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Research Article

Difference Sequence Spaces Generated by $\varphi-$Function and Their Bi-complex Extension

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Abstract

Classical sequence spaces such as l_∞, c, and c_0 have been extensively studied and recognized as fundamental in the advancement of functional analysis and related areas of mathematics. In this article, we investigate the algebraic properties together with paranorm structures of the sequence spaces $W_0(\Delta, f)(\mathbb{C}_2)$, $W(\Delta, f)(\mathbb{C}_2)$, and $W_\infty(\Delta, f)(\mathbb{C}_2)$ in a bi-complex setting, which are induced by a non-negative real-valued function φ.
Keywords: $\phi-$function, Orlicz function, Bi-complex number, Paranormed space.

1 INTRODUCTION

A sequence space refers to a set of sequences, generally consisting of real, complex, or bi-complex numbers, which forms a vector space under the standard operations of addition term by term and multiplication by a scalar. The set of all sequences is denoted by ω. Any non-empty linear subspace of ω is referred to as a sequence space.

In particular, the spaces l_∞, c, and c_0 correspond respectively to the families of bounded, convergent, and null sequences, and they can be described as follows:

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$$l_\infty = \{y = (y_k) \in \omega : \sup_k |y_k| < \infty\}$$
$$c = \{y = (y_k) \in \omega : \exists\, l \in \mathbb{C} : |y_k - l| \to 0 \text{ as } k \to \infty\}$$
$$c_0 = \{y = (y_k) \in \omega : |y_k| \to 0 \text{ as } k \to \infty\}$$

and the norm associated with these spaces is defined as

$$\|y\|_\infty = \sup_k |y_k|, \text{for all natural numbers } k.$$

2 DEFINITIONS AND PRELIMINARIES

Definition 2.1. Let X be a linear space, and $h : X \to \mathbb{R}$ denotes a function that assigns real values. The function h is said to be a paranorm on X [2] if it satisfies the following conditions:

1. $h(x) \geq 0$ for all $x \in X$.
2. h $(\theta) = 0$, where $\theta = (0, 0, ...)$ is the zero vector in X.
3. h (-x) = h (x), i.e h is an even function.
4. The function h satisfies the subadditivity property, i.e. for all x, y ∈ X,

$$\mathrm{h}\,(x + y) \leq \mathrm{h}\,(x) + \mathrm{h}\,(y)\,.$$

5. h is continuous with respect to scalar multiplication.

The space (X, h) is called paranormed space. Furthermore, a paranorm h is called a total paranorm on X if the condition $\mathrm{h}\,(x) = 0$ implies $x = \theta$.

The notion of paranormed spaces was first introduced by Nakano [11] and Simmons [23], and was later studied and generalized by Tripathy and Sen [3], Maddox [12] as well as Pahari [17, 18].

Definition 2.2. A mapping M: $[0, \infty) \to [0, \infty)$ is called an Orlicz function [16] if it is continuous, non-decreasing and convex, and it satisfies the following conditions:

(a) $\mathrm{M}\,(0) = 0$,

(b) $\mathrm{M}(x) > 0$ for all $x > 0$,

(c) $M(x) \to \infty$ as $x \to \infty$.

Definition 2.3. An Orlicz function M is said to satisfy Δ_2 - condition [16] provided that there exists a constant $L > 0$ such that

$$M(2x) \leq L\,M(x) \text{ for all } x \geq 0.$$

The concept of Orlicz function was employed by Lindenstrauss and Tzafriri [13] in defining the Orlicz sequence space

$l_M = \{\ y = (y_k) \in \omega\colon \sum_{k=1}^{\infty} M(\frac{|y_k|}{\sigma}) < \infty \text{ for some } \sigma > 0\ \}$, where ω represents the space of all sequences of real numbers.

This space l_M becomes a Banach space [13] when equipped with the Luxemburg norm

$$\|y\| = \inf\{\ \sigma > 0\colon \sum_{k=1}^{\infty} M(\frac{|y_k|}{\sigma}) \leq 1\}.$$

If the Orlicz function is given by $M(x) = x^p$ for $(1 \leq p < \infty)$, then the Orlicz sequence space l_M reduces to the classical sequence space l_p.

The geometric and topological aspects of Orlicz sequence spaces have been widely investigated by several researchers, such as Kolk [8], Parashar and Chaudhary [22], Khan [24].

