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ESCAPING SET AND BOUNDED ORBIT SET OF HOLOMORPHIC SEMIGROUP ON $\mathbb{C}$

NIRMAL GURUNG [1,*] AND BISHNU HARI SUBEDI [2]

[1 2] *Central Department of Mathematics, Tribhuvan University, Kathmandu, Nepal*
[1] *gurungnirmal476@gmail.com and* [2] *subedi.abs@gmail.com*
[*] *Correspondence to: Bishnu Hari Subedi, Email: subedi.abs@gmail.com*

Abstract: In this paper, we study the structure and properties of escaping sets of holomorphic semigroups. In particular, we study the relationship between escaping set of holomorphic semigroup and escaping set of each function that lies in that semigroup. We also study about the invariantness of escaping sets.
Also, in this paper, we define the term bounded orbit set $K(H)$ and the set $K'(H)$ of holomorphic semigroup H. Then we study their invariantness and their relations with escaping sets. We also construct a particular class of holomorphic semigroups generated by two holomorphic functions such that bounded orbit set of holomorphic semigroup is equal to bounded orbit set of its generators.
Key Words: Holomorphic Function, Holomorphic Semigroup, Escaping set of Holomorphic Semigroup, Bounded Orbit Set of Holomorphic Semigroup.
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1. Introduction

Let $\mathbb{C}$ denotes the set of complex numbers, $\mathbb{Z}$ denotes the set of integer numbers, $\mathbb{Z}_{\geq 0}$ denotes the set of non-negative integer numbers and $\mathbb{N}$ denotes the set of natural numbers. Let $h : \mathbb{C} \to \mathbb{C}$ be a holomorphic function on $\mathbb{C}$, an entire function which is either polynomial or transcendental entire. Let $h^{\circ}(z) = z$ and define

$$h^n(z) = h(h^{n-1}(z)),\ n \in \mathbb{N}.$$

Then $h^n(z)$ is called n^{th} iteration of h with itself.
We say a holomorphic family $\mathcal{H}$ of holomorphic functions is normal in some domain $D \subset \mathbb{C}$ if every sequence in $\mathcal{H}$ has a subsequence that locally uniformly converges to a holomorphic function or locally uniformly diverges to ∞ on D. We say $\mathcal{H}$ is normal at $z \in D$, if there exists a neighborhood $N(z)$ such that throughout the neighborhood the family $\mathcal{H}$ is normal [12].
The dynamics of holomorphic function was originated in early 20th century with the independent work of Pierre Fatou and Gaston Julia [3, 12]. Both of them were motivated from

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Montel's theory of normal family [12]. Due to Fatou and Julia [3, 12], the Fatou set and Julia set of holomorphic function h is defined as

$$F(h) = \{z \in \mathbb{C} : \{h^n : n \in \mathbb{N}\} \textit{ is normal at } z\},\ J(h) = \mathbb{C} - F(h) \textit{ respectively.}$$

The escaping set and bounded set of holomorphic function h are denoted by $I(h)$, $K(h)$ and defined by

$$I(h) = \{z : h^n(z) \to \infty \textit{ as } n \to \infty\},$$
$$K(h) = \{z : \{h^n(z) : n \in \mathbb{Z}_{\geq 0}\} \textit{ is bounded}\}$$

respectively [1, 11].

