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Research Article

On Generalized Difference Sequences of Bicomplex Numbers and its Extension to n-Normed Related to $m(\phi)$

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Abstract

In this work, we study some of the properties related to difference sequences of bicomplex numbers. Besides of these, we also introduce a new class of n-normed generalized difference sequence over bicomplex numbers associated with ℓ_p space.This work extends classical results on generalized difference sequence spaces associated with $m(\phi)$ to the bicomplex framework.We introduce few topological and algebraic properties including completeness, convergence-free property, symmetricity and solidness.

Keywords: Bicomplex numbers, Generalized difference operator, n-Norm, n-Banach Space, Symmetricity.

AMS(MOS) Subject Classification: 40A05, 40A25, 40A30, 40C05, 46A45, 46B20.

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1 Introduction

The study of n-normed space was first initiated by Gähler [**?**] and further developed by Misiak [6], Gunawan [2]. Kizmaz [5] introduced difference sequence space $\ell_\infty(\Delta)$, $c(\Delta)$, $c_0(\Delta)$, Tripathy and Esi [16] generalized these to Δ_m and Tripathy, Altin, and Et [15] introduced Δ_m^n. Sargent [13] defined the space $m(\phi)$. Tripathy, Rana, and Borgohain [17] combined these ideas into an n-normed generalized difference sequence space $m(\phi, \Delta_p^q)$ over real scalars.
Recent studies by Jaiswal, *et al.* [3, 4, 7, 8] have significantly contributed to the theory of bicomplex numbers by extending the sequence of complex numbers. Kizmaz [5] investigated difference sequence spaces defined as follows:

$$S(\Delta) = \{y = (y_k) : (\Delta y_k) \in S\}$$

for $S = l_\infty, c$ and c_0, where $\Delta y = (y_k - y_{k+1})$, for all $k \in \mathbb{N}$, which is Banach space and norm is defined by $\|y\|_\Delta = \|y_1\| + \sup_k \|\Delta y_k\|$.
Tripathy and Esi [16] introduced a new type of difference sequence spaces, for fixed $m \in \mathbb{N}$, and $S(\Delta_m) = \{y = (y_k) : (\Delta_m y_k) \in S\}$, for $S = l_\infty, c$ and c_0, where $\Delta_m y = (\Delta_m y_k) = (y_k - y_{k+m}), \quad$ for all $k \in \mathbb{N}$.
The above spaces are Banach spaces and the norm is defined by

$$\|y\|_{\Delta_m} = \sum_{t=1}^{m} \|y_t\| + \sup_k \|\Delta_m y_k\|.$$

Tripathy, Altin, Et [15] generalized and introduced the following.
For $p, q \in \mathbb{N}$, $S(\Delta_p^q) = \{y = (y_k) : (\Delta_p^q y_k) \in S\}$, for $S = l_\infty, c$ and c_0. Which has a binomial representation

$$\Delta_p^q y_k = \sum_{t=0}^{q} (-1)^t \binom{q}{t} y_{k+tp}.$$

This extends naturally to bicomplex valued sequences. Initially, Sargent [13] introduced the sequence space $m(\phi)$. Later on it was studied from the sequence space point of view by Rath and Tripathy [10]. We extend the generalized difference sequence space associated with $m(\phi)$ into bicomplex setting.

The aim of this paper is not only replacing real or complex scalars into bicomplex scalars, but also showing several genuinely new bicomplex results based on idempotent decomposition.

2 Preliminaries

Definition 2.1. The classical complex numbers are defined as

$$\mathbb{C} = \left\{p + qi \ : \ p, q \in \mathbb{R} \text{ and } i^2 = -1\right\}.$$

