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Research Article

Theorems on Compatible Mapping Types in Complete Menger Space

Ajay Kumar Chaudhary*

*Department of Mathematics, Trichandra Multiple Campus
Tribhuvan University, Kathmandu, Nepal
akcsaurya81@gmail.com

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Abstract

This research article presents two common fixed theorems in complete Menger space in two pairs of self-mappings by using altering distance function in the context of compatible mappings of type (P) and compatible mappings of type (K). Article discusses the topological properties of Menger spaces and mappings between these spaces. This result generalizes the result of Khan et al. [18], and extends the results of [8], and [9]

Keywords: Common fixed point, Menger space, Compatible mappings of type (P), Compatible mappings of type (K), Altering distance function.
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1 Introduction

In real life measurement, assigning a fixed number to the distance between two points is very idealized thinking. In such a situation, we usually refer to the average value of several measurements to the distance of two points in space. This notion introduces the concept of statistical metric space, thereafter referred to as probabilistic metric space, to Karl Menger's [19] consciousness in 1942.

Menger introduced probabilistic metric space as a generalization of metric space by replacing metric function $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$ with distribution function $F_{p,q} : \mathbb{R} \to [0, 1]$, and then for any number x, the value $F_{p,q}(x)$ was interpreted as the probability that the distance between p and q is less than x. For more details, refer to [12, 14, 15, 22, 24, 25], [28, 29] and [31].

*Corresponding author

In 1991, S. N. Mishra [20] generalized the concept of G. Jungck's [17] compatible mapping in Menger space. Continuing this, B. Singh and S. Jain [26] gave the notion of weak compatibility in Menger space, and then various authors worked on this space, for references: [1, 2, 3], [4, 5, 6], [16], [23], and [30].

In 1984, M. S. Khan et al. [18] introduced the altering distance function, which makes changes in the distance between two points in space. Some works in this line of research are noted in [4], [11, 10], [13], and [21].

In this paper, using the concept of altering distance function, we prove common fixed point results in complete Menger space, which generalizes and extends the results of [18], [8], and [9].

2 Preliminaries

I start by reviewing a few fundamental definitions in a sequel, which will be required in Menger space.

Definition 2.1. [4] A mapping $F : \mathbb{R} \to \mathbb{R}^+$ is said to be distribution function if it is a non-decreasing function, left continuous with $inf\{F(x) : x \in \mathbb{R}\} = 0$ and $sup\{F(x) : x \in \mathbb{R}\} = 1$..

Here, we denote the set of all distribution functions by Ω, while H denotes the specific distribution function defined by:

$$H(x) = \begin{cases} 0, & \text{if } x \leq 0, \\ 1, & \text{if } x > 0. \end{cases}$$

Definition 2.2. [4] A *probabilistic metric space (pm-space)* is an ordered pair (X, F) where X is any non-empty abstract set of elements, $F : X \times X \to \Omega$ is distribution function defined by $(p, q) \mapsto F_{p,q}$, where $\Omega = \{F_{p,q} : p, q \in X\}$, $F_{p,q}$ satisfies the following conditions:

P1: $F(p, q, x) = 1$ for every $x > 0$ if and only if $p = q$.

P2: $F(p, q, 0) = 0$ for every $p, q \in X$.

P3: $F(p, q, x) = F(q, p, x)$ for every $p, q \in X$.

P4: $F(p, q, x + y) = 1$ if and only if $F(p, r, x) = 1$ and $F(r, q, y) = 1$ for all $x, y > 0$.

$F(p, q, x)$ is also denoted by $F_{p,q}(x)$ to represent the value of $F_{p,q}$ at $x \in \mathbb{R}$.

Definition 2.3. [7] A mapping $t : [0, 1] \times [0, 1] \to [0, 1]$ is called a *triangular norm (t-norm)* if it satisfies the following conditions:

T1: $t(0,0) = 0$ and $t(a,1) = a$ for all $a \in [0,1]$;

T2: $t(a,b) = t(b,a)$ for all $a, b \in [0,1]$;

T3: $t(a,b) \leq t(c,d)$, if $a \leq c$ and $b \leq d$; and

T4: $t(t(a,b),c) = t(a,t(b,c))$.

Definition 2.4. [4] A *Menger space* is a triplet (X, F, t), where X is a nonempty set, F is a function defined on $X \times X$ to the set of distribution functions, and t is a triangular norm such that followings are satisfied:

P1: $F(p,q,x) = 1$ for every $x > 0$ if and only if $p = q$.

