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Research Article

Product of two ${}_2F_2$ Generalized Hypergeometric Functions and their Special Cases

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Abstract

A remarkable number of identities have been established related to the product of generalized hypergeometric series in recent years. In the present work, we shall establish certain identities involving the product of two generalized hypergeometric functions. Bailey's theorem is used to evaluate the particular cases for the product.

Keywords : Hypergeometric series, Generalized hypergeometric function, Kummer's type I transformation, Kummer's type II transformation, Product
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1 Introduction and Preliminaries

Hypergeometric functions are transcendental functions and most of the elementary functions used in mathematics are expressed in the form of hypergeometric function in any way. The ordinary hypergeometric function ${}_2F_1[a, b; c; x]$ is called the Gaussian hypergeometic function. It is expressed in the form of the hypergeometric series.
The generalized hypergeometric function ${}_pF_q$ with h and k number of numerator and de-

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nominator parameters respectively is defined by [1, 8] is given by

$$ {}_pF_q\left[\begin{matrix} h_1, & \dots, & h_p \\ & & & ; & x \\ k_1, & \dots, & k_q \end{matrix}\right] = \sum_{n=0}^{\infty}\frac{(h_1)_n \dots (h_p)_n}{(k_1)_n \dots (k_q)_n}\frac{x^n}{n!} \tag{1.1} $$

where $(\lambda)_n$ is a Pochhammer symbol. It can be expressed in terms of gamma function as given below.

$$ \begin{aligned} (\lambda)_n := & \frac{\Gamma(\lambda+n)}{\lambda} \qquad (\alpha \in \mathbb{C}\backslash\mathbb{Z}_0^-) \\ = & \begin{cases} \lambda(\lambda+1)...(\lambda+n-1) & ; n \in \mathbb{N} \\ 1 & ; n = 0 \end{cases} \end{aligned} \tag{1.2} $$

The series (1.1) is convergent for all x in $p \leq q$. For details, see [1].
When $p = q$, then (1.1) coincides with the exponential function;

$$ {}_0F_0(-;-;z) = \sum_{n=0}^{\infty}\frac{z^n}{n!} \tag{1.3} $$

By using differential equations, Kummer[8] obtained the result(1.4), often known as Kummer type I transformation for the series ${}_1F_1$, is given below, .

$$ e^x {}_1F_1\left[\begin{matrix} \alpha \\ & ; & x \\ \beta \end{matrix}\right] = {}_1F_1\left[\begin{matrix} \beta-\alpha \\ & ; & x \\ \beta \end{matrix}\right] \tag{1.4} $$

Bailey also obtained the same result through classical Guass summation theorem[1, 11].
The Kummer second transformation is given by

$$ e_1^{-\frac{x}{2}} F_1\left[\begin{matrix} \alpha \\ & ; & x \\ 2\alpha \end{matrix}\right] = {}_0F_1\left[\begin{matrix} - \\ & ; & \frac{x^2}{16} \\ \alpha+\frac{1}{2} \end{matrix}\right] \tag{1.5} $$

In 2008, Rathie and Pogany [12] generalized the Kummer second type transformation in the form ;

$$ e^{-\frac{x}{2}} {}_1F_1\left[\begin{matrix} \alpha, & d+1 \\ & & ; x \\ 2\alpha+1, & d \end{matrix}\right] = {}_0F_1\left[\begin{matrix} - \\ & ; & \frac{x^2}{16} \\ \alpha+\frac{1}{2} \end{matrix}\right] - \frac{(2\alpha-d)}{2d(2\alpha+1)} {}_0F_1\left[\begin{matrix} - \\ & ; & \frac{x^2}{16} \\ \alpha+\frac{3}{2} \end{matrix}\right] \tag{1.6} $$