Definition 2.2. (Segre, 1892) [14] The bicomplex numbers $\mathbb{BC}$ are defined by $z = a + bi + cj + dij = z_1 + z_2 j$, where $a, b, c, d \in \mathbb{R}$, $z_1, z_2 \in \mathbb{C}$, and $i^2 = -1, j^2 = -1, ij = ji = k, k^2 = 1$. The collection of bicomplex numbers $\mathbb{BC} = \{ z_1 + z_2 j : z_1, z_2 \in \mathbb{C} \}$.
Every bicomplex number admits the idempotent representation $z = z_1 e_1 + z_2 e_2$, where $e_1 = \frac{1+ij}{2}, e_2 = \frac{1-ij}{2}$, satisfying $e_1^2 = e_1, e_2^2 = e_2, e_1 e_2 = 0, e_1 + e_2 = 1$. The set $\mathbb{BC}$ is a commutative ring under the usual operations of addition and multiplication.
Moreover, for $z = z_1 + z_2 j$, the Euclidean norm on $\mathbb{BC}$ is

$$\|z\|_{\mathbb{BC}} = \sqrt{|z_1|^2 + |z_2|^2}.$$

Definition 2.3. (Gähler [1]) Let $n \in \mathbb{N}$ and let X be a vector space over the bicomplex field $\mathbb{BC}$. A bicomplex-valued mapping $\|\cdot, \ldots, \cdot\|_n : X^n \to \mathbb{BC}$ is called a *bicomplex n-norm* on X, if for all $w_1, \ldots, w_n, s, t \in X$ and $\alpha \in \mathbb{BC}$, the following conditions hold:

1. $\|(w_1, w_2, \ldots, w_n)\|_n = 0$ if and only if $w_1, w_2, \ldots, w_n$ are linearly dependent.
2. $\|(w_1, w_2, \ldots, w_n)\|_n$ is invariant under permutation.
3. $\|(w_1, \ldots, w_{n-1}, \alpha w_n)\|_n = |\alpha| \, \|(w_1, \ldots, w_{n-1}, w_n)\|_n$.
4. $\|(w_1, \ldots, w_{n-1}, s + t)\|_n \leq \|(w_1, \ldots, w_{n-1}, s)\|_n + \|(w_1, \ldots, w_{n-1}, t)\|_n$.

In this case, the pair $(X, \| \cdots \|_n)$ is known as an n-normed space over the bicomplex numbers.

Definition 2.4. Suppose (x_k) is a sequence defined in an n-normed space X over $\mathbb{BC}$. Then (x_k) is said to converge to an element $x \in X$, if
$\lim_{k\to\infty} \|(w_1, w_2, \ldots, w_{n-1}, x_k - x)\|_n = 0, \forall\, w_1, w_2, \ldots, w_{n-1} \in X$.
Further, (x_k) is called a Cauchy sequence if
$\lim_{j,k\to\infty} \|(w_1, w_2, \ldots, w_{n-1}, x_j - x_k)\|_n = 0, \forall\, w_1, w_2, \ldots, w_{n-1} \in X$. An n-normed space over $\mathbb{BC}$ is called *complete* if every Cauchy sequence converges to a point of the space. Such a complete space is called an *n-Banach space*.

Definition 2.5. An n-normed sequence space E defined over the bicomplex numbers $\mathbb{BC}$ is called *solid* if $(y_k) \in E$ whenever $(x_k) \in E$ and

$$\|(w_1, w_2, \ldots, w_{n-1}, y_k)\|_n \leq \|(w_1, w_2, \ldots, w_{n-1}, x_k)\|_n, \forall\, k \in \mathbb{N}.$$

Definition 2.6. Consider a sequence $x = (x_k)$ and let $S(x) = \{(x_{\pi(n)}) : \pi \text{ is a permutation of } \mathbb{N}\}$. A sequence space E over $\mathbb{BC}$ is called *symmetric* if every rearrangement of any sequence in E also belongs to E.

Definition 2.7. Let E be a sequence space over the bicomplex numbers $\mathbb{BC}$. Then the space E is called *convergence-free* if $(y_k) \in E$ whenever $(x_k) \in E$ and $y_k = 0$ whenever $x_k = 0$.

Definition 2.8. The sequence space E defined over $\mathbb{BC}$ is called monotone if E includes the canonical preimages corresponding to each of its step spaces.

Lemma 2.9. *Solidity of a sequence space E over $\mathbb{BC}$ implies monotonicity.*

In the past, different classes of sequences have been introduced and their different algebraic and topological properties have been investigated by Tripathy, Altin and Et [15] and many others. For bicomplex sequence spaces, Segre [14] recent contributions include works by Sager and Sağır [12].