Also, the bungee set of h is denoted by $BU(h)$ and defined by

$$BU(h) = \mathbb{C} - \{I(h) \cup K(h)\}[9].$$

In the beginning the escaping set of polynomial function is treated as a Fatou component of the function that escapes towards infinity. But the formal definition of escaping set was firstly given by Eremenko in 1989 [1]. In the paper, it is proved that for and transcendental entire function h, $I(h) \neq \emptyset, I(h) \cap J(h) \neq \emptyset, J(h) = \partial I(h)$. In the same paper, it is proved that each component of $\overline{I(h)}$ is unbounded and formulated a conjecture known as Eremenko conjecture which states that for a transcendental entire function h, each component of $I(h)$ is unbounded. In 2022, Rempe and et al. [8], solved the conjecture negatively. The set $K(h)$ was first studied by Bergweiler in 2012 [11] and proved that $K(h) \neq \emptyset, K(h) \cap J(h) \neq \emptyset, J(h) = \partial K(h)$. In particular, the bounded orbit set of polynomial function is also known as filled Julia set[3]. In 1987, Eremenko and Lyubich [2] showed the existence of the bungee set $BU(h)$ of transcendental entire function h and later in 2015, Osborne and Sixsmith [9] formally introduced the notion of bungee set and they proved that for polynomial P, $BU(P) = \emptyset$ and for any transcendental entire function h, $BU(h) \neq \emptyset$ and $BU(h) \cap J(h) \neq \emptyset, J(h) = \partial BU(h)$. Also, the sets $I(h),\ K(h),\ BU(h)$ all are completely invariant [1, 9, 11].

One of the partition of the complex plane is Fatou set and Julia set [3,12]. Another partition of complex plane is escaping, bounded orbit and bungee sets. But for the polynomial, escaping set and bounded orbit set form a partition of complex plane [9].

The dynamics of non cyclic holomorphic semigroup was firstly studied by Hinkkanen and Martin [4] in 1996. They study the Fatou and Julia set of holomorphic semigroup generated by rational functions. Later in 1998, Poon [10] studied the Fatou and Julia sets of holomorphic semigroup generated by transcendental entire functions. Due to them [4, 10], the Fatou and Julia set of holomorphic semigroup H are denoted by $F(H)$ and $J(H)$ and defined by

$$F(H) = \{z :\ H \textit{ is nomal at } z\} \textit{ and } J(H) = \mathbb{C} - F(H) \textit{ respectively.}$$

The escaping set of holmorphic semigroup was firstly studied by Kumar and Kumar [6] in 2016. The escaping set of holomorphic semigroup H due to Kumar and Kumar [6] is defined as

$$I'(H) = \{z : \forall\, (g_n) \subset H\ \exists \textit{ subsequence } (h_{n_k}) \textit{ of } (h_n) \textit{ s.t. } h_{n_k}(z) \to \infty \textit{ as } n_k \to \infty\}$$

They [6] proved that $I'(H)$ may be empty and $I'(H) \subset \bigcap_{h \in H} I(h)$. Also, they [6] proved that for any transcendental semigroup, it is forward invariant and later Kumar et.al. [7] proved that for any abelian transcendental semigroup escaping set is backward invariant. Also, they [7] proved that $J(H) \cap I'(H) \neq \emptyset$ and $J(H) = \partial I'(H)$ where $J(H)$ is Julia set of transcendental entire semigroup H.
Later, Subedi [5] introduced another version of escaping set of holomorphic semigroup. Due to Subedi the escaping set of holomorphic semigroup H is defined by

$$I(H) = \{z : \forall h \in H \,, h^n(z) \to \infty \; as \; n \to \infty\}.$$

He [5] proved that his definition is not logically equivalent with the definition of Kumar and Kumar. He [5] also proved that $I(H) \subset \bigcap_{h \in h} I(h)$, $J(H) = \partial I(H)$, $J(H) \cap I(H) \neq \emptyset$.
Also, he [5] proved the following lemma.

Lemma 1.1. *Let H be a holomorphic semigroup such that $I(H) \neq \emptyset$ and $z \in I(H)$. Then for every non convergent sequence in S has a sub-sequence that diverges to ∞ at z.*

Then by using this lemma, he [5] proved that escaping set is forward invariant for any holomorphic semigroup and backward invariant for any abelian holomorphic semigroup.
In this paper, we critically studied the structures and properties of escaping set holomorphic semigroup due to Kumar and Kumar and due to subedi. Also, we extend the bounded orbit set of individual function into the bounded orbit set of holomorphic semigroup. Then we observe its basic structures and properties.

Definition 1.2 (Iteratively Bounded)**.** we say holomorphic semigroup H is iteratively bounded at z if for each $h \in H$ the orbit $\{h^n(z) : n \in \mathbb{Z}_{\geq 0}\}$ is bounded.