Definition 2.4. For a sequence $y = (y_k)$, its difference sequence is defined as

$$(\Delta y_k) = (y_k) - (y_{k+1}) \text{ for all } k \in \mathbb{N}.$$

Kizmaz [10] introduced the following classes of difference sequence spaces.

$$l_\infty(\Delta) = \{y = (y_k) : \Delta y_k \in l_\infty\},$$
$$c(\Delta) = \{y = (y_k) : \Delta y_k \in c\},$$
$$c_0(\Delta) = \{y = (y_k) : \Delta y_k \in c_0\}.$$

where, $\Delta y_k = y_k - y_{k+1}$.

Definition 2.5. Bi-complex number:

Bi-complex numbers are the extension of complex numbers obtained by introducing two independent imaginary units. This concept was first proposed by Corrado Segre [4] in 1892 .

In this work, the notations $\mathbb{C}_0$, $\mathbb{C}_1$ and $\mathbb{C}_2$ denote the set of real, complex and bi-complex numbers respectively.

A bi-complex number z is defined as

$$\begin{aligned} z &= (a_1 + \mathrm{i}\, b_1) + \mathrm{j}\,(\, c_1 + \mathrm{i}\, d_1) \\ &= z_1 + \mathrm{j}\, z_2 \ \text{, where } z_1 = a_1 + \mathrm{i}\, b_1,\ z_2 = c_1 + \mathrm{i} d_1 \end{aligned}$$

with $a_1,\, b_1,\, c_1,\, d_1 \in \mathbb{C}_0,\, z_1,\, z_2, \in \mathbb{C}_1$.

Here, i and j are the independent imaginary units that satisfy the relations:

$i^2 = j^2 = -1$, $ij = ji = k$ where k is a hyperbolic unit and $k^2 = 1$.

The set of all bi-complex numbers is given by

$$\mathbb{C}_2 = \{z_1 + \mathrm{j}\, z_2 : z_1, z_2 \in \mathbb{C}_1\}.$$

Within $\mathbb{C}_2$, there exist two non-trivial idempotent elements e_1 and e_2 which are defined by

$$\mathrm{e}_1 = \frac{1 + ij}{2} \text{ and } \mathrm{e}_2 = \frac{1 - ij}{2}.$$

They obey the basic defining relations

$$\mathrm{e}_1 + \mathrm{e}_2 = 1,\ \mathrm{e}_1.\mathrm{e}_2 = \mathrm{e}_2.\mathrm{e}_1 = 0,\ \mathrm{e}_1^2 = \mathrm{e}_1 \text{ and } \mathrm{e}_2^2 = \mathrm{e}_2.$$

Every bi-complex number $\mathrm{z} = z_1 + \mathrm{j}\, z_2$ (where z_1, $z_2 \in \mathbb{C}_1$) can be uniquely decomposed in the idempotent basis as

$$z = \mu_1 \mathrm{e}_1 + \mu_2\, \mathrm{e}_2,$$

where the components are given by

$$\mu_1 = z_1 - \mathrm{i}\, z_2 \text{ and } \mu_2 = z_1 + \mathrm{i}\, z_2.$$

The Euclidean norm defines a Euclidean-type norm on $\mathbb{C}_2$ which is given by

$$\|z\|_{\mathbb{C}_2} = \sqrt{a_1^2 + b_1^2 + \ c_1^2 + \ d_1^2} = \sqrt{|z_1|^2 + |z_2|^2} \ = \sqrt{\frac{|\mu_1|^2 + |\mu_2|^2}{2}}.$$

For an additional detail on bi-complex numbers and their sequence spaces, we refer [1, 5, 6, 9, 19, 20, 21].

Definition 2.6. A mapping $f : \mathbb{R} \to [0, \infty)$ is called a φ - function [7] if it fulfills the following requirements:

(a) $f(t) = 0$ if and only if t = 0,

(b) f is an even function,

(c) f is non-decreasing on $[0, \infty)$.

A φ - function is closely related to an Orlicz function M.

