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A LOWER BOUND IN A LAW OF THE ITERATED LOGARITHM FOR SUMS OF SYMMETRIC AND INDEPENDENT RANDOM VARIABLES

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Abstract: The law of the iterated logarithm for tail sum, abbreviated Tail LIL, was first introduced by R. Salem and S. Zygmund for sums of lacunary series. Tow and Teicher later introduced a corresponding result for independent random variables. Our article takes a different approach and focuses on obtaining one sided Tail LIL for the sums of independent and identically distributed symmetric random variables.

Key Words: Borel-Canttelli Lemma, law of the iterated logarithm, symmetric random variables
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1. Introduction

The law of the iterated logarithm (LIL) is a widely recognized limit law in the fields of mathematics and statistics and it describes the asymptotic behavior of different sums of random variables. Various versions of LIL exist in different mathematical contexts. The initial formulation of the law was presented by A. Khintchine [6] for Bernoulli's random variable, and later Kolmogorov [7] extended his result to independent random variables. Following the introduction of Kolmogorov's LIL, mathematicians began exploring its application in numerous other areas of analysis. There are several contexts where researchers are currently investigating the LIL. See, for example, [1] for various directions of the LIL. Considerable literature on the LIL is available, providing a wealth of information in this field. In the standard version of the LIL, we consider finite sums and the resulting LIL is called regular LIL. However, the LIL also takes into account the tail sums of a series and is popularly called the tail LIL. The first tail LIL result was obtained by R. Salem and A. Zygmund [8] for the sums of lacunary trigonometric series.

Theorem 1.1 (Salem and Zygmund, 1950)**.** *Suppose a lacunary series* $\tilde{S}_N(\theta) = \sum_{k=N}^{\infty}(a_k \cos n_k\theta + b_k \sin n_k\theta)$ *where* $c_k^2 = a_k^2 + b_k^2$ *satisfies* $\sum_{k=1}^{\infty} c_k^2 < \infty$*. Define* $\tilde{B}_N = \left(\frac{1}{2}\sum_{k=N}^{\infty} c_k^2\right)^{\frac{1}{2}}$ *and* $\tilde{M}_N = \max_{k\geq N}|c_k|$*. Suppose that* $\tilde{B}_1 < \infty$ *and that* $\tilde{M}_N^2 \leq$

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$K_N\left(\frac{\tilde{B}_N^2}{\ln\ln\frac{1}{\tilde{B}_N}}\right)$ *for some sequence of numbers* $K_N\downarrow 0$ *as* $N\to\infty$. *Then*

$$\limsup_{N\to\infty}\frac{\tilde{S}_N(\theta)}{\sqrt{2\tilde{B}_N^2\ln\ln\frac{1}{\tilde{B}_N}}}\leq 1$$

for almost every θ *in the unit circle.*

The other direction of the tail LIL for the sums of lacunary series was obtained by Ghimire and Moore [5].

Theorem 1.2 (Ghimire and Moore, 2012). *Let* $S_m(x)=\sum_{k=1}^m a_k\cos(2\pi n_k x)$ *such that* $\frac{n_{k+1}}{n_k}\geq q>1$ *and* $\sum_{k=1}^\infty a_k^2<\infty$. *Assume that* $\max_{k\geq N}a_k^2=o\left(\frac{\frac{1}{2}\sum_{k=N}^\infty a_k^2}{\ln\ln\frac{1}{\sqrt{\frac{1}{2}\sum_{k=N}^\infty a_k^2}}}\right)$. *Then for almost every* x,

$$\limsup_{n\longrightarrow\infty}\frac{|\sum_{k=n}^\infty a_k\cos(2\pi n_k x)|}{\sqrt{2\quad\frac{1}{2}\sum_{k=n}^\infty a_k^2\quad\ln\ln\frac{1}{\sqrt{\frac{1}{2}\sum_{k=n}^\infty a_k^2}}}}\geq 1.$$

In what follows, random variables means variables that are independent, symmetric and bounded and are identically distributed. Chow and Teicher [2] initially obtained a variant of the tail LIL for random variable. S. Ghimire [4] achieved the following onesided version of the tail LIL for random variables using an approach different from that of the Chow and Teicher.