Definition 1.3 (Bounded Orbit Set)**.** We denote the bounded orbit set of holomorphic semigroup H by $K(H)$ and defined by

$$K(H) = \{z \in \mathbb{C} : H \; is \; iteratively \; bounded \; at \; z\}.$$

Example 1.4. Let H be a holomorphic semigroup genrated by two holomorphic functions z^2 and z^3. Then $K(H) = \overline{\mathbb{D}}$.

Example 1.5. Let $H = < h_k >, \; h_k = e^k e^z, \; k \in \mathbb{Z}$. Then $K(H) = \emptyset$.

Example 1.6. Let $H = < sin\, z, cos\, z >$. Then the set $\{z \in \mathbb{C} : h^n(z) \in \mathbb{R}\, for\, some\, h \in H\, and\, for\, some\, n \in \mathbb{Z}_{\geq 0}\}$ lies in $K(H)$.

Again we define another set which is subset of bounded orbit set.

Definition 1.7. The set $K'(H)$ of holomorphic semigroup H is defined by

$$K'(H) = \{z : \forall \; sequence \; (h_n) \subset H, \; \{h_n(z) : n \in \mathbb{N}\} \; is \; bounded\}.$$

2. Main Result

We figure out the following results concern to escaping sets, bounded orbit set and the set $K^{'}(H)$ of holomorphic semigroup H.

Remark 2.1. Let H be a holomorphic semigroup. Then the relation $I'(H) \subset \bigcap_{h\in H} I'(h)$ may NOT hold in general.

Proof. For this, let $z \in I'(H)$. Then we need to show that $\forall\ h \in H,\ z \in I(H)$. So, let $h \in H$. Then (h^n) is a sequence in H. Then by definition there exists a subsequence (h^{n_k}) of (h^n) such that $h^{n_k}(z) \to \infty$ as $n_k \to \infty$. But this does not imply that $h^n(z) \to \infty$ as $n \to \infty$ because orbit of z under the map f may be neither escape nor bounded. That is, $z \in BU(h)$. □

Proposition 2.2. *Let H be a holomorphic semigroup. Then $I(H) = \bigcap_{h\in H} I(h)$.*

Proof. It is trivial because
$z \in I(H) \iff \forall\ h \in H,\ h^n(z) \to \infty$ as $n \to \infty$. □

Now we construct a counter example to the lemma 1.1.

Example 2.3 (A counterexample to the lemma 1.1)**.** Let $H =< h_k >$ where $h_k(z) = e^{kz},\ k \in \mathbb{N}$. Then $z = i\pi$ lies in $I(H)$ but there exists a non convergent sequence in H such that every sub-sequence of that sequence is bounded at z.

Proof. Since each sequence $(g_n) \subset H$, $g_n^k(i\pi) \to \infty$ as $k \to \infty$, $z = i\pi \in I(H)$. Let consider a sequence $(h_k(z))$. Let $z = i\pi$, then $(h_k(z)) = (h_k(i\pi)) = \{-1,\ 1\ -1,\ \dots\}$. This sequence has no sub-sequence that diverges to ∞. This contradicts lemma 1.1. □

Proposition 2.4. *Let H be an abelian transcendental holomorphic semigroup. Then $I(H)$ is backward invariant.*

Here, we prove the same result proposition 2.4 without using lemma 1.1.

Proof. It is sufficient to show that

$$\forall\ h \in H,\ h^{-1}(I(H)) \subset I(H).$$

Equivalently,

$$z \notin I(H) \Rightarrow h(z) \notin I(H).$$

So, let $z \notin I(H)$. Then there exists $g \in H$ such that $g^n(z) \not\longrightarrow \infty$ as $n \to \infty$. Then there exists a subsequence (g^{n_k}) of (g^n) such that $\{g^{n_k}(z)\}$ is bounded. Then for all $h \in H$, $\{h(g^{n_k})(z)\}$ is bounded. Since H is abelian, $h \circ g^{n_k} = g^{n_k} \circ h$. Thus $\{g^{n_k}(h(z))\}$ is also bounded. Hence $g^{n_k}(h(z)) \not\longrightarrow \infty$ as $n_k \to \infty$.
This implies that $g^n(h(z)) \not\longrightarrow \infty$ as $n \to \infty$. Thus $h(z) \notin I(H)$. □

We give a counter example to the forward invariantness of $I(H)$.