In 2013 Rakha et. al.[9] established an extension of Kummer second theorem in the form

$$e^{-\frac{x}{2}}{}_1F_1\left[\begin{matrix}\alpha, & d+2 \\ 2\alpha+2, & d\end{matrix};x\right] = {}_0F_1\left[\begin{matrix}- \\ \alpha+\frac{1}{2}\end{matrix};\frac{x^2}{16}\right] + \frac{\left(\frac{\alpha}{d}+\frac{1}{2}\right)x}{(\alpha+1)}{}_0F_1\left[\begin{matrix}- \\ \alpha+\frac{3}{2}\end{matrix};\frac{x^2}{16}\right]$$
$$+\frac{cx^2}{2(2\alpha+3)}{}_0F_1\left[\begin{matrix}- \\ \alpha+\frac{5}{2}\end{matrix};\frac{x^2}{16}\right] \tag{1.7}$$

where

$$c = \left[\frac{1}{\alpha+1}\left(\frac{1}{2}-\frac{\alpha}{d}\right)+\frac{\alpha}{d(d+1)}\right] \tag{1.8}$$

Bailey [2] has derived a number of results on the product of two generalized hypergeometric functions. Most applicable among them is given below;

$${}_0F_1\left[\begin{matrix}- \\ \alpha\end{matrix};x\right] \times {}_0F_1\left[\begin{matrix}- \\ \beta\end{matrix};x\right]$$
$$= {}_2F_3\left[\begin{matrix}\frac{1}{2}(\alpha+\beta), & \frac{1}{2}(\alpha+\beta-1), & \\ \alpha, & \beta, & \alpha+\beta-1\end{matrix};4x\right] \tag{1.9}$$

In 2019, Kim & Rathie[3] evaluated the product of two hypergeometric function ${}_2F_2$ as follows;

$$
\begin{aligned}
S = {}_2F_2\left[\begin{matrix}\alpha, & d+1 \\ & & ;x \\ 2\alpha+1, & d\end{matrix}\right] \times {}_2F_2\left[\begin{matrix}\alpha, & e+1 \\ & & ;x \\ 2\alpha+1, & e\end{matrix}\right] \\
= e\Bigg\{ {}_2F_3\left[\begin{matrix}\frac{1}{2}(\alpha+\beta+1), & \frac{1}{2}(\alpha+\beta), & \\ & & & ;\frac{x^2}{4} \\ \alpha+\frac{1}{2}, & \beta+\frac{1}{2}, & \alpha+\beta\end{matrix}\right] \\
+ \frac{x(2\alpha-d)}{2d(2\alpha+1} {}_2F_3\left[\begin{matrix}\frac{1}{2}(\alpha+\beta+1), & \frac{1}{2}(\alpha+\beta+2), & \\ & & & ;\frac{x^2}{4} \\ \alpha+\frac{1}{2}, & \beta+\frac{3}{2}, & \alpha+\beta+2\end{matrix}\right] \\
+ \frac{x(2\beta-e)}{2e(2\beta+1)} {}_2F_3\left[\begin{matrix}\frac{1}{2}(\alpha+\beta+3), & \frac{1}{2}(\alpha+\beta+2), & \\ & & & ;\frac{x^2}{4} \\ \alpha+\frac{3}{2}, & \beta+\frac{1}{2}, & \alpha+\beta+2\end{matrix}\right] \\
+ \frac{x^2(2\alpha-d)(2\beta-e)}{4de(2\alpha+1)(2\beta+1)} {}_2F_3\left[\begin{matrix}\frac{1}{2}(\alpha+\beta+3), & \frac{1}{2}(\alpha+\beta+2), & \\ & & & ;\frac{x^2}{4} \\ \alpha+\frac{3}{2}, & \beta+\frac{3}{2}, & \alpha+\beta+2\end{matrix}\right]\Bigg\}
\end{aligned} \tag{1.10}
$$

Several works have been done to evaluate the product of two hyper-geometric functions. For details, refer [2, 3, 5, 6, 7]. Here, we will establish an identity on the product of any two generalized hypergeometric functions ${}_2F_2$ in the form;

$$
{}_2F_2\left[\begin{matrix}\alpha, & d+1 \\ & & ;x \\ 2\alpha+1, & d\end{matrix}\right] \times {}_2F_2\left[\begin{matrix}\beta, & e+2 \\ & & ;x \\ 2\beta+2, & e\end{matrix}\right] \tag{1.11}
$$

We shall also obtain thirty six particular values for the product of two hypergeometric functions as mentioned in the theorem as the particular cases. For this, we will be taking $x = 1$, $\alpha = 1$ and $d, e = 1, 2, 3, 4, 5, 6$.