Theorem 1.3 (Ghimire, 2014). *Let* $\{Y_i\}_{i=1}^\infty$ *be a sequence of random variables which are independent, symmetric, bounded and identically distributed with mean zero and variance one such that* $-1\leq Y_i\leq 1$, *and suppose* $\{a_i\}_{i=1}^\infty$ *satisfies* $\sum_{i=1}^\infty a_i^2<\infty$. *Then*

$$\limsup_{m\to\infty}\frac{\sum_{i=m}^\infty a_iY_i(t)}{\sqrt{2\sum_{i=m}^\infty a_i^2\ln\ln\frac{1}{\sum_{i=m}^\infty a_i^2}}}\leq 1$$

for a.e. $t\in[0,1)$.

In this paper, we derive an onesided tail LIL with some additional hypotheses. It is worth mentioning that our approach differs from the methodology employed by Chow and Teicher. Our main result is the following theorem.

Theorem 1.4. *Let* $\{Y_i\}_{i=1}^\infty$ *be a sequence of random variables which are independent, symmetric, bounded and identically distributed with mean* 0 *and variance* 1 *such that* $-1\leq Y_i\leq 1$, $X_i=a_iY_i$ *and suppose* $\{a_i\}_{i=1}^\infty$ *be such that* $\sum_{i=1}^\infty a_i^2<\infty$ *and for all* $\varepsilon>0$, *there exists* M *such that* $m\geq M$, $a_m^2<\varepsilon^2\sum_{i=m}^\infty a_i^2$, *then for a.e.* t, *we have*

$$\limsup_{m\to\infty}\frac{\sum_{i=m}^\infty a_iY_i(t)}{\sqrt{2\sum_{i=m}^\infty a_i^2\left(\ln\ln\frac{1}{\sum_{i=m}^\infty a_i^2}\right)}}\geq 1.$$

2. Preliminaries

For the proof of our main result, some definitions and estimates are in order.

Definition 2.1 (Martingales). Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of random variables on the measure space $(\Omega, \mathfrak{F}, \mu)$ and $\{\mathfrak{F}_n\}_{n=1}^{\infty}$ be a sequence of $\sigma-$algebras such that $\mathfrak{F}_n \subseteq \mathfrak{F}$ for all n. Then the sequence $\{(X_n, \mathfrak{F}_n)\}_{n=1}^{\infty}$ is said to a martingale if it satisfies the following conditions:

(i) $\mathfrak{F}_n \subseteq \mathfrak{F}_{n+1}$ for all n;
(ii) X_n is measurable with respect to $\mathfrak{F}_n$;
(iii) $\mathbb{E}(|X_n|) < \infty$;
(iv) $\mathbb{E}(X_{n+1}|F_n) = X_n$ a.e. on Ω.

Here a.e. stands for almost everywhere equal. The sequence is called a submartingale if and only if the equality in (iv) is replaced by $\leq$ and the sequence is called a supermartingale if and only if the equality is replaced by $\geq$.

Theorem 2.2 (Hoeffding [3]). *Let $\{Y_i\}_{i=1}^{m}$ be a sequence of independent random variables with mean zero and bounded ranges such that $a_i \leq Y_i \leq b_i$ for all $i = 1, 2, ..m$. Then for each $\lambda > 0$, we have*

$$|\{t : |\textstyle\sum_{i=1}^{m} Y_i(t)| > \lambda\}| \leq 2\exp\left(\dfrac{-2\lambda^2}{\sum_{i=1}^{m}(b_i - a_i)^2}\right)$$

where $|\{.\}|$ denotes the Lebesgue measure.

Theorem 2.3 (Doob's Maximal Identity [3]). *If $\{Y_i\}_{i=1}^{\infty}$ is a sequence of submartingale, then for any $\lambda > 0$ and $k \in \mathbb{N}$, we have*

$$\left|\{t : \max_{1\leq j\leq k} Y_j(t) \geq \lambda\}\right| \leq \frac{1}{\lambda}\mathbb{E}\left(Y_k\right).$$

Theorem 2.4 (Levy's Inequality [3]). *If $X_1, X_2, ...X_n$ be independent and symmetric random variables and put $S_m = X_1 + X_2 + ... + X_m, m \leq n$. Then for all $\lambda > 0$, we have*

$$\left|\{t : \max_{1\leq k\leq m} |S_k(t)| \geq \lambda\}\right| \leq 2\left|\{t : |S_m(t)| \geq \lambda\}\right|$$

and

$$\left|\{t : \max_{1\leq k\leq m} |X_k(t)| \geq \lambda\}\right| \leq 2\left|\{t : |S_m(t)| \geq \lambda\}\right|.$$

We next establish some useful estimates.