Example 2.5 (A Counter Example). Let $G =< g, h >$ be a holomorphic semigroup where $g(z) = 1 + z + e^{-z}$ and $h(z) = e^z$. Then $I(H)$ is NOT forward invariant.

Proof. Choose a point $z = ln\pi + i\frac{\pi}{2}$. Then $z \in I(H)$ and $h(z) = i\pi$ which is a fixed point of g. So, for each $n \in \mathbb{N}$, $g^n(h(z)) = i\pi$. This implies that $h(z) \notin I(g)$. Thus $I(H)$ is NOT forward invariant. □

Now we discuss some structures and properties of $K(H)$ and $K'(H)$.

Proposition 2.6. *Let H be a holomorphic semigroup. Then $\forall\ h \in H, K(H) \subset K(h)$ and $K(H) = \bigcap_{h\in H} K(h)$.*

Proof. Let $z \in K(H)$. Then H is iteratively bounded at z. That is, for all $f \in H$, $\{h^n(z)\}_{n\geq 0}$ is bounded. Thus for all $h \in H$, $z \in K(H)$. Hence $K(H) \subset K(h)$.
This implies that $K(H) \subset \bigcap_{h\in H} K(h)$. Conversely, let $z \in \bigcap_{h\in H} K(h)$. Then for all $h \in H$, orbit of z under H is bounded. So, H is iteratively bounded at z. Hence $z \in K(H)$.Thus, $\bigcap_{h\in H} K(h) \subset K(H)$. □

Proposition 2.7. *For any holomorphic semigroup H, $K(H) \subset I(H)^c$.*

Proof. We have,

$$I(H)^c = \{z :\ \exists\ h \in H\ s.t.\ h^n(z) \not\to \infty\ as\ n \to \infty\}.$$

That is,

$$I(H)^c = \{z :\ \exists\ h \in H\ with\ z \in K(h) \cup BU(h)\}.$$

Now, let $z \in K(H)$. Then $\forall\ h \in H$, $z \in K(h)$. This implies that $z \in I(H)^c$. □

But $K(H) \neq I(H)^c$. For this, we have the following Example.

Example 2.8. Let $H =< g, h >$ where $g(z) = e^z$ and $h(z) = e^{-z}$. Then $K(H) \neq I(H)^c$.

Proof. We have, $H =< g, h >$ where $g(z) = e^z$ and $h(z) = e^{-z}$. Choose a point $z = 0$. Then, since $0 \in K(h)$, so $0 \in I(H)^c$. But $0 \in I(g)$. So $0 \notin K(H)$. Thus $K(H) \neq I(H)^c$. □

Similar result holds for $I'(H)$. That is,

Proposition 2.9. *For any holomorphic semigroup H, $K(H) \subset I'(H)^c$.*

Proof. We have,

$$I'(H)^c = \{z :\ \exists\ a\ sequence\ (h_n)\ s.t.\ \forall\ subsequences\ (h_{n_k})\ of\ (h_n),\ h_{n_k}(z) \not\to \infty\ as\ n_k \to \infty\}.$$

Now, let $z \in K(H)$. Then $\forall\ h \in H$, sequence (h^n) is bounded at z. So, every subsequence of (h^n) is bounded at z. That is, every subsequence (h^{n_k}) of (h^n) is not escape to infinity as $n_k \to \infty$. Thus, $z \in I'(H)^c$. □

But, example 2.8. show that $K(H) \neq I'(H)^c$.