2 Main Results

In this section, we will prove the results as mentioned in the following theorem.

Theorem 2.1. *The following relation holds true;*

$$
{}_2F_2\left[\begin{matrix}\alpha, & d+1 \\ & & ;x \\ 2\alpha+1, & d\end{matrix}\right]\times{}_2F_2\left[\begin{matrix}\beta, & e+2 \\ & & ;x \\ 2\beta+2, & e\end{matrix}\right]
$$

$$
\begin{aligned}
&= e^{x}\Bigg\{{}_2F_3\left[\begin{matrix}\frac{1}{2}(\alpha+\beta+2), & \frac{1}{2}(\alpha+\beta+1), & \\ & & & ;\frac{x^2}{4} \\ \alpha+\frac{1}{2}, & \beta+\frac{3}{2}, & \alpha+\beta+1\end{matrix}\right] \\
&+ xc_1\times{}_2F_3\left[\begin{matrix}\frac{1}{2}(\alpha+\beta+3), & \frac{1}{2}(\alpha+\beta+2), & \\ & & & ;\frac{x^2}{4} \\ \alpha+\frac{3}{2}, & \beta+\frac{3}{2}, & \alpha+\beta+2\end{matrix}\right] \\
&+ x.c_2\times{}_2F_3\left[\begin{matrix}\frac{1}{2}(\alpha+\beta+2), & \frac{1}{2}(\alpha+\beta+1), & \\ & & & ;\frac{x^2}{4} \\ \alpha+\frac{1}{2}, & \beta+\frac{3}{2}, & \alpha+\beta+1\end{matrix}\right] \\
&+ \frac{x^2}{2}c_2\times{}_2F_3\left[\begin{matrix}\frac{1}{2}(\alpha+\beta+3), & \frac{1}{2}(\alpha+\beta+2), & \\ & & & ;\frac{x^2}{4} \\ \alpha+\frac{3}{2}, & \beta+\frac{3}{2}, & \alpha+\beta+2\end{matrix}\right] \\
&+ \frac{x^2}{2}\frac{c_3}{(2\beta+3)}{}_2F_3\left[\begin{matrix}\frac{1}{2}(\alpha+\beta+3), & \frac{1}{2}(\alpha+\beta+2), & \\ & & & ;\frac{x^2}{4} \\ \alpha+\frac{1}{2}, & \beta+\frac{5}{2}, & \alpha+\beta+2\end{matrix}\right] \\
&+ \frac{x^3.c_3}{2(2\beta+3)}\times{}_2F_3\left[\begin{matrix}\frac{1}{2}(\alpha+\beta+4), & \frac{1}{2}(\alpha+\beta+3), & \\ & & & ;\frac{x^2}{4} \\ \alpha+\frac{3}{2}, & \beta+\frac{5}{2}, & \alpha+\beta+3\end{matrix}\right]\Bigg\}
\end{aligned}
\tag{2.1}
$$

where

$$
c_1=\frac{(2\alpha-d)}{2d(2\alpha+1)}, c_2=\frac{(2\beta-e)}{2e(2\beta+1)}
$$

$$
c_3=\left[\frac{1}{(\alpha+1)}\left(\frac{1}{2}-\frac{\alpha}{d}\right)+\frac{\alpha}{d(d+1)}\right] \tag{2.2}
$$

Proof. To prove the theorem, let us consider the sum

$$
S=e^{-x}\Bigg\{{}_2F_2\left[\begin{matrix}\alpha, & d+1 \\ & & ;x \\ 2\alpha+1, & d\end{matrix}\right]\times{}_2F_2\left[\begin{matrix}\beta, & e+2 \\ & & ;x \\ 2\beta+2, & e\end{matrix}\right]\Bigg\}
$$

$$
=e^{-\frac{-1}{2}}{}_2F_2\left[\begin{matrix}\alpha, & d+1 \\ & & ;x \\ 2\alpha+1, & d\end{matrix}\right]\times_{e}{}^{-\frac{-1}{2}}\,{}_2F_2\left[\begin{matrix}\beta, & e+2 \\ & & ;x \\ 2\beta+2, & e\end{matrix}\right]
$$

