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# AN UPPER BOUND IN A LAW OF THE ITERATED LOGARITHM FOR RADEMACHER FUNCTIONS

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**Abstract:** N. Kolmogorov introduced a law of the iterated logarithm, abbreviated LIL, in the case of independent random variables. Over the years, analog of his result has been introduced in various contexts of analysis. Here, we introduce a similar LIL in the context of sums of Rademacher functions.

**Key Words**: Rademacher functions, Law of the iterated logarithm, Borel-Cantelli Lemma
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## 1. Introduction

A law of the iterated logarithm, abbreviated LIL, is a well known limit law that has been developed in various contexts of statistics and mathematics. We note that LIL operates approximately amid central limit theorem and law of large numbers. More precisely, LIL can be taken as an improvement of these two limit laws. There are various situations where these limit laws can not be applied. In such cases, one needs to use the LIL which exhibits the importance of LIL. We begin with brief discussion of classical occurrences of LIL. Khintchine [7] originated the first LIL in order to prefect the Borel's theorem describing the long term behavior of normal numbers where he considered Bernoulli random variables. His result was further improvised by Kolmogorov [8] in the context of independent random variables, abbreviated i.r.v.. We state the celebrated LIL of Kolmogorov.

**Theorem 1.1** ([8]). *Let $S_m = \sum_{k=1}^{m} X_k$ where $\{X_k\}$ is a sequence of independent random variables. Let us denote the variance of $S_m$ by $s_m$. Assume that $s_m \to \infty$ and $|X_m|^2 \leq \dfrac{K_m s_m^2}{\ln\ln(e^e + s_m^2)}$ for $K_m \to 0$. Then, almost surely*

$$\limsup_{m\to\infty} \frac{S_m}{\sqrt{2 s_m \ln\ln s_m^2}} = 1.$$

After the introduction of this celebrated LIL, mathematician started to work to obtain similar LIL in other numerous contexts of analysis. Some of the areas where people are working on the LIL are identically distributed random variable, dyadic martingales, partial sums of lacunary series, Bloch functions, Brownian motion, linear processes, Banach spaces,

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harmonic functions, random vectors just a few to name. There are various amount of literature available in LIL. The LIL mainly takes into account of sums of initial $n-$terms of the sequence in the regular type LIL. On the other hand, it also considers the remainder after $n-$terms or the tail sums of the given sequence. The former case is commonly called regular LIL whereas the latter law is called the tail LIL. For more about the law of the iterated logarithm, the reader is referred to [1], [3], [5], and [6]. Salem and Zygmund achieved similar LIL for series of lacunary which is taken as the opening result in such an area of mathematics. We recall the classical LIL of Salem and Zygmund [10].

**Theorem 1.2.** *Let $S_m$ denote the partial sums of lacunary series and $n_k$ are positive integers. Set $B_m^2 = \frac{1}{2}\sum_{k=1}^m(|a_k|^2 + |b_k|^2)$ and $M_m = \max_{1\le k\le m}(|a_k|^2 + |b_k|^2)^{\frac{1}{2}}$. Suppose also that $B_m \longrightarrow \infty$ as $m \longrightarrow \infty$ and $S_m$ satisfies the Kolmogorov-type condition: $M_m^2 \le K_m \frac{B_m^2}{\ln\ln(e^e+B_m^2)}$ for some sequence of numbers $K_m \downarrow 0$. Then*

$$\limsup_{m\to\infty} \frac{S_m(\theta)}{\sqrt{2B_m^2 \ln\ln B_m}} \le 1$$

*for almost every $\theta \in T$, the unit circle.*

Erdös and Gál [4] improvised the above law and discovered the following result.

**Theorem 1.3.** *Let $S_m(\theta) = \sum_{k=1}^m \exp(in_k\theta)$ be partial sums of lacunary series and $n_k$ denote integers. Then $\limsup_{m\to\infty} \frac{S_m(\theta)}{\sqrt{m\ln\ln m}} = 1$ for a.e. $\theta \in T$.*

Later, M. Wiess [11] was able to obtain the final version as follows.