By extending the results of Herawati and Gulton [7], Ghimire and Pahari [14] introduced a family of difference sequence spaces and studied their structural properties. These spaces are defined as follows.

$$W_0(\Delta, f) = \{\, y = (y_{\mathbf{k}}) \in \omega : \frac{1}{m}\sum_{k=1}^{m} f\,(\frac{|\Delta y_{\mathbf{k}}|}{\sigma}) \to 0 \text{ as m } \to \infty \text{ and for some } \sigma > 0\,\},$$

$$W(\Delta, f) = \{\, y = (y_{\mathbf{k}}) \in \omega : \exists\, l > 0 : \frac{1}{m}\sum_{k=1}^{m} f\,(\frac{|\Delta y_{\mathbf{k}} - l|}{\sigma}) \to 0 \text{ as m } \to \infty \text{ and for some } \sigma > 0\,\},$$

$$W_\infty(\Delta, f) = \{\, y = (y_{\mathbf{k}}) \in \omega : \exists\, \sigma > 0 : \sup_m \frac{1}{m}\sum_{k=1}^{m} f\,(\frac{|\Delta y_{\mathbf{k}}|}{\sigma}) < \infty\,\},$$

where f is a φ-function.

We now extend these constructions to the framework of bi-complex numbers. The generalized sequence classes are given by:

$$W_0(\Delta, f)(\mathbb{C}_2) = \{\, y = (y_{\mathbf{k}}) \in \omega(\mathbb{C}_2) : \frac{1}{m}\sum_{k=1}^{m} f\,(\frac{\|\Delta y_k\|}{\sigma}) \to 0 \text{ as m } \to \infty \text{ and for some } \sigma > 0\,\},$$

$$W(\Delta, f)(\mathbb{C}_2) = \{\, y = (y_{\mathbf{k}}) \in \omega(\mathbb{C}_2) : \exists\, l > 0 : \frac{1}{m}\sum_{k=1}^{m} f\,(\frac{\|\Delta y_{\mathbf{k}} - l\|}{\sigma}) \to 0 \text{ as m } \to \infty \text{ and for some } \sigma > 0\,\},$$

$$W_\infty(\Delta, f)(\mathbb{C}_2) = \{\, y = (y_{\mathbf{k}}) \in \omega(\mathbb{C}_2) : \exists\, \sigma > 0 : \sup_m \frac{1}{m}\sum_{k=1}^{m} f\,(\frac{\|\Delta y_{\mathbf{k}}\|}{\sigma}) < \infty\,\}.$$

Definition 2.7. A sequence space X is called solid or normal [15] if for every sequence $y = (y_{\mathbf{k}}) \in$ X, there exists a real sequence $(a_{\mathbf{k}})$ with $|a_{\mathbf{k}}| \leq 1$ for all k $\in \mathbb{N}$ such that the sequence $(a_{\mathbf{k}} y_{\mathbf{k}})$ also belongs to X.

3 MAIN RESULTS

This section is devoted to establishing some results related to bi-complex sequence classes $W_0(\Delta, f)(\mathbb{C}_2), W(\Delta, f)(\mathbb{C}_2)$, and $W_\infty(\Delta, f)(\mathbb{C}_2)$.

Theorem 3.1. Suppose the φ - function f satisfies the Δ_2-condition, then the sequence classes $W_0(\Delta, f)(\mathbb{C}_2), W(\Delta, f)(\mathbb{C}_2)$, and $W_\infty(\Delta, f)(\mathbb{C}_2)$ form linear spaces over the field of complex numbers $\mathbb{C}_1$.

Proof. We first establish the linearity of $W_\infty(\Delta, f)(\mathbb{C}_2)$ and the linearity of the spaces $W_0(\Delta, f)(\mathbb{C}_2)$ and $W(\Delta, f)(\mathbb{C}_2)$ follows in the similar manner.

Let $y = (y_{\mathbf{k}}),\ \ z = (z_{\mathbf{k}}) \in W_\infty(\Delta, f)(\mathbb{C}_2)$ and $\alpha,\ \beta \in \mathbb{C}_1$. Then, there exist positive constants σ_1 and σ_2 such that

$$\sup_m \frac{1}{m}\sum_{k=1}^{m} f(\frac{\|\Delta y_k\|}{\sigma_1}) < \infty \tag{3.1}$$

and

$$\sup_m \frac{1}{m}\sum_{k=1}^{m} f\left(\frac{\|\Delta z_k\|}{\sigma_2}\right) < \infty. \tag{3.2}$$