By using the property (1.6) and (1.7), we have

$$S = \left\{ {}_0F_1\left[\begin{array}{c} - \\ \\ \alpha+\frac{1}{2} \end{array} ; \frac{1}{16}\right] + \frac{(2\alpha-d)}{2d(2\alpha+1)} {}_0F_1\left[\begin{array}{c} - \\ \\ \alpha+\frac{3}{2} \end{array} ; \frac{1}{16}\right] \right\}$$

$$\times \left\{ {}_0F_1\left[\begin{array}{c} - \\ \\ \alpha+\frac{1}{2} \end{array} ; \frac{x^2}{16}\right] + \frac{\left(\frac{\alpha}{d}+\frac{1}{2}\right)}{\alpha+1} x\, {}_0F_1\left[\begin{array}{c} - \\ \\ \alpha+\frac{3}{2} \end{array} ; \frac{x^2}{16}\right] \right.$$

$$\left. + \frac{cx^2}{2(2\alpha+3} {}_0F_1\left[\begin{array}{c} - \\ \\ \alpha+\frac{5}{2} \end{array} ; \frac{x^2}{16}\right] \right\}$$

Now, Using the Bailey's identity (1.9);

$$S = {}_2F_3\left[\begin{array}{ccc} \frac{1}{2}(\alpha+\beta+2), & \frac{1}{2}(\alpha+\beta+1), & \\ & & \\ \alpha+\frac{1}{2}, & \beta+\frac{3}{2}, & \alpha+\beta+1 \end{array} ; \frac{x^2}{4}\right]$$

$$+ xc_1 \times {}_2F_3\left[\begin{array}{ccc} \frac{1}{2}(\alpha+\beta+3), & \frac{1}{2}(\alpha+\beta+2), & \\ & & \\ \alpha+\frac{3}{2}, & \beta+\frac{3}{2}, & \alpha+\beta+2 \end{array} ; \frac{x^2}{4}\right]$$

$$+ x.c_2 \times {}_2F_3\left[\begin{array}{ccc} \frac{1}{2}(\alpha+\beta+2), & \frac{1}{2}(\alpha+\beta+1), & \\ & & \\ \alpha+\frac{1}{2}, & \beta+\frac{3}{2}, & \alpha+\beta+1 \end{array} ; \frac{x^2}{4}\right]$$

$$+ \frac{x^2}{2} c_2 \times {}_2F_3\left[\begin{array}{ccc} \frac{1}{2}(\alpha+\beta+3), & \frac{1}{2}(\alpha+\beta+2), & \\ & & \\ \alpha+\frac{3}{2}, & \beta+\frac{3}{2}, & \alpha+\beta+2 \end{array} ; \frac{x^2}{4}\right]$$

$$+ \frac{x^2}{2} \frac{c_3}{(2\beta+3)} {}_2F_3\left[\begin{array}{ccc} \frac{1}{2}(\alpha+\beta+3), & \frac{1}{2}(\alpha+\beta+2), & \\ & & \\ \alpha+\frac{1}{2}, & \beta+\frac{5}{2}, & \alpha+\beta+2 \end{array} ; \frac{x^2}{4}\right]$$

$$+ \frac{x^3.c_3}{2(2\beta+3)} \times {}_2F_3\left[\begin{array}{ccc} \frac{1}{2}(\alpha+\beta+4), & \frac{1}{2}(\alpha+\beta+3), & \\ & & \\ \alpha+\frac{3}{2}, & \beta+\frac{5}{2}, & \alpha+\beta+3 \end{array} ; \frac{x^2}{4}\right]$$

where c_1, c_2 and c_3 are defined already in the theorem statement. Now shifting e^{-x} to the RHS and after simplification, we prove theorem 2.1 □

3 Special cases

Here, we will establish some important results on the product of any two generalized hypergeometric series ${}_2F_2$

(1) Setting $\lim_{d\to\infty}$ and $\lim_{d\to\infty}$, we obtain $c_1 = -\frac{1}{2(2\alpha+1)}$, $c_2 = -\frac{1}{2(\beta+1)}$, and $c_3 = \frac{1}{2(\beta+1)}$.