**Theorem 1.4.** *Suppose $S_m(\theta) = \sum_{k=1}^m(a_k\cos n_k\theta + b_k\sin n_k\theta)$ is a $q-$ lacunary series. Set $B_m = \left(\frac{1}{2}\sum_{k=1}^m(|a_k|^2 + |b_k|^2)\right)^{\frac{1}{2}}$ and $M_m = \max_{1\le k\le m}(|a_k|^2 + |b_k|^2)^{\frac{1}{2}}$. Suppose also that $B_m \to \infty$ as $m \to \infty$ and $M_m$ satisfies the Kolmogorov-type condition: $M_m^2 \le K_m \frac{B_m^2}{\ln\ln(e^e+B_m^2)}$ for some sequence of numbers $K_m \downarrow 0$. Then*

$$\limsup_{m\to\infty} \frac{S_m(\theta)}{\sqrt{2B_m^2 \ln\ln B_m}} = 1$$

*for a.e. $\theta \in T$.*

One can easily see that sums of Rademacher functions (see Definition 2.1) behave like random variable and satisfy the condition of independence. There are various laws of LIL established for independent random variable. In all of these iterated laws of logarithm, the authors have used the probabilistic approach. Here, we introduce a similar LIL in the context of Rademacher functions in which we use measure theoretic approach without exploiting the properties of independent random variables. We obtain an upper bound in a LIL of sums of Rademacher functions. The following theorem is our main result.

**Theorem 1.5.** *Let $\{r_n\}_{n=1}^\infty$ be a sequence of Rademacher functions on the interval $[0, 1)$ defined by $r_k(x) = sgn(\sin 2^k\pi x)$ where $sgn$ denotes signum function and $\{a_n\}_{n=1}^\infty$ be a sequence of real numbers. Then*

$$\limsup_{n\to\infty} \frac{|\sum_{k=1}^n a_k r_k(x)|}{\sqrt{2\sum_{k=1}^n a_k^2 \ln\ln\sqrt{\sum_{k=1}^n a_k^2}}} \le 1$$

*a.e. on the set where* $\{\sum_{k=1}^{n} a_k r_k(x)\}$ *is unbounded.*

## 2. Preliminaries

For a proof of the theorem above, we first present some definitions. We then establish two estimates for Rademacher functions which play an integral part in our main result.

**Definition 2.1.** Rademacher functions are the functions $\{r_k\}_{k=1}^{\infty}$ on $[0,1)$ satisfying $r_k(x) = \operatorname{sgn}(\sin 2^k \pi x)$ where sgn denotes the signum function.

Consider the interval $[0,1)$. Then $Q_{nj} = \left[\frac{j}{2^n}, \frac{j+1}{2^n}\right)$ $n, j \in \mathbb{Z}$ are called $n^{\text{th}}$ generation dyadic intervals where $j = 0, 1, \cdots 2^n$. Sometimes we also use $Q_n$ to denote the $n^{\text{th}}$ generation dyadic interval.

**Lemma 2.2** (Borel-Cantelli)**.** *Suppose that* $\{A_k\}$ *is a sequence of measurable sets in* $X$ *satisfying the condition* $\sum_{k=1}^{\infty} \mu(A_k) < \infty$. *Then for almost all* $x \in X$ *lie in at most finitely many of the sets* $A_k$.

For a proof of Lemma 2.2, the reader is referred to [9]. We now prove our estimates.

**Lemma 2.3** (Estimate 1)**.** *Suppose* $\{r_n\}$ *is a sequence of Rademacher functions. For any real number sequence* $\{a_n\}$, *we have*

$$\int_0^1 \exp\left(\alpha \sum_{k=1}^{n} a_k r_k(x) - \frac{\alpha^2}{2} \sum_{k=1}^{n} a_k^2\right) dx \leq 1$$

*for any real number* $\alpha$.