P2: $F(p,q,0) = 0$.

P3: $F(p,q,x) = F(q,p,x)$.

P4: $F(p,q,x+y) \geq t(F(p,r,x), F(r,q,y))$, for all $x, y \geq 0$ and $p, q, r \in X$.

Definition 2.5. [5] A mapping $S : X \to X$ in Menger space (X, F, t) is said to be *continuous* at a point $p \in X$ if for every $\epsilon > 0$ and $\lambda > 0$, there exists $\epsilon_1 > 0$ and $\lambda_1 > 0$ such that:

$$F(p,q,\epsilon_1) > 1 - \lambda_1 \implies F(Sp, Sq, \epsilon) > 1 - \lambda.$$

Definition 2.6. [5] Let (X, F, t) be a Menger space and t be a continuous t-norm. Then,

(a) A sequence $\{y_n\}$ in X is said to converge to a point y in X if and only if, for every $\epsilon > 0$ and $\lambda > 0$, there exists an integer $N = N(\epsilon, \lambda)$ such that $F_{y_n,y}(\epsilon) > 1 - \lambda$ for all $n \geq N$. In this case, we write:

$$\lim_{n\to\infty} y_n = y.$$

(b) A sequence $\{y_n\}$ in X is said to be a *Cauchy sequence* if, for every $\epsilon > 0$ and $\lambda > 0$, there exists an integer $N = N(\epsilon, \lambda)$ such that $F_{y_n,y_m}(\epsilon) > 1 - \lambda$ for all $m, n \geq N$.

(c) A Menger space (X, F, t) is said to be *complete* if every Cauchy sequence in X converges to a point in X.

Definition 2.7. [7] Common fixed point of self-mapping functions $S, T : X \to X$ is a point $x \in X$ if:

$$S(x) = T(x) = x.$$

Example 2.8. Let $S, T : \mathbb{R} \to \mathbb{R}$ be functions such that $S(x) = \frac{x^2}{4}$ and $T(x) = 2x - 4$. Then $x = 4$ is a common fixed point of S and T.

Definition 2.9. [20] Two mappings $S, T : X \to X$ are said to be *compatible mappings* in Menger space (X, F, t) if:

$$\lim_{n\to\infty} F(STx_n, TSx_n, x) = 1 \quad \forall x > 0,$$

whenever the sequence $\{x_n\}$ in X satisfies $\lim\limits_{n\to\infty} Sx_n = \lim\limits_{n\to\infty} Tx_n = y$ for some $y \in X$.

Definition 2.10. [18] A function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is called an *altering distance function* if the following properties are satisfied:

(i) ψ is continuous.

(ii) ψ is non-decreasing.

(iii) $\psi(t) = 0$ if and only if $t = 0$.

(iv) $\psi(t) \geq Mt^{\mu}$, for every $t > 0$, where $M > 0$ and $\mu > 0$ are constants.

We denote by Ψ the set of all altering distance functions. It is also called a control function.

2.1 Variants of Compatible Mappings in Menger Space

Definition 2.11. [8] Two mappings $S, T : X \to X$ are said to be *compatible mappings of type (P)* in Menger space (X, F, t) if:

$$\lim_{n\to\infty} F(SSx_n, TTx_n, x) = 1 \quad \forall x > 0,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = y$ for some $y \in X$.

Example 2.12. Let (X, d) be a metric space where $X = [0, \infty)$ with the usual metric $d(x, y) = |x - y|$, and $t(a, b) = ab$. Defining the distribution function as:

$$F_{x,y}(t) = \begin{cases} \frac{t}{t+|x-y|}, & \text{if } t > 0, \\ 0, & \text{if } t = 0. \end{cases}$$

Let (X, F, t) be a Menger space. Define mappings $S, T : X \to X$ by:

$$S(x) = \begin{cases} 5, & \text{for } x \in [0, 1), \\ x, & \text{for } x \in [1, \infty), \end{cases} \quad \text{and} \quad T(x) = \begin{cases} 1, & \text{for } x \in [0, 1), \\ \frac{1}{x}, & \text{for } x \in [1, \infty). \end{cases}$$

Take the sequence $\{x_n\}$ in X where $x_n = 1 + \frac{1}{n}$, $n \in \mathbb{N}$. Then (S, T) are compatible mappings of type (P) in the Menger space, but (S, T) are not compatible mappings.