Lemma 2.5. *Let $\{Y_i\}_{i=1}^{\infty}$ be as in Theorem 1.4. Then for all $\alpha > 0, \beta > 0$, we have*

$$\left|\left\{t : \sup_{n\geq 1} |\textstyle\sum_{k=1}^{n} X_i(t)| > \beta\right\}\right| \leq AM(\alpha)\exp\left(\dfrac{(-1+\alpha)\beta^2}{2\sum_{i=1}^{\infty} a_i^2}\right).$$

Proof. Let $\eta > 0$ and $\beta > 0$ be any number. Then we have

$$\begin{aligned}
&\left|\left\{t : \sup_{1\le n\le m} \left|\textstyle\sum_{k=1}^n X_k(t)\right| > \beta\right\}\right| \\
&\le \left|\left\{t : \sup_{1\le n\le m} \textstyle\sum_{k=1}^n X_k(t) > \beta\right\}\right| + \left|\left\{t : \sup_{1\le n\le m} -\textstyle\sum_{k=1}^n X_k(t) > \beta\right\}\right| \\
&= \left|\left\{t : \sup_{1\le n\le m} \exp\left(\textstyle\sum_{k=1}^n \eta X_k(t)\right) > e^{\eta\beta}\right\}\right| + \left|\left\{t : \sup_{1\le n\le m} \exp\left(-\textstyle\sum_{k=1}^n \eta X_k(t)\right) > e^{\eta\beta}\right\}\right|.
\end{aligned}$$

Using Doob's maximal inequality, one can show that

$$\left|\left\{t : \sup_{1\le n\le m} \left|\textstyle\sum_{k=1}^n X_k(t)\right| > \beta\right\}\right| \le \frac{2}{e^{\eta\beta}} \int_I \exp\left(\eta\left|\textstyle\sum_{k=1}^m X_k(t)\right|\right) d\mu. \tag{2.1}$$

where μ denotes the Lebesgue measure. Employing Hoeffding's theorem, we have

$$\left|\left\{t : \left|\textstyle\sum_{k=1}^m X_k(t)\right| \ge \beta\right\}\right| \le 2\exp\left(\frac{-\beta^2}{2\sum_{i=1}^m a_i^2}\right). \tag{2.2}$$

For all $\alpha > 0$ and for $I = [0, 1)$, we claim that

$$\int_I \exp\left(\eta\left|\textstyle\sum_{k=1}^m X_k(t)\right|\right) d\mu \le 2\sqrt{2\pi} M(\alpha) \exp\left((1/2 + \alpha)\eta^2 \textstyle\sum_{i=1}^m a_i^2\right).$$

Using Fubini's theorem, one can easily show that

$$\int_I e^g d\mu = \int_{-\infty}^{\infty} e^\beta \mu(\{t : g(t) > \beta\}) d\beta. \tag{2.3}$$

Note that for given $\alpha > 0$, we can find $M(\alpha) > 0$ such that for all $V > 0$,

$$V \exp\left(\frac{1}{2}V^2\right) \le N \exp\left(\left[\frac{1}{2} + \alpha\right] V^2\right). \tag{2.4}$$

Then (2.2) and (2.4) give

$$\int_I \exp\left(\eta\left|\textstyle\sum_{k=1}^m X_k(t)\right|\right) d\mu \le 2\sqrt{2\pi}\eta\sqrt{\textstyle\sum_{i=1}^m a_i^2} \exp\left(\frac{\eta^2 \sum_{i=1}^m a_i^2}{2}\right) \tag{2.5}$$

Let us choose $\eta = \dfrac{\beta}{\sum_{i=1}^m a_i^2}$. Then, from (2.1) and (2.5), we have

$$\begin{aligned}
\left|\left\{t : \sup_{1\le n\le m} \left|\textstyle\sum_{k=1}^n X_k(t)\right| > \beta\right\}\right| &\le \frac{2}{e^{\eta\beta}} \int_I \exp\left(\eta\left|\textstyle\sum_{k=1}^m X_k(t)\right|\right) d\mu \\
&\le \frac{2}{e^{\eta\beta}} 2\sqrt{2\pi} M(\alpha) \exp\left((1/2 + \alpha)\eta^2 \textstyle\sum_{i=1}^m a_i^2\right) \\
&= 4\sqrt{2\pi} M(\alpha) \exp\left((-1/2 + \alpha)\frac{\beta^2}{\sum_{i=1}^m a_i^2}\right).
\end{aligned}$$