Setting, $\sigma_3 = \max\{2|\alpha|\sigma_1, 2|\beta|\sigma_2\}$ and using the results (3.1) and (3.2) we have

$$\begin{aligned}
\sup_m \frac{1}{m}\sum_{k=1}^{m} f\left(\frac{\|\Delta(\alpha y_k + \beta z_k)\|}{\sigma_3}\right) &= \sup_m \frac{1}{m}\sum_{k=1}^{m} f\left(\frac{1}{2}\frac{\|2(\alpha\Delta y_k + \beta\Delta z_k)\|}{\sigma_3}\right) \\
&\le \frac{1}{2}\sup_m \frac{1}{m}\sum_{k=1}^{m} f\left(2\frac{\|\alpha\Delta y_k\|}{\sigma_3}\right) + \frac{1}{2}\sup_m \frac{1}{m}\sum_{k=1}^{m} f\left(2\frac{\|\beta\Delta z_k\|}{\sigma_3}\right) \\
&= \frac{1}{2}\sup_m \frac{1}{m}\sum_{k=1}^{m} f\left(\frac{2|\alpha|\ \|\Delta y_k\|}{\sigma_3}\right) + \frac{1}{2}\sup_m \frac{1}{m}\sum_{k=1}^{m} f\left(\frac{2|\beta|\ \|\Delta z_k\|}{\sigma_3}\right) \\
&\le \frac{1}{2}\sup_m \frac{1}{m}\sum_{k=1}^{m} f\left(\frac{\|\Delta y_k\|}{\sigma_1}\right) + \frac{1}{2}\sup_m \frac{1}{m}\sum_{k=1}^{m} f\left(\frac{\|\Delta z_k\|}{\sigma_2}\right) \\
&< \infty + \infty = \infty.
\end{aligned}$$

$$\therefore\ \sup_m \frac{1}{m}\sum_{k=1}^{m} f\left(\frac{\|\Delta(\alpha y_k + \beta z_k)\|}{\sigma_3}\right) < \infty.$$

Hence, the linear combination $\alpha y + \beta z \in W_\infty(\Delta, f)(\mathbb{C}_2)$ for all $y = (y_k)$ and $z = (z_k)$ $\in W_\infty(\Delta, f)(\mathbb{C}_2)$ and for all scalars $\alpha, \beta \in \mathbb{C}_1$.

Therefore, the sequence class $W_\infty(\Delta, f)(\mathbb{C}_2)$ is a linear space.

Theorem 3.2. The space $W_0(\Delta, f)(\mathbb{C}_2)$ can be equipped with a paranorm, making it a paranormed space h: $W_0(\Delta, f)(\mathbb{C}_2) \to \mathbb{R}$ defined by

$$\text{h}(x) = \inf\{\sigma > 0 : \frac{1}{m}\sum_{k=1}^{m} f\left(\frac{\|\Delta x_k\|}{\sigma}\right) \le 1\ \forall\ \text{m} \in \mathbb{N}\}$$

where, $x = (x_k) \in W_0(\Delta, f)(\mathbb{C}_2)$.

Proof.

Let $x = (x_k) \in W_0(\Delta, f)(\mathbb{C}_2)$. Then,

(i) $\mathrm{h}(x) = \inf\{\sigma > 0 : \frac{1}{m}\sum_{k=1}^{m} f\left(\frac{\|\Delta x_k\|}{\sigma}\right) \leq 1\}$

$$\therefore \quad h(0) = 0 \text{ and } h(x) \geq 0 \text{ for all } x \in W_0(\Delta, f)(\mathbb{C}_2).$$

(ii) $$\begin{aligned}\mathrm{h}(-x) &= \inf\{\sigma > 0 : \frac{1}{m}\sum_{k=1}^{m} f\left(\frac{\|-\Delta x_k\|}{\sigma}\right) \leq 1\}\\ &= \inf\{\sigma > 0 : \frac{1}{m}\sum_{k=1}^{m} f\left(\frac{\|\Delta x_k\|}{\sigma}\right) \leq 1\}\\ &= \mathrm{h}(x).\end{aligned}$$

$$\therefore \quad \mathrm{h}(-x) = \mathrm{h}(x) \text{ for all } x \in W_0(\Delta, f)(\mathbb{C}_2).$$