Let P_t represent the family of each subset of $\mathbb{N}$ containing at most t elements. Let $\{\phi_t\}$ be a non-decreasing sequence of positive real numbers satisfying $(t+1)\phi_t \geq t\phi_{t+1}$,for each $t \in \mathbb{N}$. The sequence space $m(\phi)$ introduced by Sargent (1960) [13] is defined as

$m(\phi) = \left\{(x_k) \in X : \|x\|_{m(\phi)} = \sup_{\substack{t\geq 1\\ \alpha\in P_t}} \frac{1}{\phi_t} \sum_{k\in\alpha} \|x_k\| < \infty\right\}.$

The following work extended by Tripathy, Rana and Borgohain [17], to n-normed space is as follows:

$$\left(m(\phi, \Delta_p^q), \|\cdots\|_n\right)$$

$$= \left\{(x_k) \in X : \|x_k\|_{n,m(\phi,\Delta_p^q)} = \sup_{\substack{t\geq 1\\ \alpha\in P_t}} \frac{1}{\phi_t} \sum_{k\in\alpha} \left\|\left(w_1, w_2, \ldots, w_{n-1}, \Delta_p^q x_k\right)\right\|_n < \infty\right\}.$$

In the following, we extend the above n-normed space into bicomplex setting.

$$\left(m_{\mathbb{BC}}(\phi, \Delta_p^q), \|\cdots\|_n\right)$$

$$= \left\{(x_k) \in \omega(C_2) : \|x_k\|_{n,m(\phi,\Delta_p^q)} = \sup_{\substack{t\geq 1\\ \alpha\in P_t}} \frac{1}{\phi_t} \sum_{k\in\alpha} \left\|\left(w_1, w_2, \ldots, w_{n-1}, \Delta_p^q x_k\right)\right\|_n < \infty\right\}.$$

3 Some Decomposition Properties

In this section, we study some of the algebraic and structural properties related to the bicomplex sequence space $m_{\mathbb{BC}}(\phi, \Delta_p^q)$. We also establish the idempotent decomposition, norm decomposition, and a few geometric properties.

Theorem 3.1 (Idempotent Decomposition Property)**.** *Let $y_k = y_k^{(1)} e_1 + y_k^{(2)} e_2$, where e_1 and e_2 are the idempotent elements of the bicomplex field. Then*

$$(y_k) \in m_{\mathbb{BC}}(\phi, \Delta_p^q) \quad iff \quad (y_k^{(1)}), (y_k^{(2)}) \in m(\phi, \Delta_p^q).$$

Proof. Let $(y_k) \in m_{\mathbb{BC}}(\phi, \Delta_p^q)$. Every bicomplex number has the unique idempotent representation $y_k = y_k^{(1)} e_1 + y_k^{(2)} e_2$, where $e_1 = \frac{1+ij}{2}$ and $e_2 = \frac{1-ij}{2}$. Using

$$\Delta_p^q y_k = \sum_{t=0}^{q} (-1)^t \binom{q}{t} y_{k+tp},$$

we obtain,

$$\begin{aligned}\Delta_p^q y_k &= \sum_{t=0}^{q}(-1)^t\binom{q}{t}\left(y_{k+tp}^{(1)}e_1 + y_{k+tp}^{(2)}e_2\right)\\ &= \left(\sum_{t=0}^{q}(-1)^t\binom{q}{t}y_{k+tp}^{(1)}\right)e_1 + \left(\sum_{t=0}^{q}(-1)^t\binom{q}{t}y_{k+tp}^{(2)}\right)e_2\\ &= (\Delta_p^q y_k^{(1)})e_1 + (\Delta_p^q y_k^{(2)})e_2.\end{aligned}$$

Since $\|z\|_{\mathbb{BC}} = \max\{|z_1|, |z_2|\}$,

$$\|\Delta_p^q y_k\|_{\mathbb{BC}} = \max\{|\Delta_p^q y_k^{(1)}|, |\Delta_p^q y_k^{(2)}|\}.$$