(2) If $\alpha = \beta$, and $d = e$ then $c_1 = c_2$.

(3) For the particular cases, we will be replacing $\alpha = 1$ and $d, e = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$ and 11.

If $\alpha = 1$, $d = 1$ and $e = 1$ in (2.1), then $c_1 = \frac{1}{6}$, $c_2 = \frac{1}{6}$ and $c_3 = 0$ and the product is as follows;

$$ {}_2F_2\left[\begin{matrix} 1, & 2 \\ & & ;1 \\ 3, & 1 \end{matrix}\right] \times {}_1F_1\left[\begin{matrix} 1, & 1 \\ & & ;1 \\ 1, & 1 \end{matrix}\right] = 2e \tag{3.1} $$

Similarly, we obtain other thirty five new results involving the product of two generalised hypergeometric functions as given below. We prefer to omit the details.

$$ {}_2F_2\left[\begin{matrix} 1, & 2 \\ & & ;1 \\ 3, & 1 \end{matrix}\right] \times {}_2F_2\left[\begin{matrix} 1, & 4 \\ & & ;1 \\ 3, & 2 \end{matrix}\right] = \frac{2}{3}(3e-2) \tag{3.2} $$

$$ {}_2F_2\left[\begin{matrix} 1, & 2 \\ & & ;1 \\ 3, & 1 \end{matrix}\right] \times {}_2F_2\left[\begin{matrix} 1, & 5 \\ & & ;1 \\ 3, & 3 \end{matrix}\right] = \frac{1}{3}(7e-8) \tag{3.3} $$

$$ {}_2F_2\left[\begin{matrix} 1, & 2 \\ & & ;1 \\ 3, & 1 \end{matrix}\right] \times {}_2F_2\left[\begin{matrix} 1, & 6 \\ & & ;1 \\ 3, & 4 \end{matrix}\right] = \frac{1}{5}(13e-18) \tag{3.4} $$

$$ {}_2F_2\left[\begin{matrix} 1, & 2 \\ & & ;1 \\ 3, & 1 \end{matrix}\right] \times {}_2F_2\left[\begin{matrix} 1, & 7 \\ & & ;1 \\ 3, & 5 \end{matrix}\right] = \frac{2}{15}(21e-32) \tag{3.5} $$

$$ {}_2F_2\left[\begin{matrix} 1, & 2 \\ & & ;1 \\ 3, & 1 \end{matrix}\right] \times {}_2F_2\left[\begin{matrix} 1, & 8 \\ & & ;1 \\ 3, & 6 \end{matrix}\right] = \frac{2}{21}(31e-50) \tag{3.6} $$

$$ {}_1F_1\left[\begin{matrix} 1 \\ \\ 2 \end{matrix};1\right] \times {}_1F_1\left[\begin{matrix} 1 \\ \\ 1 \end{matrix};1\right] = e(e-1) \tag{3.7}$$

$$ {}_1F_1\left[\begin{matrix} 1 \\ \\ 2 \end{matrix};1\right] \times {}_2F_2\left[\begin{matrix} 1, & 4 \\ \\ 3, & 2 \end{matrix};1\right] = \frac{1}{3}(3e^2-5e+2) \tag{3.8}$$

$$ {}_1F_1\left[\begin{matrix} 1 \\ \\ 2 \end{matrix};1\right] \times {}_2F_2\left[\begin{matrix} 1, & 5 \\ \\ 3, & 3 \end{matrix};1\right] = \frac{1}{6}(7e^2-15e+8) \tag{3.9}$$