(iii) Let $x = (x_k),\ y = (y_k)$ be any two sequences in the space $W_0(\Delta, f)(\mathbb{C}_2)$. Then, there exist positive constants $\sigma_1,\ \sigma_2$ such that

$$\begin{aligned}\frac{1}{m}\sum_{k=1}^{m} f\left(\frac{\|\Delta x_k\|}{\sigma_1}\right) &\to 0 \text{ as m} \to \infty,\\ \text{and } \frac{1}{m}\sum_{k=1}^{m} f\left(\frac{\|\Delta y_k\|}{\sigma_2}\right) &\to 0 \text{ as m} \to \infty.\end{aligned}$$

Let $\sigma = \max\{\sigma_1,\ \sigma_2\}$. As f is a non-decreasing function on $[0, \infty)$, we have

$$\begin{aligned}\mathrm{h}\,(\mathrm{x} + y) &= \inf\left\{\sigma > 0 : \frac{1}{m}\sum_{k=1}^{m} f\left(\frac{\|\Delta x_k + \Delta y_k\|}{\sigma}\right) \leq 1\right\}\\ &\leq \inf\left\{\sigma_1 > 0 : \frac{1}{m}\sum_{k=1}^{m} f\left(\frac{\|\Delta x_k\|}{\sigma_1}\right) \leq 1\right\} + \inf\left\{\sigma_2 > 0 : \frac{1}{m}\sum_{k=1}^{m} f\left(\frac{\|\Delta y_k\|}{\sigma_2}\right) \leq 1\right\}\\ &\leq \mathrm{h}\,(\mathrm{x}) + \mathrm{h}\,(y)\,.\end{aligned}$$

$$\therefore \quad \mathrm{h}\,(\mathrm{x} + y) \leq \mathrm{h}\,(\mathrm{x}) + \mathrm{h}\,(y)\ \ \forall\ \mathrm{x},\ y \in W_0(\Delta, f)(\mathbb{C}_2).$$

(iv) Finally, it is established that the operation of scalar multiplication is continuous.

Let $z = (z_k) \in W_0(\Delta, f)(\mathbb{C}_2)$ and $\mathrm{h}(z_k^{(m)} - z_k) \to 0$ as $m \to \infty$. Also, let (a_m) be a sequence of scalars which converges to α, i.e. $a_m \to \alpha$ as $m \to \infty$.

Then,

$$\begin{aligned}
\mathrm{h}\left(\alpha_m z_k^{(m)} - \alpha z_k\right) &= \inf\left\{\sigma > 0 : \frac{1}{m}\sum_{k=1}^{m} f\left(\frac{\|\alpha_m \Delta z_k^{(m)} - \alpha \Delta z_k\|}{\sigma}\right) \leq 1\right\} \\
&\leq \inf\left\{\sigma > 0 : \frac{1}{m}\sum_{k=1}^{m} f\left(\frac{\|\alpha_m \Delta z_k^{(m)} - \alpha \Delta z_k^{(m)}\|}{\sigma}\right) \leq 1\right\} + \\
&\quad \inf\left\{\sigma > 0 : \frac{1}{m}\sum_{k=1}^{m} f\left(\frac{\|\alpha \Delta z_k^{(m)} - \alpha \Delta z_k\|}{\sigma}\right) \leq 1\right\} \\
&= |\alpha_m - \alpha| \inf\left\{\sigma_1 = \frac{\sigma}{|\alpha_m - \alpha|} > 0 : \frac{1}{m}\sum_{k=1}^{m} f\left(\frac{\|\Delta z_k^{(m)}\|}{\sigma_1}\right) \leq 1\right\} + |\alpha| \\
&\quad \inf\left\{\sigma_2 = \frac{\sigma}{|\alpha|} > 0 : \frac{1}{m}\sum_{k=1}^{m} f\left(\frac{\|\Delta z_k^{(m)} - \Delta z_k\|}{\sigma_2}\right) \leq 1\right\} \\
&= |\alpha_m - \alpha|\, \mathrm{h}\left(z_k^{(m)}\right) + |\alpha|\, \mathrm{h}\left(z_k^{(m)} - z_k\right) \to 0 \text{ as } \alpha_m \to \alpha \text{ and } \mathrm{h}\left(z_k^{(m)} - z_k\right) \to 0.
\end{aligned}$$

$\therefore \mathrm{h}\left(\alpha_m z_k^{(m)} - \alpha z_k\right) \to 0$ as $m \to \infty$. Hence, scalar multiplication is continuous and $W_0(\Delta, f)(\mathbb{C}_2)$ is a paranormed space.