As $(y_k) \in m_{\mathbb{BC}}(\phi, \Delta_p^q)$,

$$\sup_{\substack{t\geq 1\\ \alpha\in P_t}} \frac{1}{\phi_t}\sum_{k\in\alpha}\|\Delta_p^q y_k\|_{\mathbb{BC}} < \infty.$$

Hence

$$\sup_{\substack{t\geq 1\\ \alpha\in P_t}} \frac{1}{\phi_t}\sum_{k\in\alpha}\|\Delta_p^q y_k^{(i)}\|_{\mathbb{BC}} < \infty, \qquad i = 1, 2,$$

which implies,

$$(y_k^{(1)}), (y_k^{(2)}) \in m(\phi, \Delta_p^q).$$

Conversely, suppose

$$(y_k^{(1)}), (y_k^{(2)}) \in m(\phi, \Delta_p^q).$$

Then

$$\sup_{\substack{t\geq 1\\ \alpha\in P_t}} \frac{1}{\phi_t}\sum_{k\in\alpha}\|\Delta_p^q y_k^{(i)}\|_{\mathbb{BC}} < \infty, \qquad i = 1, 2.$$

Since

$$\|\Delta_p^q y_k\|_{\mathbb{BC}} = \max\{|\Delta_p^q y_k^{(1)}|, |\Delta_p^q y_k^{(2)}|\} \leq |\Delta_p^q y_k^{(1)}| + |\Delta_p^q y_k^{(2)}|,$$

we have

$$\frac{1}{\phi_t}\sum_{k\in\alpha}\|\Delta_p^q y_k\|_{\mathbb{BC}} \leq \frac{1}{\phi_t}\sum_{k\in\alpha}|\Delta_p^q y_k^{(1)}| + \frac{1}{\phi_t}\sum_{k\in\alpha}|\Delta_p^q y_k^{(2)}|.$$

Taking supremum on both sides,

$$\sup_{\substack{t\geq 1\\ \alpha\in P_t}} \frac{1}{\phi_t}\sum_{k\in\alpha}\|\Delta_p^q y_k\|_{\mathbb{BC}} < \infty.$$

Therefore,

$$(y_k) \in m_{\mathbb{BC}}(\phi, \Delta_p^q),$$

and hence

$$(y_k) \in m_{\mathbb{BC}}(\phi, \Delta_p^q) \iff (y_k^{(1)}), (y_k^{(2)}) \in m(\phi, \Delta_p^q).$$

Hence the proof. $\square$

Theorem 3.2 (Norm Decomposition Property)**.** *For every $y = y^{(1)}e_1 + y^{(2)}e_2$, we have*

$$\|y\|_{\mathbb{BC}} = \max\left\{\|y^{(1)}\|_m, \|y^{(2)}\|_m\right\}.$$

Proof. Let $y = y^{(1)}e_1 + y^{(2)}e_2$, where $y^{(1)} = (y_k^{(1)})$ and $y^{(2)} = (y_k^{(2)})$.

Using

$$\Delta_p^q y_k = \sum_{t=0}^{q}(-1)^t\binom{q}{t}y_{k+tp},$$

and substituting $y_k = y_k^{(1)}e_1 + y_k^{(2)}e_2$, we obtain

$$\begin{aligned}\Delta_p^q y_k &= \sum_{t=0}^{q}(-1)^t\binom{q}{t}\left(y_{k+tp}^{(1)}e_1 + y_{k+tp}^{(2)}e_2\right)\\ &= \left(\sum_{t=0}^{q}(-1)^t\binom{q}{t}y_{k+tp}^{(1)}\right)e_1 + \left(\sum_{t=0}^{q}(-1)^t\binom{q}{t}y_{k+tp}^{(2)}\right)e_2\\ &= (\Delta_p^q y_k^{(1)})e_1 + (\Delta_p^q y_k^{(2)})e_2.\end{aligned}$$

Since $\|z\|_{\mathbb{BC}} = \max\{|z_1|, |z_2|\}$,

$$\|\Delta_p^q y_k\|_{\mathbb{BC}} = \max\{|\Delta_p^q y_k^{(1)}|, |\Delta_p^q y_k^{(2)}|\}.$$