Using the continuity of Lebesgue measure, we get

$$\left|\left\{t : \sup_{n\ge 1} \left|\textstyle\sum_{k=1}^n X_k(t)\right| > \beta\right\}\right| \le A(\alpha) \exp\left(\frac{(-1 + \alpha)\beta^2}{2\sum_{i=1}^\infty a_i^2}\right)$$

for some constant $A(\alpha)$ depending on α. □

Lemma 2.6. *Let $\{X_i\}_{i=1}^{\infty}$ be as in Theorem 1.4. Then for $\alpha > 0$, $\beta > 0$, we have*

$$\left|\left\{t : \sup_{n\geq m} \left|\sum_{i=n+1}^{\infty} X_i(t)\right| > \lambda\right\}\right| \leq A(\alpha)\exp\left(\frac{(-1+\alpha)\beta^2}{2\sum_{i=m+1}^{\infty} a_i^2}\right)$$

for some constant $A(\alpha)$.

Proof. Let m be fixed. Define

$$b_i = \begin{cases} 0 & \text{if } i \leq m, \\ a_i & \text{if } i > m. \end{cases}$$

Applying Lemma 2.5 with $\{\sum_{i=1}^{n} b_i Y_i\}$ and using $b_i = 0$ for $i \leq m$, we have

$$\left|\left\{t : \sup_{n\geq m} |\sum_{i=m+1}^{n} a_i Y_i(t)| > \beta\right\}\right| \leq A(\alpha)\exp\left(\frac{(-1+\alpha)\beta^2}{2\sum_{i=m+1}^{\infty} a_i^2}\right).$$

For $n \geq m$, it follows that

$$\left|\left\{t : \sup_{n\geq m} |\sum_{i=1}^{n} X_i(t) - \sum_{i=1}^{m} X_i(t)| > \beta\right\}\right| \leq A(\alpha)\exp\left(\frac{(-1+\alpha)\beta^2}{2\sum_{i=m+1}^{\infty} a_i^2}\right). \tag{2.6}$$

Let $M >> m$. Then, using Levy's inequality, we get

$$\left|\left\{t : \max_{m\leq n\leq M-1} |\sum_{i=0}^{n-m} X_{M-i}(t)| > \beta\right\}\right| \leq 2\left|\left\{t : |\sum_{i=0}^{M-m-1} X_{M-i}(t)| > \beta\right\}\right|.$$

Here

$$\left|\left\{t : \max_{m\leq n\leq M-1} |\sum_{i=0}^{n-m} X_{M-i}(t)| > \beta\right\}\right| = \left|\left\{t : \max_{m\leq n\leq M-1} |\sum_{i=1}^{M} X_i(t) - \sum_{i=1}^{n} X_i(t)| > \beta\right\}\right|$$

and

$$\left|\left\{t : |\sum_{i=0}^{M-m-1} X_{M-i}(t)| > \beta\right\}\right| = \left|\left\{t : |\sum_{i=1}^{M} X_i(t) - \sum_{i=1}^{m} X_i(t)| > \beta\right\}\right|.$$

Thus, we have

$$\left|\left\{t : \max_{m\leq n\leq M-1} |\sum_{i=1}^{M} X_i(t) - \sum_{i=1}^{n} X_i(t)| > \beta\right\}\right| \leq 2\left|\left\{t : |\sum_{i=1}^{M} X_i(t) - \sum_{i=1}^{m} X_i(t)| > \beta\right\}\right|.$$

Consequently, we get

$$\left|\left\{t : \max_{m\leq n\leq M} |\sum_{i=1}^{M} X_i(t) - \sum_{i=1}^{n} X_i(t)| > \beta\right\}\right| \leq 2\left|\left\{t : |\sum_{i=1}^{M} X_i(t) - \sum_{i=1}^{m} X_i(t)| > \beta\right\}\right|. \tag{2.7}$$

Since $M >> m$, it follows from (2.6) and (2.7) that

$$\left|\left\{t : |\sum_{i=1}^{M} X_i(t) - \sum_{i=1}^{m} X_i(t)| > \beta\right\}\right| \leq A(\alpha)\exp\left(\frac{(-1+\alpha)\beta^2}{2\sum_{i=m+1}^{\infty} a_i^2}\right). \tag{2.8}$$

