathcal{D}}_q}$, such that*

$$w \in \mathrm{RH}^{\mathcal{D}}_{q+\varepsilon}\,.$$

Proof. This is the dyadic form of the reverse Hölder self-improvement theorem; see, for example, Anderson and Weirich [1]. Gehring's original theorem and standard weighted formulations can be found in [8, 7]. □

Combining Lemma 3.1 with Lemma 3.2, we see that if S_w is bounded on $L^p(\mathbb{R})$ for some $p > 2$, then

$$w \in \mathrm{RH}^{\mathcal{D}}_s$$

for some $s > p/2$. We now prove the converse.

Theorem 3.3 (Sufficiency). *Let $p > 2$. If $w \in \mathrm{RH}^{\mathcal{D}}_s$ for some $s > p/2$, then S_w is bounded on $L^p(\mathbb{R})$.*

Proof. Fix $p > 2$ and assume that $w \in \mathrm{RH}^{\mathcal{D}}_s$ for some $s > p/2$. We first prove the estimate for finite truncations of S_w. This avoids any a priori assumption that $S_w f$ is finite or belongs to $L^p(\mathbb{R})$. For $N \in \mathbb{N}$, define

$$\mathcal{D}_N = \{I \in \mathcal{D} : 2^{-N} \le |I| \le 2^N,\ I \cap [-N, N] \ne \varnothing\}.$$

The family $\mathcal{D}_N$ is finite. Let $S_{w,N}$ be the truncated square function

$$S_{w,N} f(x) = \left(\sum_{I \in \mathcal{D}_N} \frac{w(x)}{\langle w \rangle_I} \frac{|\langle f, h_I \rangle|^2}{|I|} \mathbf{1}_I(x) \right)^{1/2}.$$

Since the families $\mathcal{D}_N$ increase to $\mathcal{D}$ and all summands are nonnegative, the partial sums defining $S_{w,N} f(x)^2$ converge pointwise to $S_w f(x)^2$ as $N \to \infty$. It is enough to prove

$$\|S_{w,N} f\|^2_{L^p} \le C \|f\|^2_{L^p}$$

with a constant independent of N.
Set

$$a = \frac{p}{2}.$$

Since $p > 2$, we have $a > 1$. By duality in $L^a(\mathbb{R})$,

$$\|(S_{w,N} f)^2\|_{L^a} = \sup_{\substack{g \ge 0 \\ \|g\|_{L^{a'}} = 1}} \int_{\mathbb{R}} (S_{w,N} f(x))^2 g(x)\, dx.$$

Thus it suffices to estimate this integral uniformly over all nonnegative $g \in L^{a'}(\mathbb{R})$ with $\|g\|_{L^{a'}} = 1$. Fix such a function g. Since $S_{w,N}$ is defined by a finite sum, Tonelli's theorem gives

$$\begin{aligned}\int_{\mathbb{R}} (S_{w,N} f(x))^2 g(x)\,dx &= \int_{\mathbb{R}} \sum_{I\in\mathcal{D}_N} \frac{w(x)}{\langle w\rangle_I} \frac{|\langle f, h_I\rangle|^2}{|I|} \mathbf{1}_I(x) g(x)\,dx \\ &= \sum_{I\in\mathcal{D}_N} \frac{|\langle f, h_I\rangle|^2}{\langle w\rangle_I |I|} \int_I w(x) g(x)\,dx.\end{aligned}$$

We estimate the inner average. By Hölder's inequality with exponents s and s',

$$\int_I w(x) g(x)\,dx \le \left(\int_I w(x)^s\,dx\right)^{1/s} \left(\int_I g(x)^{s'}\,dx\right)^{1/s'}.$$

Dividing by $|I|\langle w\rangle_I$, this becomes

$$\begin{aligned}\frac{1}{|I|\langle w\rangle_I} \int_I w(x) g(x)\,dx &\le \frac{\langle w^s\rangle_I^{1/s}}{\langle w\rangle_I} \langle g^{s'}\rangle_I^{1/s'} \\ &\le [w]_{\mathrm{RH}_s^{\mathcal{D}}} \langle g^{s'}\rangle_I^{1/s'}.\end{aligned}$$

Since I is dyadic, for every $y \in I$ we have

$$\langle g^{s'}\rangle_I \le M_{\mathcal{D}}(g^{s'})(y).$$

Therefore

$$\langle g^{s'}\rangle_I^{1/s'} \le \inf_{y\in I} M_{\mathcal{D}}(g^{s'})(y)^{1/s'}.$$

Define

$$F(x) = M_{\mathcal{D}}(g^{s'})(x)^{1/s'}.$$

Then

$$\int_{\mathbb{R}} (S_{w,N} f(x))^2 g(x)\,dx \le C \sum_{I\in\mathcal{D}_N} |\langle f, h_I\rangle|^2 \inf_{y\in I} F(y).$$