Theorem 2.13. *[7] Let (X, F, t) be a Menger space with the continuous t-norm t, and let $S : X \to X$. Then S is continuous at a point $y \in X$ if and only if for every sequence $\{y_n\}$ in X converging to a point y, the sequence $\{Sy_n\}$ converges to the point Sy, i.e., if $\{y_n\} \to y$ then $\{Sy_n\} \to Sy$.*

Proposition 2.1. [8] In a Menger space (X, F, t), if $t(k, k) \geq k$ for all $k \in [0, 1]$, then $t(a, b) = \min\{a, b\}$ for all $a, b \in [0, 1]$.

Proposition 2.2. [8] Let (X, F, t) be a Menger space such that the t-norm t is continuous and $t(x, x) \geq x$ for all $x \in [0, 1]$, and let $S, T : X \to X$ be mappings. If S and T are compatible mappings of type (P) and $Sk = Tk$ for some $k \in X$, then

$$SSk = STk = TSk = TTk.$$

Proposition 2.3. [8] Let (X, F, t) be a Menger space such that the t-norm t is continuous and $t(x, x) \geq x$ for all $x \in [0, 1]$, and let $S, T : X \to X$ be mappings. Let S and T be compatible mappings of type (P) and

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = y \quad \text{for some } y \in X.$$

Then:

(i) $\lim_{n\to\infty} TTx_n = Sy$ if S is continuous at y.

(ii) $\lim_{n\to\infty} SSx_n = Ty$ if T is continuous at y.

(iii) $STy = TSy$ and $Sy = Ty$ if S and T are continuous at y.

Definition 2.14. [9] Two self-mappings $S, T : X \to X$ are said to be *compatible mappings of type (K)* in a Menger space (X, F, t) if:

$$\lim_{n\to\infty} F(SSx_n, Tz, t) = 1 \quad \text{and} \quad \lim_{n\to\infty} F(TTx_n, Sz, t) = 1, \quad \forall t > 0,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some $z \in X$.

Example 2.15. Let (X, d) be a metric space where $X = [0, 2]$, and (X, F, t) be a Menger space with:

$$F(x, y, t) = \begin{cases} \frac{t}{t+d(x,y)}, & \text{for } t > 0, \\ 0, & \text{for } t = 0, \end{cases}$$

for all $x, y \in X$, and $t > 0$. Define S and T as:

$$S(x) = \begin{cases} 2, & \text{for } x \in [0, 1] \setminus \left\{\frac{1}{2}\right\}, \\ 0, & \text{for } x = \frac{1}{2}, \\ \frac{2-x}{2}, & \text{for } x \in (1, 2], \end{cases}$$

and:

$$T(x) = \begin{cases} 0, & \text{for } x \in [0,1] \setminus \left\{\frac{1}{2}\right\}, \\ 2, & \text{for } x = \frac{1}{2}, \\ \frac{x}{2}, & \text{for } x \in (1,2]. \end{cases}$$

Take $\{x_n\}$ in X, where $x_n = 1 + \frac{1}{n}$, $n \in \mathbb{N}$. Then S and T are compatible mappings of type (K) but neither compatible mappings of type (P) nor type (A).

We need the following lemmas for the establishment of main results in the Menger space.

Lemma 2.16. *[26] Let (X, F, t) be a Menger space. If there exists $k \in (0,1)$ such that for all $p, q \in X$, $F(p, q, kx) \geq F(p, q, t)$ then $p = q$.*

Lemma 2.17. *[27] Let $\{k_n\}$ be a sequence in Menger space (X, F, t), where t is continuous $t-$norm and $t(x, x) \geq x$ for all $x \in [0,1]$. If there exists a constant $k \in [0,1]$ such that $lim_{n\to\infty} F(k_n, k_n+1, kx) \geq F(k_n-1, k_n, x)$, for all $x > 0$ and $n \in N$, then $\{k_n\}$ is a Cauchy sequence in X.*

3 Main Theorems

Theorem 3.1. *Let (X, F, t) be a complete Menger space with $t(x, y) = \min\{x, y\}$ for all $x, y \in [0,1]$, and let $Q, R, S, T : X \to X$ be mappings such that:*

(3.1.1) $Q(X) \subset T(X)$ *and* $R(X) \subset S(X)$,

(3.1.2) the pairs (Q, S) and (R, T) are compatible mappings of type (P),

(3.1.3) one of Q, S, R, T is continuous,

(3.1.4) there exists a constant $k \in (0,1)$ such that:

$$F(Qx, Ry, kz) \geq \psi\{\min\{F(Sx, Qx, z), F(Ty, Ry, z), F(Ty, Qx, rz), F(Sx, Ry, (2-r)z), F(Sx, Ty, z)\}\}$$

for all $x, y \in X$, $r \in (0,2)$, and $z > 0$, where $\phi : [0,1] \to [0,1]$ satisfies:

- *ψ is continuous and non-decreasing on $[0,1]$,*
- *$\psi(n) > n$ for all $n \in [0,1]$.*

noting that if $\psi \in \Psi$, class of all mappings $\psi : [0,1] \to [0,1]$ then $\psi(0) = 0$, $\psi(1) = 1$, $\psi(n) \geq n$, for all $n \in [0,1]$.