In a similar manner, it can be shown that $W(\Delta, f)(\mathbb{C}_2)$ and $W_\infty(\Delta, f)(\mathbb{C}_2)$ also form paranormed spaces.

Theorem 3.3. The linear space $W(\Delta, f)(\mathbb{C}_2)$ is a complete paranormed space under the assumption that φ - function f is convex and satisfies the Δ_2-condition.

Proof. Let $(z_k^n) = (z_1^{(n)}, z_2^{(n)}, z_3^{(n)}, ...)$ be any Cauchy sequence in $W(\Delta, f)(\mathbb{C}_2)$. Then, for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $m, n \geq n_0$ we have

$$\mathrm{h}\left(z^{(m)} - z^{(n)}\right) < \varepsilon \text{ and } \quad \frac{1}{r}\sum_{k=1}^{r} f\left(\frac{\|\Delta z_k^{(m)} - \Delta z_k^{(n)}\|}{\varepsilon}\right) \leq 1.$$

As f is convex, we get

$$\frac{1}{r}\sum_{k=1}^{r} f\left(\|\Delta z_k^{(m)} - \Delta z_k^{(n)}\|\right) \leq \varepsilon \cdot \frac{1}{r}\sum_{k=1}^{r} f\left(\frac{\|\Delta z_k^{(m)} - \Delta z_k^{(n)}\|}{\varepsilon}\right) \leq \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, $f\left(\|\Delta z_k^{(m)} - \Delta z_k^{(n)}\|\right) = 0$ for all $m, n \geq n_0$.

Thus, $\left\|\Delta z_{\mathbf{k}}^{(m)} - \Delta z_{\mathbf{k}}^{(n)}\right\| < \varepsilon$ for every $m, n \geq n_0$. Hence $(z_{\mathbf{k}}^n)$ is a Cauchy sequence in $\mathbb{R}$ for all $\mathbf{k} \in \mathbb{N}$. From the completeness property of $\mathbb{R}$, there exists $z_{\mathbf{k}} \in \mathbb{R}$ such that

$$\lim_{n\to\infty} z_{\mathbf{k}}^n = z_{\mathbf{k}}.$$

Thus, for all $n \geq n_0$ we get

$$\left\| z_{\mathbf{k}}^{(m)} - z_{\mathbf{k}} \right\| = \left\| z_{\mathbf{k}}^{(m)} - \lim_{n\to\infty} z_{\mathbf{k}}^{(n)} \right\| = \lim_{n\to\infty} \left\| z_{\mathbf{k}}^{(m)} - z_{\mathbf{k}}^{(n)} \right\| < \varepsilon^2.$$

Since $(z_{\mathbf{k}}^n) \in W(\Delta, f)(\mathbb{C}_2)$, there exists $l > 0$ so that $\frac{1}{r}\sum_{\mathbf{k}=1}^{r} f\left(\frac{\|\Delta z_{\mathbf{k}}^{(n)} - l\|}{\sigma}\right) \to 0$ as $r \to \infty$ and for $\sigma > 0$.

The continuity of f implies that

$$\frac{1}{r}\sum_{\mathbf{k}=1}^{r} f\left(\frac{\|\Delta z_{\mathbf{k}} - l\|}{\sigma}\right) = \frac{1}{r}\sum_{\mathbf{k}=1}^{r} f\left(\frac{\left\| \lim_{n\to\infty} \Delta z_{\mathbf{k}}^{(n)} - l \right\|}{\sigma}\right) = \lim_{n\to\infty} \frac{1}{r}\sum_{\mathbf{k}=1}^{r} f\left(\frac{\|\Delta z_{\mathbf{k}}^{(n)} - l\|}{\sigma}\right) = 0 \text{ as } r \to \infty.$$

$\therefore \frac{1}{r}\sum_{\mathbf{k}=1}^{r} f\left(\frac{\|\Delta z_{\mathbf{k}} - l\|}{\sigma}\right) \to 0$ as $r \to \infty$. Hence $(z_{\mathbf{k}}) \in W(\Delta, f)(\mathbb{C}_2)$.

Finally, we show that $\mathrm{h}(z^{(n)} - z) \to 0$ as $n \to \infty$.