Using the definition of the norm,

$$\|y\|_{\mathbb{BC}} = \sup_{\substack{t\geq 1\\ \alpha\in P_t}}\frac{1}{\phi_t}\sum_{k\in\alpha}\max\left\{|\Delta_p^q y_k^{(1)}|, |\Delta_p^q y_k^{(2)}|\right\}.$$

Hence,

$$\|y\|_{\mathbb{BC}} \geq \|y^{(1)}\|_m, \qquad \|y\|_{\mathbb{BC}} \geq \|y^{(2)}\|_m,$$

which implies

$$\|y\|_{\mathbb{BC}} \geq \max\left\{\|y^{(1)}\|_m, \|y^{(2)}\|_m\right\}.$$

Using $\max\{a, b\} \leq a + b$, we obtain

$$\|\Delta_p^q y_k\|_{\mathbb{BC}} \leq |\Delta_p^q y_k^{(1)}| + |\Delta_p^q y_k^{(2)}|,$$

and therefore

$$\|y\|_{\mathbb{BC}} \leq \|y^{(1)}\|_m + \|y^{(2)}\|_m.$$

Hence,

$$\|y\|_{\mathbb{BC}} = \max\left\{\|y^{(1)}\|_m, \|y^{(2)}\|_m\right\}.$$

□

Example 3.3. The space $m_{\mathbb{BC}}(\phi, \Delta_p^q)$ is not solid.

Proof. On taking example we prove it. Consider $(x_k) = (k)$. For first difference,

$$\Delta x_k = x_k - x_{k+1} = k - (k+1) = -1.$$

Hence $(x_k) \in m_{\mathbb{BC}}(\phi, \Delta_p^q)$. Now consider the sequence (a_k) of scalars defined by

$$a_k = \begin{cases} 1, & \text{if } k \text{ is even,} \\ 0, & \text{if } k \text{ is odd.} \end{cases}$$

Then

$$a_k x_k = \begin{cases} 2k, & \text{if } k \text{ is even,} \\ 0, & \text{otherwise.} \end{cases}$$

So $(a_k x_k) = (0, 2, 0, 4, 0, 6, \ldots)$. Now $\Delta(a_k x_k) = (-2, 2, -4, 4, -6, 6, \ldots)$. Which is unbounded and hence $(a_k x_k) \notin m_{\mathbb{BC}}(\phi, \Delta)$. So the space is not solid. □

Example 3.4. The space $m_{\mathbb{BC}}(\phi, \Delta_p^q)$ is not symmetric.

Proof. We prove it by taking an example. Consider $(x_k) = (k)$. Then for first difference,

$$\Delta x_k = x_k - x_{k+1} = k - (k+1) = -1.$$

Hence $(x_k) \in m_{\mathbb{BC}}(\phi, \Delta_p^q)$. Now rearrange the sequence as

$$y = (1, 3, 2, 5, 4, 7, 6, \ldots).$$

Then the difference $\Delta y = (y_k - y_{k+1}) = (-2, 1, -3, 1, -4, 1, \ldots)$. Which is unbounded and hence $(y_k) \notin m_{\mathbb{BC}}(\phi, \Delta)$. So the space is not symmetric. □

Example 3.5. The space $m_{\mathbb{BC}}(\phi, \Delta_p^q)$ is not convergence free.

Proof. On taking example we prove it. Let $(x_k) = (k)$.

Then the difference $\Delta_4 x_k = -4$ and hence $(x_k) \in m_{\mathbb{BC}}(\phi, \Delta_p^q)$.

Define $y_k = k^2$. Then

$$\Delta_4 y_k = -8k - 16.$$

Which is unbounded and hence $(y_k) \notin m_{\mathbb{BC}}(\phi, \Delta_4)$. So the space is not convergence free. □

4 Main Results

In this section, we present some theorems related to the generalized difference sequence space related to $m(\phi)$ over bicomplex numbers in n-normed setting. This result extends few classical properties from real and complex sequence spaces to the bicomplex setting.