Then, Q, R, S, T have a unique common fixed point in X.

Proof. Consider $u_0 \in X$. Since $Q(X) \subset T(X)$, so there exists a point $u_1 \in X$ such that $Qu_0 = Tu_1 = v_0$. Again, since $R(X) \subset S(X)$, for u_1, we may choose $u_2 \in X$ such that $Ru_1 = Su_2 = v_1$. Repeating this process, we inductively construct sequences $\{u_n\}$ and $\{v_n\}$ in X such that:

$$Qu_{2n} = Tu_{2n+1} = v_{2n}, \quad Ru_{2n+1} = Su_{2n+2} = v_{2n+1}, \quad n = 0, 1, 2, \dots$$

By substituting $x = u_{2n}$, $y = u_{2n+1}$, $r = 1 - p$ with $p \in (0, 1)$ in (3.1.4), we obtain:

$F(Qu_{2n}, Ru_{2n+1}, kz) \geq \psi\{\min\{F(Su_{2n}, Qu_{2n}, z), F(Tu_{2n+1}, Ru_{2n+1}, z), F(Tu_{2n+1}, Qu_{2n}, (1-p)z), F(Su_{2n}, Ru_{2n+1}, (1+p)z), F(Su_{2n}, Tu_{2n+1}, z)\}\}$

$F(v_{2n}, v_{2n+1}, kz) \geq \psi\{\min\{F(v_{2n-1}, v_{2n}, z), F(v_{2n}, v_{2n+1}, z), F(v_{2n}, v_{2n}, (1-p)z), F(v_{2n-1}, v_{2n+1}, (1+p)z), F(v_{2n-1}, v_{2n}, z)\}\}$

$\geq \psi\{\min\{F(v_{2n-1}, v_{2n}, z), F(v_{2n}, v_{2n+1}, z), F(v_{2n-1}, v_{2n+1}, (1+p)z), F(v_{2n-1}, v_{2n}, z)\}\}$

$\geq \psi\{\min\{F(v_{2n-1}, v_{2n}, z), F(v_{2n}, v_{2n+1}, z), F(v_{2n-1}, v_{2n}, z), F(v_{2n}, v_{2n+1}, pz), F(v_{2n-1}, v_{2n}, z)\}\}$

$\geq \psi\{\min\{F(v_{2n-1}, v_{2n}, z), F(v_{2n}, v_{2n+1}, z), F(v_{2n}, v_{2n+1}, pz)\}\}$

Simplifying further as $p \to 1$ gives:

$F(v_{2n}, v_{2n+1}, kz) \geq \psi\{\min\{F(v_{2n-1}, v_{2n}, z), F(v_{2n}, v_{2n+1}, z), F(v_{2n}, v_{2n+1}, z)\}\}$

$\geq \psi\{\min\{F(v_{2n-1}, v_{2n}, z), F(v_{2n}, v_{2n+1}, z)\}\}$

or, $F(v_{2n}, v_{2n+1}, kz) \geq \psi\{F(v_{2n-1}, v_{2n}, z)\} > F(v_{2n-1}, v_{2n}, z)$, by property of ψ

Thus:

$$F(v_{2n}, v_{2n+1}, kz) \geq F(v_{2n-1}, v_{2n}, z).$$

Similarly, we derive:

$$F(v_{2n+1}, v_{2n+2}, kz) \geq F(v_{2n}, v_{2n+1}, z).$$

Therefore, for every $n \in N$,

$$F(v_n, v_{n+1}, kz) \geq F(v_{n-1}, v_n, z).$$

By Lemma 2.20, $\{v_n\}$ is a Cauchy sequence in X. Since (X, F, t) is complete, $\{v_n\}$ converges to a point $q \in X$. Consequently, the sub-sequences $\{Qu_{2n}\}, \{Ru_{2n+1}\}, \{Su_{2n}\}, \{Tu_{2n+1}\}$ of $\{v_n\}$ also converge to q.