From the equations (2.7) and (2.8), we have

$$\left|\left\{t : \sup_{m\leq n\leq M} |\sum_{i=1}^{M} X_i(t) - \sum_{i=1}^{n} X_i(t)| > \beta\right\}\right| \leq A(\alpha)\exp\left(\frac{(-1+\alpha)\beta^2}{2\sum_{i=m+1}^{\infty} a_i^2}\right).$$

Define $A_M := \{t : \sup_{m\le n\le M} |\sum_{i=1}^M X_i(t) - \sum_{i=1}^n X_i(t)| > \beta\}$ and $A := \bigcup_{M=1}^\infty A_M$. By the continuity of Lebesgue measure, we have $\lim_{M\to\infty} |A_M| = |A|$. Then we have

$$\left|\left\{t : \sup_{n\ge m} |\textstyle\sum_{i=1}^\infty X_i(t) - \sum_{i=1}^n X_i(t)| > \beta\right\}\right| \le 2A(\alpha)\exp\left(\frac{(-1+\alpha)\beta^2}{2\sum_{i=m+1}^\infty a_i^2}\right).$$

$$\left|\left\{t : \sup_{n\ge m} \left|\textstyle\sum_{i=n+1}^\infty X_i(t)\right| > \beta\right\}\right| \le A(\alpha)\exp\left(\frac{(-1+\alpha)\beta^2}{2\sum_{i=m+1}^\infty a_i^2}\right).$$

□

3. Proof of the Main Result

In this section, we provide a proof for our main theorem.
Let ϵ, α be such that $0 < \epsilon << 1$, $0 < \alpha << 1$ and choose θ sufficiently large. Take $\xi > 0$ satisfying $(1+\xi)(1-\epsilon^2)(1-\alpha) > 1$. Let us define stopping time $m_k \to \infty$ with $m_1 \le m_2 \le \cdots$ by

$$m_k = \min\left(m : \sum_{i=m+1}^\infty a_i^2 < \frac{1}{\theta^k}\right).$$

If m_k is sufficiently large, then by assumption we have $a_{m_k}^2 < \epsilon^2 \sum_{i=m_k}^\infty a_i^2$. Using this together with the definition of m_k, we have

$$(1-\epsilon^2)\textstyle\sum_{i=m_k}^\infty a_i^2 < \sum_{i=m_k+1}^\infty a_i^2 < \frac{1}{\theta^k}. \tag{3.1}$$

Again we have

$$(1-\epsilon^2)\frac{1}{\theta^k} \le (1-\epsilon^2)\textstyle\sum_{i=m_k}^\infty a_i^2. \tag{3.2}$$

Using (3.1) and (3.2), we have

$$(1-\epsilon^2)\frac{1}{\theta^k} < \textstyle\sum_{i=m_k+1}^\infty a_i^2 < \frac{1}{\theta^k}.$$

Consequently,

$$(1-\epsilon^2)\theta \le \frac{\sum_{i=m_k+1}^\infty a_i^2}{\sum_{i=m_{k+1}+1}^\infty a_i^2} \le \frac{\theta}{1-\epsilon^2}. \tag{3.3}$$

Using the above inequality and Lemma 2.6, we get

$$\left|\left\{t : \sup_{m\geq m_{k+1}}\left|\sum_{i=m+1}^{\infty} a_iY_i(t)\right| > \sqrt{\frac{4(1+\xi)}{\theta}\sum_{i=m_k+1}^{\infty} a_i^2 \ln\ln\left(\frac{1}{\sum_{i=m_k+1}^{\infty} a_i^2}\right)}\right\}\right|$$

$$\leq \left|\left\{t : \sup_{m\geq m_{k+1}}\frac{\left|\sum_{i=m+1}^{\infty} a_iY_i(t)\right|}{\sqrt{\sum_{i=m_{k+1}+1}^{\infty} a_i^2}} > \sqrt{\frac{4(1+\xi)}{\theta}\theta(1-\epsilon^2)\ln\ln\left(\frac{1}{\sum_{i=m_{k+1}+1}^{\infty} a_i^2}\right)}\right\}\right|$$

$$= \left|\left\{t : \sup_{m\geq m_{k+1}}\left|\sum_{i=1}^{\infty} a_iY_i(t) - \sum_{i=1}^{m} a_iY_i(t)\right| > \sqrt{4(1+\xi)(1-\epsilon^2)\sum_{i=m_{k+1}+1}^{\infty} a_i^2 \ln\ln\left(\frac{1}{\sum_{i=m_k+1}^{\infty} a_i^2}\right)}\right\}\right|$$