$$ {}_1F_1\left[\begin{matrix} 1 \\ \\ 2 \end{matrix};1\right] \times {}_2F_2\left[\begin{matrix} 1, & 6 \\ \\ 3, & 4 \end{matrix};1\right] = \frac{1}{10}(13e^2-31e+18) \tag{3.10}$$

$$ {}_1F_1\left[\begin{matrix} 1 \\ \\ 2 \end{matrix};1\right] \times {}_2F_2\left[\begin{matrix} 1, & 7 \\ \\ 3, & 5 \end{matrix};1\right] = \frac{1}{15}(21e^2-53e+32) \tag{3.11}$$

$$ {}_1F_1\left[\begin{matrix} 1 \\ \\ 2 \end{matrix};1\right] \times {}_2F_2\left[\begin{matrix} 1, & 8 \\ \\ 3, & 6 \end{matrix};1\right] = \frac{1}{21}(31e^2-81e+50) \tag{3.12}$$

$$ {}_2F_2\left[\begin{matrix} 1, & 4 \\ \\ 3, & 3 \end{matrix};1\right] \times {}_1F_1\left[\begin{matrix} 1 \\ \\ 1 \end{matrix};1\right] = \frac{2e}{3}(2e-3) \tag{3.13}$$

$$ {}_2F_2\left[\begin{matrix} 1, & 4 \\ \\ 3, & 3 \end{matrix};1\right] \times {}_2F_2\left[\begin{matrix} 1, & 4 \\ \\ 3, & 2 \end{matrix};1\right] = \frac{2}{3}(6e^2-13e+6) \tag{3.14}$$

$$ {}_2F_2\left[\begin{matrix} 1, & 4 \\ & & ;1 \\ 3, & 3 \end{matrix}\right] \times {}_2F_2\left[\begin{matrix} 1, & 5 \\ & & ;1 \\ 3, & 3 \end{matrix}\right] = \frac{1}{9}(14e^2 - 37e + 24) \tag{3.15} $$

$$ {}_2F_2\left[\begin{matrix} 1, & 4 \\ & & ;1 \\ 3, & 3 \end{matrix}\right] \times {}_2F_2\left[\begin{matrix} 1, & 6 \\ & & ;1 \\ 3, & 4 \end{matrix}\right] = \frac{1}{15}(26e^2 - 75e + 54) \tag{3.16} $$

$$ {}_2F_2\left[\begin{matrix} 1, & 4 \\ & & ;1 \\ 3, & 3 \end{matrix}\right] \times {}_2F_2\left[\begin{matrix} 1, & 7 \\ & & ;1 \\ 3, & 5 \end{matrix}\right] = \frac{2}{45}(42e^2 - 127e + 96) \tag{3.17} $$

$$ {}_2F_2\left[\begin{matrix} 1, & 4 \\ & & ;1 \\ 3, & 3 \end{matrix}\right] \times {}_2F_2\left[\begin{matrix} 1, & 8 \\ & & ;1 \\ 3, & 6 \end{matrix}\right] = \frac{2}{63}(62e^2 - 193e + 150) \tag{3.18} $$

$$ {}_2F_2\left[\begin{matrix} 1, & 5 \\ & & ;1 \\ 3, & 4 \end{matrix}\right] \times {}_2F_2\left[\begin{matrix} 1 \\ & ;1 \\ 1 \end{matrix}\right] = \frac{e}{2}(3e - 5) \tag{3.19} $$

$$ {}_2F_2\left[\begin{matrix} 1, & 5 \\ & & ;1 \\ 3, & 4 \end{matrix}\right] \times {}_2F_2\left[\begin{matrix} 1, & 4 \\ & & ;1 \\ 3, & 2 \end{matrix}\right] = \frac{1}{6}(9e^2 - 21e + 10) \tag{3.20} $$

$$ {}_2F_2\left[\begin{matrix} 1, & 5 \\ & & ;1 \\ 3, & 4 \end{matrix}\right] \times {}_2F_2\left[\begin{matrix} 1, & 5 \\ & & ;1 \\ 3, & 3 \end{matrix}\right] = \frac{1}{12}(21e^2 - 59e + 40) \tag{3.21} $$