For each interval I, we have

$$\inf_{y\in I} F(y) \le \frac{1}{|I|} \int_I F(x)\,dx.$$

Hence

$$\begin{aligned}\int_{\mathbb{R}} (S_{w,N} f(x))^2 g(x)\,dx &\le C \sum_{I\in\mathcal{D}_N} |\langle f, h_I\rangle|^2 \frac{1}{|I|} \int_I F(x)\,dx \\ &= C \int_{\mathbb{R}} \sum_{I\in\mathcal{D}_N} \frac{|\langle f, h_I\rangle|^2}{|I|} \mathbf{1}_I(x) F(x)\,dx \\ &\le C \int_{\mathbb{R}} Sf(x)^2 F(x)\,dx,\end{aligned}$$

where

$$Sf(x) = \left(\sum_{I \in \mathcal{D}} \frac{|\langle f, h_I \rangle|^2}{|I|} \mathbf{1}_I(x) \right)^{1/2}$$

is the standard dyadic square function.
Applying Hölder's inequality with exponents $a = p/2$ and $a' = (p/2)'$, we obtain

$$\int_{\mathbb{R}} Sf(x)^2 F(x)\, dx \le \|Sf\|_{L^p}^2 \|F\|_{L^{a'}}.$$

It remains to estimate $\|F\|_{L^{a'}}$. Since

$$F = M_{\mathcal{D}}(g^{s'})^{1/s'},$$

we have

$$\|F\|_{L^{a'}} = \|M_{\mathcal{D}}(g^{s'})\|_{L^{a'/s'}}^{1/s'}.$$

Let

$$r = \frac{a'}{s'}.$$

The dyadic maximal operator is bounded on $L^r(\mathbb{R})$ for $r > 1$. Since $s > a = p/2$, we have

$$s' < a'.$$

Thus $r = a'/s' > 1$, and the boundedness of $M_{\mathcal{D}}$ on $L^r(\mathbb{R})$ gives

$$\begin{aligned} \|F\|_{L^{a'}} &= \|M_{\mathcal{D}}(g^{s'})\|_{L^r}^{1/s'} \\ &\le C\|g^{s'}\|_{L^r}^{1/s'} \\ &= C\|g\|_{L^{rs'}}. \end{aligned}$$

Since $rs' = a'$, and $\|g\|_{L^{a'}} = 1$, we conclude that

$$\|F\|_{L^{a'}} \le C.$$

Consequently,

$$\int_{\mathbb{R}} (S_{w,N} f(x))^2 g(x)\, dx \le C\|Sf\|_{L^p}^2.$$

Using the boundedness of the standard dyadic square function on $L^p(\mathbb{R})$ for $1 < p < \infty$, we then have

$$\int_{\mathbb{R}} (S_{w,N} f(x))^2 g(x)\, dx \le C_p \|f\|_{L^p}^2,$$

where the constant is independent of N and of the choice of g. Taking the supremum over all nonnegative $g \in L^{a'}(\mathbb{R})$ with $\|g\|_{L^{a'}} = 1$, we obtain

$$\|(S_{w,N} f)^2\|_{L^a} \le C_p \|f\|_{L^p}^2.$$

Equivalently,

$$\|S_{w,N}f\|_{L^p}^2 \leq C_p\|f\|_{L^p}^2.$$

Finally, the nonnegative partial sums defining $S_{w,N}f(x)^2$ converge pointwise to $S_wf(x)^2$ as $N \to \infty$. Hence, by the monotone convergence theorem,

$$\|S_wf\|_{L^p}^p = \lim_{N\to\infty} \|S_{w,N}f\|_{L^p}^p \leq C_p^{p/2}\|f\|_{L^p}^p.$$

Taking the p-th root proves that S_w is bounded on $L^p(\mathbb{R})$. □

Corollary 3.4. *Let $p > 2$. Then S_w is bounded on $L^p(\mathbb{R})$ if and only if*

$$w \in \bigcup_{s>p/2} \mathrm{RH}_s^{\mathcal{D}}.$$

Proof. If S_w is bounded on $L^p(\mathbb{R})$, then Lemma 3.1 gives

$$w \in \mathrm{RH}_{p/2}^{\mathcal{D}}.$$

By the dyadic Gehring self-improvement lemma, Lemma 3.2, there exists $\varepsilon > 0$ such that

$$w \in \mathrm{RH}_{p/2+\varepsilon}^{\mathcal{D}}.$$

Thus

$$w \in \bigcup_{s>p/2} \mathrm{RH}_s^{\mathcal{D}}.$$

Conversely, if $w \in \mathrm{RH}_s^{\mathcal{D}}$ for some $s > p/2$, then Theorem 3.3 implies that S_w is bounded on $L^p(\mathbb{R})$. □

Together, Theorem 2.2 and Corollary 3.4 prove Theorem 1.4.

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