$$\leq A(\alpha)\exp\left(\frac{(-1+\alpha)4(1+\xi)(1-\epsilon^2)\sum_{i=m_{k+1}+1}^{\infty} a_i^2 \quad \ln\ln\left(\frac{1}{\sum_{i=m_k+1}^{\infty} a_i^2}\right)}{2\sum_{i=m_{k+1}+1}^{\infty} a_i^2}\right)$$

$$\leq A(\alpha)\left(\ln\theta^k\right)^{2(\alpha-1)(\xi+1)(1-\epsilon^2)}.$$

Thus,

$$\left|\left\{t : \sup_{m\geq m_{k+1}}\left|\sum_{i=1}^{\infty} a_iY_i(t) - \sum_{i=1}^{m} a_iY_i(t)\right| > \sqrt{4(1+\xi)(1-\epsilon^2)\sum_{i=m_{k+1}+1}^{\infty} a_i^2 \ln\ln\frac{1}{\sum_{i=m_k+1}^{\infty} a_i^2}}\right\}\right|$$
$$\leq A(\alpha)\frac{1}{k^{2(1-\alpha)(1+\xi)(1-\epsilon^2)}}\frac{1}{(\ln\theta)^{2(1-\alpha)(1+\xi)(1-\epsilon^2)}}.$$

This gives

$$\sum_{k=1}^{\infty}\left|\left\{t : \sup_{m\geq m_{k+1}}\left|\sum_{i=1}^{\infty} a_iY_i(t) - \sum_{i=1}^{m} a_iY_i(t)\right| > \sqrt{\frac{4(1+\xi)(1-\epsilon^2)}{\theta}\sum_{i=m_k+1}^{\infty} a_i^2 \ln\ln\left(\frac{1}{\sum_{i=m_k+1}^{\infty} a_i^2}\right)}\right\}\right|$$
$$< \infty.$$

Applying Borel-Cantelli Lemma for a.e. t,

$$\sup_{m\geq m_{k+1}}\left|\sum_{i=1}^{\infty} a_iY_i(t) - \sum_{i=1}^{m} a_iY_i(t)\right| \leq \sqrt{\frac{4(1+\xi)(1-\epsilon^2)}{\theta}\sum_{i=m_k+1}^{\infty} a_i^2 \ln\ln\left(\frac{1}{\sum_{i=m_k+1}^{\infty} a_i^2}\right)}$$

which is true for significantly large k, say $k \geq M$.
Hence for a.e. t,

$$\sup_{m\geq m_{k+1}}\frac{\left|\sum_{i=1}^{\infty} a_iY_i(t) - \sum_{i=1}^{m} a_iY_i(t)\right|}{\sqrt{2\sum_{i=m_k+1}^{\infty} a_i^2 \ln\ln\frac{1}{\sum_{i=m_k+1}^{\infty} a_i^2}}} \leq \sqrt{\frac{2(1+\xi)(1-\epsilon^2)}{\theta}} \tag{3.4}$$

for significantly large $k \geq M$.
We recall the following result on exponential bounds whose proof can be found in [9].

Theorem 3.1. *Let $\{X_k\}$ be a sequence of i.r.v. having mean 0 and variance σ_k^2. Let $S_n = \sum_{k=1}^{n} X_k$, $s_n^2 = \sum_{k=1}^{n}\sigma_k^2$. Let $d = \max_{k\leq n}\left|\frac{X_k}{s_n}\right|$. Then, given $\gamma > 0$, if $d = d(\gamma)$ is* *significantly small and $\alpha = \alpha(\gamma)$ is for significantly large, then*

$$P\left(\frac{S_n}{s_n} > \alpha\right) > \exp\left(-\frac{\alpha^2}{2}(1+\gamma)\right).$$

Set $S_m = \sum_{i=n}^m a_i Y_i$ and $s_m^2 = \sum_{i=n}^m a_i^2$. Then fix $\gamma > 0$ choose $d(\gamma)$ accordingly. Assume that m_k is significantly large. Then $\forall l \geq m_k + 1$, the assumption implies

$$|a_l| \leq \frac{d(\gamma)}{2}\sqrt{\sum_{i=l}^\infty a_i^2} \leq \frac{d(\gamma)}{2}\sqrt{\sum_{i=m_k+1}^\infty a_i^2}. \tag{3.5}$$