Now, suppose that T is continuous. Then, since R and T are compatible mappings of type (P), then by proposition 2.16, $RR_{u_{2n+1}}, TR_{u_{2n+1}} \to Tz$ as $n \to \infty$. Putting $x = u_{2n}$ and $y = R_{u_{2n+1}}$ in relation (3.1.4), we get

$$F(Qu_{2n}, RRu_{2n+1}, kz) \geq \psi \left\{ \min \begin{bmatrix} F(Su_{2n}, Qu_{2n}, z), \\ F(TRu_{2n+1}, RRu_{2n+1}, z), \\ F(TRu_{2n+1}, Qu_{2n}, rz), \\ F(Su_{2n}, RRu_{2n+1}, (2-r)z), \\ F(Su_{2n}, TRu_{2n+1}, z) \end{bmatrix} \right\}.$$

Taking $n \to \infty$, we have

$$F(q, Tq, kz) \geq \psi \left\{ \min \begin{bmatrix} F(q, q, z), \\ F(Tq, Tq, z), \\ F(Tq, q, rz), \\ F(q, Tq, (2-r)z), \\ F(q, Tq, z) \end{bmatrix} \right\}.$$

Letting $r = 1 - p$ with $p \in (0, 1)$, then

$$F(q, Tq, kz) \geq \psi \left\{ \min \begin{bmatrix} F(Tq, q, (1-p)z), \\ F(q, Tq, (2-(1-p))z), \\ F(q, Tq, z) \end{bmatrix} \right\}.$$

Or,

$$F(q, Tq, kz) \geq \psi \left\{ \min \begin{bmatrix} F(Tq, q, (1-p)z), \\ F(q, Tq, (1+p)z), \\ F(q, Tq, z) \end{bmatrix} \right\}.$$

$$\geq \psi \left\{ \min \begin{bmatrix} F(Tq, q, (1-p+1+p)z), \\ F(q, Tq, z) \end{bmatrix} \right\}.$$

$$\geq \psi \left\{ \min \left[F(Tq, q, 2z), F(q, Tq, z) \right] \right\}.$$

$$\geq \psi \min \{F(q, Tq, z)\}.$$

Therefore,

$$F(q, Tq, kz) \geq \psi \left\{ F(q, Tq, z) \right\}.$$

Or,

$$F(q, Tq, kz) \geq F(q, Tq, z), \text{ by property of } \psi.$$

which implies $q = Tq$ by Lemma 2.19
Similarly, replacing x by u_{2n} and y by q in relation (3.1.4), we have

$$F(Qu_{2n}, Rq, kz) \geq \psi\left\{\min\begin{bmatrix}F(Su_{2n}, Qu_{2n}, z),\\ F(Tq, Rq, z),\\ F(Tq, Qrz),\\ F(Su_{2n}, Rq, (2-r)z),\\ F(Su_{2n}, Tq, z)\end{bmatrix}\right\}.$$

Taking $n \to \infty$, we get

$$F(q, Rq, kz) \geq \psi\left\{\min\begin{bmatrix}F(q, q, z),\\ F(q, Rq, z),\\ F(q, q, rz),\\ F(q, Rq, (2-r)z),\\ F(q, q, z)\end{bmatrix}\right\}.$$

$$\geq \psi\left\{\min\left[F(q, Rq, z), F(q, Rq, (2-(1-p))z)\right]\right\}.$$

$$\geq \psi\left\{\min\left[F(q, Rq, z), F(q, Rq, (1+p)z)\right]\right\}.$$

$$\geq \psi\left\{\min\left[F(q, Rq, z), F(q, q, z), F(q, Rq, pz)\right]\right\}.$$

as $p \to 1$

$$\geq \psi\left\{\min\left[F(q, Rq, z), F(q, Rq, z)\right]\right\}.$$

so that $F(q, Rq, kz) \geq \psi\{F(q, Rq, z)\}$
Or,

$$F(q, Rq, kz) \geq F(q, Rq, z), \text{ by property of } \psi.$$

which implies $q = Rq$ by Lemma 2.19.
Since, $R(X) \subset S(X)$, so there exist a point w in X such that $Rq = Sw = q$.