Theorem 4.1. *If X is a bicomplex n-Banach space, then the space*

$$\left(m_{\mathbb{BC}}(\phi, \Delta_p^q), \|\cdots\|_n\right)$$

is a bicomplex n-Banach space with respect to the norm defined by

$$\|x\|_{n,m(\phi,\Delta_p^q)} = \sum_{r=1}^{pq} \|(w_1, w_2, \ldots, w_{n-1}, x_r)\|_n + \sup_{\substack{t\geq 1\\ \alpha\in P_t}} \frac{1}{\phi_t} \sum_{k\in\alpha} \left\|(w_1, w_2, \ldots, w_{n-1}, \Delta_p^q x_k)\right\|_n,$$

for all $w_1, w_2, \ldots, w_{n-1} \in X$.

Proof. Suppose that $x^{(i)} = \left(x_k^{(i)}\right)_{i=1}^{\infty}$ be a Cauchy sequence in the space $(m_{\mathbb{BC}}(\phi, \Delta_p^q), \|\cdots\|_n)$. Then for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ so that, $\|x^i - x^j\|_{n,m(\phi,\Delta_p^q)} < \varepsilon$, for each $i, j \geq n_0$. Hence, for all $i, j \geq n_0$.

$$\sum_{r=1}^{pq} \left\|\left(w_1, w_2, \ldots, w_{n-1}, x_r^{(i)} - x_r^{(j)}\right)\right\|_n + \sup_{\substack{t\geq 1\\ \alpha\in P_t}} \frac{1}{\phi_t} \sum_{k\in\alpha} \left\|\left(w_1, w_2, \ldots, w_{n-1}, \Delta_p^q \left(x_k^{(i)} - x_k^{(j)}\right)\right)\right\|_n < \varepsilon.$$

Hence

$$\sum_{r=1}^{pq} \left\|\left(w_1, w_2, \ldots, w_{n-1}, x_r^{(i)} - x_r^{(j)}\right)\right\|_n < \varepsilon, \qquad \text{for all } i, j \geq n_0.$$

Therefore $\left\|\left(w_1, w_2, \ldots, w_{n-1}, x_r^{(i)} - x_r^{(j)}\right)\right\|_n < \varepsilon, \qquad \text{for all } i, j \geq n_0.$

i.e.,$\left(x_r^{(i)}\right)$ is a Cauchy sequence in X, and it is a bicomplex n-Banach space, hence it is convergent in X, for $i = 1, 2, 3, \ldots, pq$. Therefore, $\lim_{i\to\infty} x_r^{(i)} = x_r$. Also, for all $i, j \geq n_0$.

$$\sup_{\substack{t\geq 1\\ \alpha\in P_t}} \frac{1}{\phi_t} \sum_{k\in\alpha} \left\|\left(w_1, w_2, \ldots, w_{n-1}, \Delta_p^q \left(x_k^{(i)} - x_k^{(j)}\right)\right)\right\|_n < \varepsilon,$$

Let $t = 1$, we get

$$\sum_{k\in\alpha} \left\|\left(w_1, w_2, \ldots, w_{n-1}, \Delta_p^q \left(x_k^{(i)} - x_k^{(j)}\right)\right)\right\|_n < \varepsilon,$$

for all $i, j \geq n_0$. Hence $\left(\Delta_p^q x_k^{(i)}\right)$ is a Cauchy in X and it is bicomplex n-Banach space. So it is convergent in X. Let $\lim_{i\to\infty} \Delta_p^q x_k^{(i)} = y_k$ in X, for all $k \in \mathbb{N}$. Now we show that $\lim_{i\to\infty} x^{(i)} = x$ and $x \in (m(\phi, \Delta_p^q), \|\cdots\|_n)$. If $k = 1$, we get $\lim_{i\to\infty} x_{pq+1}^{(i)} = x_{pq+1}$, for $p, q \geq 1$. We have $\lim_{i\to\infty} x_k^{(i)} = x_k, \qquad$ for all $k \in \mathbb{N}$.