Proposition 2.10. *$K(H)$ is forward invariant if H is abelian holomorphic semigroup.*

Proof. We need to show $\forall\ h \in H$, $h(K(H)) \subset K(H)$. That is, $z \in K(H) \Rightarrow h(z) \in K(H)$. For this, let $z \in K(H)$.Then $\forall g \in H$ such that (g^n) is bounded at z. This implies that $\forall h \in H$, the sequence $(h \circ g_n)$ is bounded at z since each $h \in H$ is a holomorphic function in $\mathbb{C}$. Since H is abelian, $h \circ g^n = g^n \circ h$. So, $(g^n \circ h)$ is bounded at z. That is, $g^n(h(z))$ is bounded. Thus $h(z) \in K(H)$. □

Proposition 2.11. *Let H be a holomorphic semigroup. Then $K'(H) \subset K(H)$.*

Proof. For this, let $z \in K'(H)$. Then by definition, $\forall\ (g_n) \subset H$, $\{g_n(z)\}$ is bounded. In particular, $\forall h \in H$, (h^n) is a sequence in H. So, by definition $\{h^n(z)\}$ is bounded. So, $z \in K(H)$. □

But $K'(H)$ is proper subset of $K(H)$.

Example 2.12. Let $H =< \{h_k\} >$ where $h_k(z) = e^{-k\,z},\ k \in \mathbb{N}$. Then $-1 \in K(H)$ but $-1 \notin K'(H)$.

Proof. Since at $z = -1$, $\forall\ g \in H$, $\{g^n(-1)\}$ is bounded. So, $-1 \in K(H)$. But at $z = -1$, the sequence

$$\{h_k(-1)\} = \{e^1, e^2, e^3, e^4, \dots\}$$

is unbounded. So, $-1 \notin K'(H)$. □

Proposition 2.13. *For any holomorphic semigroup H, $K'(H) \subset I'(H)^c$ and $K'(H) \subset I(H)^c$.*

Proof. Since $K'(H) \subset K(H)$ and $K(H) \subset I'(H)^c$. Thus, $K'(H) \subset I(H)^c$. Similarly, since $K(H) \subset I(H)^c$, so $K'(H) \subset I(H)^c$. □

But $K'(H) \neq I'(H)^c$ and $K'(H) \neq I(H)^c$. For this, choose the Example 2.8
Now we discuss about the invariant properties of $K'(H)$.

Proposition 2.14. *Let H be a holomorphic semigroup. Then $K'(H)$ is forward invariant.*

Proof. We need to that $\forall h \in H$, $h(K'(H)) \subset K'(H)$. For this, let $z \in K'(H)$ and (g_n) be a sequence in H. Then $\{g_n \circ h : h \in H\}$ is sequence in H. Then by definition, $(g_n \circ h)$ is bounded at z. That is, $\{g_n \circ h)(z)\} = \{g_n(h(z))\}$ is bounded. Since (g_n) and h were chosen arbitrary, so $\forall\ h \in h$, $h(z) \in K'(H)$. □

We investigate the relationship between the bounded orbit set and its generators for some special type of functions as follows.

Lemma 2.15. *For any holomorphic function h, $K(h) \subset K(h^k),\ k \in \mathbb{N}$.*

Proof. Let $z \in K(h)$. Then $\{h^n(z) : n \in \mathbb{Z}_{\geq 0}\}$ is bounded. This implies that for every $k \in \mathbb{N}$, $\{h^{kn}(z) = (h^k)^n(z) : n \in \mathbb{Z}_{\geq 0}\}$ is bounded because it is a subsequence of bounded sequence. Thus $z \in K(h^k)$. □

Proposition 2.16. *For any transcendental holomorphic semigroup H generated by two holomorphic functions g and h where g is periodic with periodicity p and $h = g + p$, we have*

$$K(H) = k(g) = k(h).$$

Proof. Since $\forall\ z \in \mathbb{C},\ h(z) = g(z) + p$, we have $h^2(z) = g^2(z) + p$. This implies that

$$\forall\ m \in \mathbb{N},\ h^m(z) = g^m(z) + p.$$

This implies that

$$z \in K(g) \Leftrightarrow z \in K(h).$$

Thus $K(g) = K(h)$. Also
$\forall\ n_1, n_2 \in \mathbb{Z}_{\geq 0}$ with $n_1 = 0$ & $n_2 = 0$ does NOT hold simultaneously,

$$(g^{n_1} \circ h^{n_2})(z) = g^{n_1+n_2}(z)\ or\ (h^{n_1} \circ g^{n_2})(z) = g^{n_1+n_2}(z) + p.$$