We take m large enough so that

$$\sqrt{\sum_{i=m_k+1}^\infty a_i^2} \leq 2\sqrt{\sum_{i=m_k+1}^m a_i^2}. \tag{3.6}$$

From (3.5) and (3.6), we have

$$\frac{|a_l|}{\sqrt{\sum_{i=m_k+1}^m a_i^2}} \leq d(\gamma).$$

Then

$$\max_{m_k+1 \leq l \leq m} \frac{|a_l|}{\sqrt{\sum_{i=m_k+1}^m a_i^2}} \leq d(\gamma).$$

Using Theorem 3.1, we have,

$$\left|\left\{t : \frac{|\sum_{i=m_k+1}^m a_i Y_i(t)|}{\sqrt{\sum_{i=m_k+1}^m a_i^2}} > \alpha\right\}\right| > \exp\left(-\frac{\alpha^2}{2}(1+\gamma)\right).$$

Choose $\alpha = \sqrt{\frac{2(1-\xi/2)}{(1+\gamma)}\ln\ln\left(\frac{1}{\sum_{i=m_k+1}^\infty a_i^2}\right)}$ such that $\xi > 0$. Then for significantly large m_k, α is large enough as needed in the Theorem 3.1, we get

$$\begin{aligned}
&\left|\left\{t : \frac{|\sum_{i=m_k+1}^m a_i Y_i(t)|}{\sqrt{\sum_{i=m_k+1}^n a_i^2}} > \sqrt{\frac{2(1-\xi/2)}{(1+\gamma)}\ln\ln\left(\frac{1}{\sum_{i=m_k+1}^\infty a_i^2}\right)}\right\}\right| \\
&> \exp\left(\frac{-2(1-\xi/2)}{(1+\gamma)}\ln\ln\left(\frac{1}{\sum_{i=m_k+1}^\infty a_i^2}\right)\frac{(1+\gamma)}{2}\right) \\
&\geq \frac{1}{(k\ln\theta - \ln(1-\epsilon^2))^{1-\frac{\xi}{2}}} \\
&> \frac{1}{2(k\ln\theta)^{1-\frac{\xi}{2}}}.
\end{aligned}$$

Thus for a large k, we have

$$\left|\left\{t : \frac{\left|\sum_{i=m_k+1}^\infty a_i Y_j(t) - \sum_{i=m+1}^\infty a_i Y_j(t)\right|}{\sqrt{2\sum_{i=m_k+1}^n a_i^2 \ln\ln\frac{1}{\sum_{i=m_k+1}^\infty a_i^2}}} > \sqrt{\frac{(1-\xi/2)}{(1+\gamma)}}\right\}\right| > \frac{1}{2(k\ln\theta)^{1-\frac{\xi}{2}}}. \tag{3.7}$$

Moreover, we also have

$$\sum_{i=m_k+1}^m a_i^2 \geq \sum_{i=m_k+1}^\infty a_i^2(1-\epsilon^2 - 1/\theta). \tag{3.8}$$

From (3.8) and (3.7) we get,

$$\left|\left\{t:\frac{\left|\sum_{i=m_k+1}^{\infty}a_iY_i(t)-\sum_{i=m+1}^{\infty}a_iY_i(t)\right|}{\sqrt{\left(2\sum_{i=m_k+1}^{\infty}a_i^2\left(1-\epsilon^2-\frac{1}{\theta}\right)\ln\ln\left(\frac{1}{\sum_{i=m_k+1}^{\infty}a_i^2}\right)\right)}}>\sqrt{\frac{(1-\xi/2)}{(1+\gamma)}}\right\}\right|>\frac{1}{2(k\ln\theta)^{1-\frac{\delta}{2}}}.$$

One can easily show that

$$\left|\left\{t:\frac{\left|\sum_{i=m_{k+1}+1}^{\infty}a_iY_i(t)-\sum_{i=m_k+1}^{\infty}a_iY_i(t)\right|}{\sqrt{2\sum_{i=m_k+1}^{\infty}a_i^2\ln\ln\frac{1}{\sum_{i=m_k+1}^{\infty}a_i^2}}}>\sqrt{\frac{(1-\xi/2)}{(1+\gamma)}\left(1-\epsilon^2-\frac{1}{\theta}\right)}-2\sqrt{\frac{(1+\xi)(1-\epsilon^2)}{\theta}}\right\}\right|>$$
$$\left|\left\{t:\frac{\left|\sum_{i=m_k+1}^{\infty}a_iY_i(t)-\sum_{i=m+1}^{\infty}a_iY_i(t)\right|}{\sqrt{2\sum_{i=m_k+1}^{\infty}a_i^2\ln\ln\left(\frac{1}{\sum_{i=m_k+1}^{\infty}a_i^2}\right)}}>\sqrt{\frac{(1-\xi/2)}{(1+\gamma)}\left(1-\epsilon^2-\frac{1}{\theta}\right)}\right\}\right|.$$