Hence $\lim_{i\to\infty} \Delta_p^q x_k^{(i)} = \Delta_p^q x_k$, and $\lim_{i\to\infty} x_k^{(i)} = x_k$, for all $k \in \mathbb{N}$. Which shows that $\left(x^{(i)}\right)$ is a Cauchy sequence in $(m(\phi, \Delta_p^q), \|\cdots\|_n)$. So

$$\sup_{\substack{t\geq 1\\ \alpha\in P_t}} \frac{1}{\phi_t} \sum_{k\in\alpha} \left\|\left(w_1, w_2, \ldots, w_{n-1}, \Delta_p^q \left(x_k^{(i)} - x_k^{(j)}\right)\right)\right\|_n < \varepsilon,$$

for all $i, j \geq n_0$. As $j \to \infty$, we have

$$\sum_{r=1}^{pq} \left\|\left(w_1, w_2, \ldots, w_{n-1}, x_r^{(i)} - x_r\right)\right\|_n + \sup_{\substack{t\geq 1\\ \alpha\in P_t}} \frac{1}{\phi_t} \sum_{k\in\alpha} \left\|\left(w_1, w_2, \ldots, w_{n-1}, \Delta_p^q \left(x_k^{(i)} - x_k\right)\right)\right\|_n < \varepsilon.$$

Hence by definition, $\|x^i - x\|_{n,m(\phi,\Delta_p^q)} < \varepsilon$, for all $i \geq n_0$. Here $\lim_{i\to\infty} x_k^{(i)} = x_k$, for all $i \geq n_0$.

$$\begin{aligned}\|x\|_{n,m(\phi,\Delta_p^q)} &= \|x - x^{(i)} + x^{(i)}\|_{n,m(\phi,\Delta_p^q)} \\ &\leq \|x - x^{(i)}\|_{n,m(\phi,\Delta_p^q)} + \|x^{(i)}\|_{n,m(\phi,\Delta_p^q)} < \varepsilon + M,\end{aligned}$$

where

$$M = \sup_i \|x^{(i)}\|_{n,m(\phi,\Delta_p^q)} < \infty.$$

Thus,

$$\|x\|_{n,m(\phi,\Delta_p^q)} < \infty,$$

and hence

$$x \in (m_{\mathbb{BC}}(\phi, \Delta_p^q), \|\cdots\|_n)$$

Therefore,

$$\left(m_{\mathbb{BC}}(\phi, \Delta_p^q), \|\cdots\|_n\right)$$

is a bicomplex n-Banach space. □

Theorem 4.2. *The space* $(m_{\mathbb{BC}}(\phi, \Delta_p^q), \|\cdots\|_n) \subseteq (m_{\mathbb{BC}}(\psi, \Delta_p^q), \|\cdots\|_n)$ *if and only if* $\sup_{t\geq 1} \frac{\phi_t}{\psi_t} < \infty$.

Proof. Let us suppose that $\sup_{t\geq 1} \frac{\phi_t}{\psi_t} = k < \infty$. So that $\phi_t \leq k\psi_t$. Now, if $(x_k) \in (m_{\mathbb{BC}}(\phi, \Delta_p^q), \|\cdots\|_n)$.

Then $\sup_{\substack{t\geq 1\\ \alpha\in P_t}} \frac{1}{\phi_t} \sum_{k\in\alpha} \|(w_1, w_2, \ldots, w_{n-1}, \Delta_p^q x_k)\|_n < \infty$.

Therefore, $\sup_{\substack{t\geq 1\\ \alpha\in P_t}} \frac{1}{k\psi_t} \sum_{k\in\alpha} \|(w_1, w_2, \ldots, w_{n-1}, \Delta_p^q x_k)\|_n < \infty$.

i.e., $(x_k) \in (m_{\mathbb{BC}}(\psi, \Delta_p^q), \|\cdots\|_n)$. Hence $(m_{\mathbb{BC}}(\phi, \Delta_p^q), \|\cdots\|_n) \subseteq (m_{\mathbb{BC}}(\psi, \Delta_p^q), \|\cdots\|_n)$.