$$ {}_2F_2\left[\begin{matrix} 1, & 5 \\ & & ;1 \\ 3, & 4 \end{matrix}\right] \times {}_2F_2\left[\begin{matrix} 1, & 6 \\ & & ;1 \\ 3, & 4 \end{matrix}\right] = \frac{1}{25}(52e^2 - 163e + 126) \tag{3.22} $$

$$ {}_2F_2\left[\begin{matrix}1, & 5\\ & & ;1\\ 3, & 4\end{matrix}\right]\times {}_2F_2\left[\begin{matrix}1, & 7\\ & & ;1\\ 3, & 5\end{matrix}\right]=\frac{1}{30}(63e^2-201e+160) \tag{3.23}$$

$$ {}_2F_2\left[\begin{matrix}1, & 5\\ & & ;1\\ 3, & 4\end{matrix}\right]\times {}_2F_2\left[\begin{matrix}1, & 8\\ & & ;1\\ 3, & 6\end{matrix}\right]=\frac{1}{42}(93e^2-305e+105) \tag{3.24}$$

$$ {}_2F_2\left[\begin{matrix}1, & 6\\ & & ;1\\ 3, & 5\end{matrix}\right]\times {}_2F_2\left[\begin{matrix}1 & \\ & ;1\\ 1 & \end{matrix}\right]=\frac{2e}{5}(4e-7) \tag{3.25}$$

$$ {}_2F_2\left[\begin{matrix}1, & 6\\ & & ;1\\ 3, & 5\end{matrix}\right]\times {}_2F_2\left[\begin{matrix}1, & 4\\ & & ;1\\ 3, & 2\end{matrix}\right]=\frac{2}{15}(12e^2-29e+14) \tag{3.26}$$

$$ {}_2F_2\left[\begin{matrix}1, & 6\\ & & ;1\\ 3, & 5\end{matrix}\right]\times {}_2F_2\left[\begin{matrix}1, & 5\\ & & ;1\\ 3, & 3\end{matrix}\right]=\frac{1}{15}(28e^2-81e+56) \tag{3.27}$$

$$ {}_2F_2\left[\begin{matrix}1, & 6\\ & & ;1\\ 3, & 5\end{matrix}\right]\times {}_2F_2\left[\begin{matrix}1, & 6\\ & & ;1\\ 3, & 4\end{matrix}\right]=\frac{1}{25}(52e^2-163e+126) \tag{3.28}$$

$$ {}_2F_2\left[\begin{matrix}1, & 6\\ & & ;1\\ 3, & 5\end{matrix}\right]\times {}_2F_2\left[\begin{matrix}1, & 7\\ & & ;1\\ 3, & 5\end{matrix}\right]=\frac{2}{75}(84e^2-275e+224) \tag{3.29}$$

$$ {}_2F_2\left[\begin{matrix}1, & 6\\ & & ;1\\ 3, & 5\end{matrix}\right]\times {}_2F_2\left[\begin{matrix}1, & 8\\ & & ;1\\ 3, & 6\end{matrix}\right]=\frac{2}{105}(124e^2-417e+350) \tag{3.30}$$

$$
{}_2F_2\left[\begin{matrix} 1, & 7 \\ & & ;1 \\ 3, & 6 \end{matrix}\right] \times {}_2F_2\left[\begin{matrix} 1 \\ & ;1 \\ 1 \end{matrix}\right] = \frac{e}{3}(5e-9) \tag{3.31}
$$

$$
{}_2F_2\left[\begin{matrix} 1, & 7 \\ & & ;1 \\ 3, & 6 \end{matrix}\right] \times {}_2F_2\left[\begin{matrix} 1, & 4 \\ & & ;1 \\ 3, & 2 \end{matrix}\right] = \frac{1}{9}(15e^2-37e+18) \tag{3.32}
$$