Using this, we get

$$\sum_{k=1}^{\infty}\left|\left\{t:\frac{\left|\sum_{i=m_{k+1}+1}^{\infty}a_iY_i(t)-\sum_{i=m_k+1}^{\infty}a_iY_i(t)\right|}{\sqrt{2\sum_{i=m_k+1}^{\infty}a_i^2\ln\ln\left(\frac{1}{\sum_{i=m_k+1}^{\infty}a_i^2}\right)}}>\sqrt{\frac{(1-\xi/2)}{(1+\gamma)}\left(1-\epsilon^2-\frac{1}{\theta}\right)}-2\sqrt{\frac{(1+\xi)(1-\epsilon^2)}{\theta}}\right\}\right|$$
$$\geq\sum_{k=1}^{\infty}\frac{1}{2(k\ln\theta)^{1-\frac{\xi}{2}}}$$
$$=\infty.$$

Here each term, $\sum_{i=m_{k+1}+1}^{\infty}a_iY_i(t)-\sum_{i=m_k+1}^{\infty}a_iY_i(t)$ is an independent random variables. Using Borel-Cantelli Lemma for a.e. t, we find $m_1<m_2<m_3<\cdots$ such that

$$\left|\left\{t:\frac{\left|\sum_{i=m_{k+1}+1}^{\infty}a_iY_i(t)-\sum_{i=m_k+1}^{\infty}a_iY_i(t)\right|}{\sqrt{2\sum_{i=m_k+1}^{\infty}a_i^2\ln\ln\left(\frac{1}{\sum_{i=m_k+1}^{\infty}a_i^2}\right)}}>\sqrt{\frac{(1-\xi/2)}{(1+\gamma)}\left(1-\epsilon^2-\frac{1}{\theta}\right)}-2\sqrt{\frac{(1+\xi)(1-\epsilon^2)}{\theta}}\right\}\right|$$

Using (3.4) we get,

$$\sqrt{\frac{(1+\xi)(1-\epsilon^2)}{\theta}}+\frac{\left|\sum_{i=m_k+1}^{\infty}a_iY_i(t)\right|}{\sqrt{2\sum_{i=m_k+1}^{\infty}a_i^2\ln\ln\left(\frac{1}{\sum_{i=m_k+1}^{\infty}a_i^2}\right)}}>\sqrt{\frac{(1-\xi/2)}{(1+\gamma)}\left(1-\epsilon^2-\frac{1}{\theta}\right)}-2\sqrt{\frac{(1+\xi)(1-\epsilon^2)}{\theta}}$$

This gives

$$\frac{\left|\sum_{i=m_k+1}^{\infty}a_iY_i(t)\right|}{\sqrt{2\sum_{i=m_k+1}^{\infty}a_i^2\ln\ln\left(\frac{1}{\sum_{i=m_k+1}^{\infty}a_i^2}\right)}}>\sqrt{\frac{(1-\xi/2)}{(1+\gamma)}\left(1-\epsilon^2-\frac{1}{\theta}\right)}-3\sqrt{\frac{(1+\xi)(1-\epsilon^2)}{\theta}}.$$

Finally we let $\theta\nearrow\infty$, $\epsilon\searrow 0$, $\xi\searrow 0$, and $\gamma\searrow 0$. This gives

$$\frac{\left|\sum_{i=m_k+1}^{\infty} a_i Y_i(t)\right|}{\sqrt{2\sum_{i=m_k+1}^{\infty} a_i^2 \ln\ln\left(\frac{1}{\sum_{i=m_k+1}^{\infty} a_i^2}\right)}} \geq 1.$$

Consequently for almost every t, we have

$$\limsup_{m\to\infty} \frac{\left|\sum_{i=m}^{\infty} a_i Y_i(t)\right|}{\sqrt{2\sum_{i=m}^{\infty} a_i^2 \ln\ln\left(\frac{1}{\sum_{i=m}^{\infty} a_i^2}\right)}} \geq 1.$$

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