Conversely, suppose that,

$$(m_{\mathbb{BC}}(\phi, \Delta_p^q), \|\cdots\|_n) \subseteq (m_{\mathbb{BC}}(\psi, \Delta_p^q), \|\cdots\|_n)$$

Then we have to prove that $\sup_{t\geq 1}\frac{\phi_t}{\psi_t} = \sup_{t\geq 1} a_t < \infty$. Let $\sup_{t\geq 1}(a_t) = \infty$.
Then there exists a subsequence (a_{t_i}) of (a_t), therefore $\lim_{i\to\infty} a_{t_i} = \infty$.
For $(x_k) \in (m_{\mathbb{BC}}(\phi, \Delta_p^q), \|\cdots\|_n)$.
We have,

$$\sup_{\substack{t\geq 1\\ \alpha\in P_t}} \frac{1}{\psi_t}\sum_{k\in\alpha} \left\|(w_1, w_2, \ldots, w_{n-1}, \Delta_p^q x_k)\right\|_n \geq \sup_{\substack{t\geq 1\\ \alpha\in P_t}} \frac{a_{t_i}}{\phi_{t_i}}\sum_{k\in\alpha} \left\|(w_1, w_2, \ldots, w_{n-1}, \Delta_p^q x_k)\right\|_n = \infty.$$

Hence $\sup_{\substack{t\geq 1\\ \alpha\in P_t}} \frac{1}{\psi_t}\sum_{k\in\alpha} \|(w_1, w_2, \ldots, w_{n-1}, \Delta_p^q x_k)\|_n = \infty$.
Hence $(x_k) \in (m_{\mathbb{BC}}(\psi, \Delta_p^q), \|\cdots\|_n)$ is a contradiction. Hence the proof. □

Theorem 4.3. *For the bicomplex sequence space $(m_{\mathbb{BC}}(\phi, \Delta_p^q), \|\cdots\|_n)$, the following relations hold:*

$$(l_p^{\mathbb{BC}}(\Delta_p^q), \|\cdots\|_n) \subseteq (m_{\mathbb{BC}}(\phi, \Delta_p^q), \|\cdots\|_n) \subseteq (l_\infty^{\mathbb{BC}}(\Delta_p^q).\|\cdots\|_n).$$

Where $(l_p^{\mathbb{BC}}(\Delta_p^q), \|\cdots\|_n)$ is the bicomplex analog defined by

$$\|x\|_{n,l_p} = \left(\sum_{k=1}^{\infty} \left\|(w_1, w_2, \ldots, w_{n-1}, \Delta_p^q x_k)\right\|_n^p\right)^{\frac{1}{p}}.$$

Proof. For the first relation assuming $\phi_t = 1$, for all $t \in \mathbb{N}$.

$$(m_{\mathbb{BC}}(\phi, \Delta_p^q), \|\cdots\|_n) = l_p^{\mathbb{BC}}(\Delta_p^q), \|\cdots\|_n).$$

So the first relation is proved. For next let $y_k \in (m_{\mathbb{BC}}(\phi, \Delta_p^q), \|\cdots\|_n)$.

Then by definition,

$$\sup_{\substack{t\geq 1\\ \alpha\in P_t}} \frac{1}{\psi_t}\sum_{k\in\alpha} \left\|(w_1, w_2, \ldots, w_{n-1}, \Delta_p^q y_k)\right\|_n = k < \infty.$$

For the second relation, take $t = 1$,

$$\left\|(w_1, w_2, \ldots, w_{n-1}, \Delta_p^q y_k)\right\|_n \leq k\phi_1,$$

.

for k. Which implies that

$$\sup_{k\geq 1} \left\|(w_1, w_2, \ldots, w_{n-1}, \Delta_p^q y_k)\right\|_n < \infty.$$

Thus we have $y_k \in l_\infty^{\mathbb{BC}}(\Delta_p^q), \|\cdots\|_n)$. This completes the proof. □

5 Conclusion

In this work, we have introduced a bicomplex generalized difference sequence space $m_{\mathbb{BC}}(\phi, \Delta_p^q)$ and investigated several of its algebraic and structural properties. The classical generalized difference sequence space associated with $m(\phi)$ has been extended to the bicomplex setting. Furthermore, the idempotent decomposition property, norm decomposition property, and several topological and geometric properties, including completeness, convergence-freeness, symmetry, and solidity, have been studied to provide a deeper understanding of the structure of the proposed bicomplex sequence space.

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