This implies that

$$\forall\ f \in H,\ f = g^k\ or\ f = g^k + p = h^k\ for\ some\ k \in \mathbb{N}.$$

So,

$$\forall f \in H,\ K(f) = K(g^k) =\ or\ K(f) = K(h^k) = K(g^k)\ for\ some\ k \in \mathbb{N}$$

Thus

$$K(H) = \bigcap_{k\in\mathbb{N}} K(g^k) = \bigcap_{k\in\mathbb{N}} K(h^k).$$

Hence from lemma 2.1

$$K(H) = \bigcap_{k\in\mathbb{N}} K(g^k) = K(g) = \bigcap_{k\in\mathbb{N}} K(h^k) = K(h).$$

□

We can generalize proposition 2.16 as follows.

Proposition 2.17. *For any holomorphic semigroup H generated by two holomorphic functions g and h where g is periodic with periodicity p and $h = g^n + p,\ n \in \mathbb{N}$, we have*

$$K(h) = K(g^n)\ \&\ K(H) = K(g).$$

Proof. Since $\forall\ z \in \mathbb{C},\ h(z) = g^n(z) + p$, we have $h^2(z) = g^{2n}(z) + p$. This implies that

$$\forall\ m \in \mathbb{N},\ h^m(z) = g^{nm}(z) + p.$$

So

$$z \in K(h) \Leftrightarrow z \in K(g^n).$$

Thus $K(h) = K(g^n)$. Also

$$\forall\ a, b \in \mathbb{Z}_{\geq 0},\ (g^a \circ h^b)(z) = g^{a+bn}(z)\ or\ (h^a \circ g^b)(z) = g^{an+b}(z) + p.$$

This implies that

$$\forall\ f \in H,\ f = g^{a+bn}\ or\ f = g^{a+bn} + p = h^{a+bn}\ for\ some\ a, b \in \mathbb{Z}_{\geq 0}.$$

So

$$\forall\ f \in H,\ K(f) = K(g^{a+bn})\ or\ K(f) = K(h^{a+bn})\ for\ some\ a, b \in \mathbb{Z}_{\geq 0}$$

Since, from lemma 2.1 $K(g) \subset K(g^{a+bn}) \subset K(g^{(a+bn)n}) = K(h^{a+bn})$, we have

$$K(H) = \bigcap_{a,b \in \mathbb{Z}_{\geq 0}} K(g^{a+bn}) = K(g).$$

$\square$

3. Conclusion

We showed that the escaping set due to Kumar and Kumar may not be subset of intersection of escaping sets of each holomorphic function of that semigroup. Also, We concluded that lemma 1.1 does not hold in general. Similarly, we investigated that escaping set of abelian holomorphic semigroup is backward invariant and escaping set of holomorphic semigroup may not be forward invariant. We also generalized the concept of bounded orbit set of holomorphic function into the bounded orbit set of holomorphic semigroup. Then we investigated that bounded orbit set of holomorphic semigroup is equal to the intersection of bounded orbit of set of each element of the semigroup. Also we investigated that bounded orbit set of abelian holomorphic semigroup is forward invariant. We concluded that bounded orbit set is proper subset of complement of escaping set due to Subedi and due to Kumar of holomorphic semigroup. We also defined a proper subset $K'(H)$ of bounded orbit set and concluded that this set is forward invariant. We investigated that this subset is also proper subset of complement of escaping set due to subedi and Kumar of holomorphic semigroup. At last, we constructed a particular class of holomorphic semigroups generated by two holomorphic functions such that bounded orbit set of holomorphic semigroup is equal to bounded orbit set of its generators.

4. Further Plan

In the near future, we will study modern holomorphic dynamical system more critically. Our research will focus to response the following queries:

(1) What about the backward invariant of $K(H)$ and $K'(H)$?
(2) Does $K(H)$ intersect $J(H)$?
(3) Does boundary of $K(H)$ equal $J(H)$?
(4) What about the relationship between bounded orbit set, escaping set, Fatou set and Julia set of holomorphic semigroup ?
(5) What about the structures and properties of bungee set of holomorphic semigroup ?

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