$$
{}_2F_2\left[\begin{matrix} 1, & 7 \\ & & ;1 \\ 3, & 6 \end{matrix}\right] \times {}_2F_2\left[\begin{matrix} 1, & 5 \\ & & ;1 \\ 3, & 3 \end{matrix}\right] = \frac{1}{18}(35e^2-103e+72) \tag{3.33}
$$

$$
{}_2F_2\left[\begin{matrix} 1, & 7 \\ & & ;1 \\ 3, & 6 \end{matrix}\right] \times {}_2F_2\left[\begin{matrix} 1, & 6 \\ & & ;1 \\ 3, & 4 \end{matrix}\right] = \frac{1}{30}(65e^2-207e+162) \tag{3.34}
$$

$$
{}_2F_2\left[\begin{matrix} 1, & 7 \\ & & ;1 \\ 3, & 6 \end{matrix}\right] \times {}_2F_2\left[\begin{matrix} 1, & 7 \\ & & ;1 \\ 3, & 5 \end{matrix}\right] = \frac{1}{45}(105e^2-349e+288) \tag{3.35}
$$

$$
{}_2F_2\left[\begin{matrix} 1, & 7 \\ & & ;1 \\ 3, & 6 \end{matrix}\right] \times {}_2F_2\left[\begin{matrix} 1, & 8 \\ & & ;1 \\ 3, & 6 \end{matrix}\right] = \frac{2}{63}(155e^2-529e+450) \tag{3.36}
$$

4 Conclusion

In this work, we have obtained the product of two ${}_2F_2(1)$ generalized hypergeometric functions. Also we have established the specific values of altogether thirty six products as the special cases. All the products are expressed in terms of exponential function. All of the above results obtained are verified through mathematica software. These results may be useful in mathematics, engineering and some branches of applied sciences. Particularly the product of generalised hypergeometric function frequently appears in Physics, Applied mathematics, Stastistics, Multivariate analysis. in the theory of partitions, combinatorial identities, number theory and lie algebra.

References

[1] A. K.Rathie, and J. Choi. Another proof of Kummer's second theorem. *Communications of the Korean Mathematical Society*, 13(4),(1998) 933-936.

[2] A. K. Rathie, and J. Pogany. New summation formula for ${}_3F_2(\frac{1}{2})$ and a Kummer type II transformation of ${}_2F_2(x)$, *Math. Commun.*, 13(1),(2008). 63-66.

[3] C. T. Preece. The product of two hypergeometric functions. *Proc. London Math. Soc.(2)* 22, (1924) 370-380.

[4] E. D.Rainville. *Special functions*, Chelsea Publishing Company, Macmillan Co. New York, (1971).

[5] M. P. Poudel, Harsh V. H., N. P.Pahari,. A note on an extension of Bailey's formula involving product of two generalized hypergeometric function. *Bulletin of Karela Mathematics Association.*(19) 1,(2022) 1-7.

[6] M. A. Rakha, and A. K. Rathie. On an extension of Kummer-type II transformation. *TWMS Journal of Applied and Engineering Mathematics*, 4(1), (2014), 80-85.

[7] M. A.Rakha, M. M.Awad, and A. K. Rathie. On an extension of Kummer's second theorem. *In Abstract and Applied Analysis*, 2013(1), Hindawi.

[8] W. N. Bailey. Product of generalized hypergeometric series, *Proc. London Math. Soc (2)* 28(4), (1928), 242-254.

[9] W. N.Bailey. Generalised Hypergeometric Series, *Stechert-Hafner*, Inc., New York, (1964).

[10] Y. S. Kim, and A. K.Rathie. On generalization of Bailey's identity involving product of generalized hypergeometric series.(2017). *arXiv preprint arXiv:1702.05855.*

[11] Y. S. Kim, J. Choi, and A. K.Rathie. An Identity Involving Product of Generalized Hypergeometric Series ${}_2F_2$ *Kyungpook Mathematical Journal*, 59(2),(2019), 293-299.

[12] Y. S.Kim, M. A. Rakha, and A. K. Rathie . Generalization of Kummer's second theorem with applications, *Comput. Math. Phys.*, Russia, Volume 50(3), (2010).387-402.