By using relation (3.1.4) with $x = w, y = q$, we have

$$F(Qw, q, kz) \geq \psi\left\{\min\begin{bmatrix}F(Sw, Qw, z),\\ F(Tq, Rq, z),\\ F(Tq, Qq, rz),\\ F(Sw, Rq, (2-r)z),\\ F(Sw, Tq, z)\end{bmatrix}\right\}.$$

$$\geq \psi \left\{ \min \begin{bmatrix} F(q, Qw, z), \\ F(Tq, q, z), \\ F(q, Qw, (1-p)z), \\ F(Sw, q, (1+p)z), \\ F(q, Tq, z) \end{bmatrix} \right\}.$$

$$\geq \psi \left\{ \min \begin{bmatrix} F(q, Qw, z), \\ F(Tq, q, z), \\ F(Qw, q, (1-p)z), \\ F(Sw, q, (1+p)z), \\ F(q, Tq, z) \end{bmatrix} \right\}.$$

$$\geq \psi \left\{ \min \begin{bmatrix} F(q, Qw, z), \\ F(q, q, z), \\ F(Qw, Sw, (1-p+1+p)z) \end{bmatrix} \right\}.$$

$$\geq \psi \left\{ \min \left[F(q, Qw, z), F(Qw, q, 2z) \right] \right\}.$$

Therefore,

$$F(Qw, q, kz) \geq \psi \left\{ F(q, Qw, z) \right\}.$$

Or,

$$F(Qw, q, kz) \geq F(q, Qw, z), \text{ by property of } \psi.$$

which implies $Qw = q$ by Lemma 2.19.

Again, since Q and S are compatible mappings of type (P) and $Qw = Sw = q$, by proposition 2.15, we have for every $\epsilon > 0$

$$1 = F(QQw, SSw, \epsilon) \geq F(Qw, Sw, \epsilon).$$

Hence $Qw = QQw = SSw = Sw$.

Finally, by relation (3.1.4) with $x = q, y = Rq = q$, we have

$$F(Qq, q, kz) = F(Qq, Rq, kz) \geq \psi \left\{ \min \begin{bmatrix} F(Sq, Qq, z), \\ F(Tq, q, z), \\ F(Tq, Qq, rz), \\ F(Sq, q, (2-r)z), \\ F(Sq, Tq, z) \end{bmatrix} \right\}.$$

$$\geq \psi \left\{ \min \begin{bmatrix} F(Qq, Qq, z), \\ F(q, q, z), \\ F(q, Qq, rz), \\ F(Qq, q, (2-r)z), \\ F(Qq, q, z) \end{bmatrix} \right\}.$$

$$\geq \psi \left\{ \min \begin{bmatrix} F(Qq, q, rz), \\ F(q, Qq, (2-r)z), \\ F(Qq, q, z) \end{bmatrix} \right\}.$$

$$\geq \psi \left\{ \min \left[F(Qq, Qq, rz + (2-r)z), F(Qq, q, z) \right] \right\}.$$

$$\geq \psi \left\{ \min \left[F(Qq, q, z) \right] \right\}.$$

$$\geq \psi \left\{ F(Qq, q, z) \right\}.$$

Or,

$$F(Qq, q, kz) \geq F(Qq, q, z), \text{ by property of } \psi.$$

Thus, $Qq = q$, by Lemma 2.19.

Hence,

$$Qq = Rq = Sq = Tq = q.$$

That is, q is a common fixed point of the given mappings $Q, R, S,$ and T.

Uniqueness: Suppose z_1 is another point in X such that

$$z_1 = Qz_1 = Rz_1 = Sz_1 = Tz_1.$$

Then, putting $x = q$ and $y = z_1, r = 1$ in (3.1.4), we get

$$F(Qq, Rz_1, kz) = F(q, z_1, kz) \geq \phi \left\{ \min \begin{bmatrix} F(Sq, Qq, z), \\ F(Tz_1, Rz_1, z), \\ F(Tz_1, Qq, z), \\ F(Sq, Tz_1, z) \end{bmatrix} \right\}.$$

Or,

$$F(q, z_1, kz) \geq \phi \left\{ \min \left[F(q, z_1, z), F(q, q, z) \right] \right\}.$$

Or,

$$F(q, z_1, kz) \geq \phi \{F(q, z_1, z)\}.$$

$$F(q, z_1, kz) \geq F(q, z_1, z), \text{ by property of } \phi.$$

Thus, $q = z_1$, by Lemma 2.19.

Hence,

$$q = Qq = Rq = Sq = Tq,$$

and q is the unique common fixed point for $Q, R, S,$ and T in X.