From the continuity of φ - function f, it follows that

$$\begin{aligned}\frac{1}{r}\sum_{\mathbf{k}=1}^{r} f\left(\frac{\|\Delta z_{\mathbf{k}}^{(n)} - \Delta z_{\mathbf{k}}\|}{\sigma}\right) &= \frac{1}{r}\sum_{\mathbf{k}=1}^{r} f\left(\frac{\|\Delta z_{\mathbf{k}}^{(n)} - \lim_{m\to\infty} \Delta z_{\mathbf{k}}^{(m)}\|}{\sigma}\right)\\ &= \frac{1}{r}\sum_{\mathbf{k}=1}^{r} f\left(\frac{\|\Delta z_{\mathbf{k}}^{(m)} - \Delta z_{\mathbf{k}}^{(n)}\|}{\sigma}\right) \leq 1\end{aligned}$$

Thus, $\mathrm{h}(z^{(n)} - z) = \inf\{\sigma > 0 : \frac{1}{r}\sum_{\mathbf{k}=1}^{r} f\left(\frac{\|\Delta z_{\mathbf{k}}^{(n)} - \Delta z_{\mathbf{k}}\|}{\sigma}\right) \leq 1\}$.

Hence, we get $\mathrm{h}(z^{(n)} - z) < \sigma$ for every $\sigma > 0$.

Therefore, there exists a real sequence $(\frac{p}{2^q})$, where p, q are real numbers such that

$$\mathrm{h}(z^{(n)} - z) < \frac{p}{2^q}, q \geq 1. \text{ Hence, we get } \mathrm{h}(z^{(n)} - z) \to 0 \text{ as } n \to \infty.$$

So, $W(\Delta, f)(\mathbb{C}_2)$ is a complete paranormed space. Similarly, we can show that $W_\infty(\Delta, f)(\mathbb{C}_2)$ is also a complete paranormed space equipped with the same paranorm defined on $W(\Delta, f)(\mathbb{C}_2)$.

Example 1. The sequence spaces $W_0(\Delta, f)(\mathbb{C}_2)$, $W(\Delta, f)(\mathbb{C}_2)$, $W_\infty(\Delta, f)(\mathbb{C}_2)$ are not normal.

Consider the sequence space $W_\infty(\Delta, f)(\mathbb{C}_2)$ and let $z = (z_k) = (kj)$ for all k. Then, $(\Delta z_k) = (-j)$

Now, $\sup_m \frac{1}{m} \sum_{k=1}^{m} f\left(\frac{\|\Delta z_k\|}{\sigma}\right) = \sup_m \frac{1}{m} \sum_{k=1}^{m} f\left(\frac{1}{\sigma}\right) = \sup_m f\left(\frac{1}{\sigma}\right) < \infty.$

Hence $z = (z_k) \in W_\infty(\Delta, f)(\mathbb{C}_2)$

Let $a_k = \{(-1)^k\}$ be any sequence of scalars satisfying $|a_k| \leq 1$ for all natural numbers k.

Then, $\Delta a_k z_k = (-1)^k z_k - (-1)^{k+1} z_{k+1} = (-1)^k (z_k + z_{k+1}) = (-1)^k (2k+1)j$

$\therefore \|\Delta a_k z_k\| = 2k + 1.$

Now for any positive real number σ, we have

$$\sup_m \frac{1}{m} \sum_{k=1}^{m} f\left(\frac{\|\Delta a_k z_k\|}{\sigma}\right) = \sup_m \frac{1}{m} \sum_{k=1}^{m} f\left(\frac{2k+1}{\sigma}\right) \to \infty, \text{ as } m \to \infty$$

This is because φ-function f is increasing and $f(t) \to \infty$ as $t \to \infty$.

$\therefore (a_k z_k) \notin W_\infty(\Delta, f)(\mathbb{C}_2)$.

Hence, the space $W_\infty(\Delta, f)(\mathbb{C}_2)$ is not normal.

4 CONCLUSION

In this article, we examined the sequence spaces $W_0(\Delta, f)(\mathbb{C}_2)$, $W(\Delta, f)(\mathbb{C}_2)$, and $W_\infty(\Delta, f)(\mathbb{C}_2)$ focusing on their algebraic properties together with paranorm structures. A natural direction for future research will be to extend these spaces to double sequences of bi-complex numbers.

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