This completes the proof. □

Theorem 3.2. *Let (X, F, t) be a complete Menger space with continuous $t(x, y) = \min\{x, y\}$ for all $x, y \in [0, 1]$, and let $Q, R, S, T : X \to X$ be four self-mappings such that:*

(i) $Q(X) \subset T(X)$ and $R(X) \subset S(X)$,

(ii) the pairs (Q, S) and (R, T) are compatible mappings of type (K),

(iii) S and T are continuous,

(iv) there exists a constant $k \in (0, 1)$ such that for every $\epsilon \in (0, 1)$, there exists $\delta \in (0, \epsilon]$ such that:

$$\epsilon - \delta < F(x, y, t) < \epsilon \implies F(Qx, Ry, kt) \geq \epsilon \quad \text{and} \quad F(Qx, Ry, kt) \geq F(x, y, t),$$

where:

$$F(x, y, t) \geq \psi\{\min\{F(Sx, Ty, t), F(Qx, Sx, t), F(Ry, Ty, t), F(Qx, Ty, t)\}\}$$

for all $x, y \in X$, and $t > 0$, where $\psi : [0, 1] \to [0, 1]$ satisfies:

- *ψ is continuous and non-decreasing on $[0, 1]$,*
- *$\psi(n) > n$ for all $n \in [0, 1]$.*

noting that if $\psi \in \Psi$, class of all mappings $\psi : [0, 1] \to [0, 1]$ then $\psi(0) = 0$, $\psi(1) = 1$, $\psi(n) \geq n$, for all $n \in [0, 1]$.

Then, Q, R, S, T have a unique common fixed point in X.

Proof. Consider $x_0 \in X$. From condition (i), we have $Q(X) \subset T(X)$ and $R(X) \subset S(X)$. Thus, there exists a point $x_1 \in X$ such that $Qx_0 = Tx_1$. Similarly, for $x_1 \in X$, there exists

$x_2 \in X$ such that $Rx_1 = Sx_2$ and so on. Inductively, we construct a sequence $\{y_n\}$ in X such that:

$$y_{2n-1} = Qx_{2n-2} = Tx_{2n-1}, \quad y_{2n} = Rx_{2n-1} = Sx_{2n}, \quad \text{for all } n = 1, 2, 3, \ldots$$

Substituting $x = x_{2n}$ and $y = x_{2n+1}$ in condition (iv), we get:
$F(y_{2n+1}, y_{2n+2}, kt) = F(Qx_{2n}, Rx_{2n+1}, kt)$
$\geq \psi\{\min\{F(Sx_{2n}, Tx_{2n+1}, t), F(Qx_{2n}, Sx_{2n}, t), F(Rx_{2n+1}, Tx_{2n+1}, t), F(Qx_{2n}, Tx_{2n+1}, t)\}\}$

$\geq \psi\{\min\{F(y_{2n}, y_{2n+1}, t), F(y_{2n+1}, y_{2n}, t), F(y_{2n+2}, y_{2n+1}, t), F(y_{2n+1}, y_{2n+1}, t)\}\}$

$\geq \psi\{\min\{F(y_{2n}, y_{2n+1}, t), F(y_{2n+1}, y_{2n+2}, t)\}\}$

or, $F(y_{2n+1}, y_{2n+2}, kt) \geq \psi\{F(y_{2n}, y_{2n+1}, t)\} > F(y_{2n}, y_{2n+1}, t)$, by property of ψ

Thus for every $n \in N$,

$$F(y_n, y_{n+1}, kt) \geq F(y_{n-1}, y_n, t).$$

By Lemma 2.20, $\{y_n\}$ is a Cauchy sequence in X. Since (X, F, t) is complete, $\{y_n\}$ converges to a point $z \in X$. Consequently, the subsequences $\{Qx_{2n-2}\}, \{Rx_{2n-1}\}, \{Sx_{2n}\}, \{Tx_{2n-1}\}$ also converge to z.

Since S and T are continuous, and (Q, S) and (R, T) are compatible mappings of type (K), we deduce:

$$QQx_{2n-2} \to Sz \quad \text{and} \quad SSx_{2n} \to Qz, \quad RRx_{2n-1} \to Tz \quad \text{and} \quad TTx_{2n-1} \to Rz. \qquad ...(1)$$

And from condition (iv), we get

$F(QQx_{2n-2}, RRx_{2n-1}, kt) \geq$
$\psi\{\min\{F(SQx_{2n-2}, TRx_{2n-1}, t), F(QQx_{2n-2}, SQx_{2n-2}, t), F(RRx_{2n-1}, TRx_{2n-1}, t),$
$F(QQx_{2n-2}, TRx_{2n-1}, t)\}\}$

As $n \to \infty$, and by using (1), we have

$$F(Sz, Tz, kt) \geq \psi\{\min\{F(Sz, Tz, t), F(Sz, Sz, t), F(Tz, Tz, t), F(Sz, Tz, t)\}\}$$

or, $F(Sz, Tz, kt) \geq \psi(F(Sz, Tz, t)) \geq F(Sz, Tz, t)$, by the properties of ψ
From lemma 2.19, we get $Sz = Tz$ ...(2)
Again, from condition (iv), we have
$F(Qz, RRx_{2n-1}, kt) \geq$
$\psi\{\min\{F(Sz, TRx_{2n-1}, t), F(Qz, Sz, t), F(Rz, TRx_{2n-1}, t), F(Qz, TRx_{2n-1}, t)\}\}$

Again taking $n \to \infty$, and using (1) and (2), we get

$$F(Qz, Tz, kt) \geq \psi\{\min\{F(Sz, Sz, t), F(Qz, Tz, t), F(Tz, Tz, t), F(Qz, Tz, t)\}\}$$

$or, F(Qz, Tz, kt) \geq \psi(F(Qz, Tz, t)) \geq F(Qz, Tz, t)$, by the properties of ψ
From lemma 2.19, we get $Qz = Tz$ ...(3)
From (2) and (3), we get

$$F(Qz, Rz, kt) \geq \psi\{\min\{F(Sz, Tz, t), F(Qz, Sz, t), F(Rz, Tz, t), F(Qz, Tz, t)\}\}$$

or,

$$F(Qz, Rz, kt) \geq \psi\{\min\{F(Qz, Qz, t), F(Qz, Qz, t), F(Rz, Qz, t), F(Qz, Qz, t)\}\}$$

$or, F(Qz, Rz, kt) \geq \psi(F(Qz, Rz, t)) \geq F(Qz, Rz, t)$, by the properties of ψ
From lemma 2.19, we get $Qz = Rz$ ...(4)
From (2), (3), and (4) we get
$Sz = Qz = Tz = Rz$ ...(5)
Now, we have to show that $Qz = z$
From condition (iv), we have
$F(Qz, Rx_{2n-1}, kt) \geq$
$\psi\{\min\{F(Sz, Tx_{2n-1}, t), F(Qz, Sz, t), F(Rx_{2n-1}, Tx_{2n-1}, t), F(Qz, Tx_{2n-1}, t)\}\}$
taking $n \to \infty$, and using (2) and (3), we get

$$F(Qz, z, kt) \geq \psi\{\min\{F(Sz, z, t), F(Qz, Sz, t), F(z, z, t), F(Qz, z, t)\}\}$$

or,

$$F(Qz, z, kt) \geq \psi\{\min\{F(Qz, z, t), F(Qz, Qz, t), F(z, z, t), F(Qz, z, t)\}\}$$

$or, F(Qz, z, kt) \geq \psi(F(Qz, z, t)) \geq F(Qz, z, t)$, by the properties of ψ
From lemma 2.19, we get $Qz = z$.
Hence, from (5), we get
$z = Qz = Rz = Tz = Sz$, and z is a common fixed point of Q, R, S, T.
Uniqueness: Suppose $w \neq z$ is another common fixed point of Q, R, S, T. Then, $Qw = Rw = Sw = Tw = w$.
Therefore, from condition (iv)
$F(z, w, kt) =$

$$F(Qz, Rw, kt) \geq \psi\{\min\{F(Sz, Tw, t), F(Qz, Sz, t), F(Rw, Tw, t), F(Qz, Tw, t)\}\}$$

or,

$$F(z, w, kt) \geq \psi\{\min\{F(z, w, t), F(z, z, t), F(w, w, t), F(z, w, t)\}\}$$

$or, F(z, w, kt) \geq \psi(F(z, w, t)) \geq F(z, w, t)$, by the properties of ψ
From lemma 2.19, we get $z = w$.
Hence, $z = Qz = Rz = Tz = Sz$, and z is unique in X. This completes the proof. □

4 Conclusion

In this paper, we have explored the concept of Menger spaces and their applications in proving fixed point theorems. We generalized and extended the results of several previous studies, including those by Khan et al. [?] and others [?], [?]. Specifically, we introduced and applied the notion of compatible mappings of types (P) and (K) to derive new fixed point theorems in complete Menger spaces.

These results contribute to the understanding of the topological properties of Menger spaces and provide a framework for future research in metric fixed point theory and its applications in probabilistic spaces.

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