*Proof.* Define $f_n(x) = \sum_{k=1}^{n} a_k r_k(x), \quad d_k(x) = f_k(x) - f_{k-1}(x)$. We have $f_n(x) = \sum_{k=1}^{n} d_k(x)$ with $f_0 = 0$. Let

$$g(n) = \int_0^1 \exp\left(f_n(x) - \frac{1}{2}\sum_{k=1}^{n} d_k^2(x)\right) dx = \int_0^1 \exp\left(\sum_{k=1}^{n} d_k(x) - \frac{1}{2}\sum_{k=1}^{n} d_k^2(x)\right) dx.$$

We first claim that $g(n+1) \leq g(n)$. One can see that the function $f_n(x) = \sum_{k=1}^{n} d_k(x)$ is constant on each dyadic interval $Q_{nj}$ with $j = 0, 1, ...2^n$. In what follows, we use $e^x$ and $\exp(x)$ interchangeably. Then

$$\begin{aligned} g(n+1) &= \int_0^1 e^{\sum_{k=1}^{n+1} d_k(x) - \frac{1}{2}\sum_{k=1}^{n+1} d_k^2(x)} dx \\ &= \sum_{j=0}^{2^n} \int_{Q_{nj}} e^{\sum_{k=1}^{n} d_k(x) - \frac{1}{2}\sum_{k=1}^{n} d_k^2(x)} e^{d_{n+1}(x) - \frac{1}{2}d_{n+1}^2(x)} dx \\ &= \sum_{j=0}^{2^n} \left[e^{\sum_{k=1}^{n} d_k(x) - \frac{1}{2}\sum_{k=1}^{n} d_k^2(x)}\right]_{Q_{nj}} \int_{Q_{nj}} e^{d_{n+1}(x) - \frac{1}{2}d_{n+1}^2(x)} dx \end{aligned}$$

Here $d_{n+1}(x) = f_{n+1}(x) - f_n(x) = a_{n+1}r_{n+1}(x)$. We claim that $g(1) \leq 1$. Employing $\cosh x \leq e^{x^2/2}$, we have

$$\begin{aligned}
g(1) &= \int_0^1 e^{d_1(x)-\frac{1}{2}d_1^2(x)}dx \\
&= \int_{[0,1/2)} e^{d_1(x)-\frac{1}{2}d_1^2(x)}dx + \int_{[1/2,1)} e^{d_1(x)-\frac{1}{2}d_1^2(x)}dx \\
&= \int_0^{1/2} e^{a_1-\frac{1}{2}a_1^2}dx + \int_{1/2}^1 e^{-a_1-\frac{1}{2}a_1^2}dx \\
&= e^{a_1-\frac{1}{2}a_1^2}\frac{1}{2} + e^{-a_1-\frac{1}{2}a_1^2}\frac{1}{2} \\
&= \frac{1}{2}(e^{a_1}+e^{-a_1})e^{-\frac{1}{2}a_1^2} \\
&= \cosh(a_1)e^{-\frac{1}{2}a_1^2} \\
&\leq e^{\frac{1}{2}a_1^2}e^{-\frac{1}{2}a_1^2} \\
&= 1.
\end{aligned}$$

Hence, $g(1) \leq 1$. Let $Q_{nj}^{'}$ and $Q_{nj}^{''}$ denote the next generation subintervals of $Q_{nj}$. Here $d_{n+1}(x) = a_{n+1}r_{n+1}(x)$ takes the value $a_{n+1}$ on $Q_{nj}^{'}$ and $-a_{n+1}$ on $Q_{nj}^{''}$. Then we have

$$\begin{aligned}
g(n+1) &= \sum_{j=0}^{2^n} \left[e^{\sum_{k=1}^n d_k(x)-\frac{1}{2}\sum_{k=1}^n d_k^2(x)}\right]_{Q_{nj}} \int_{Q_{nj}} e^{d_{n+1}(x)-\frac{1}{2}d_{n+1}^2(x)}dx \\
&= \sum_{j=0}^{2^n} \left[e^{\sum_{k=1}^n d_k(x)-\frac{1}{2}\sum_{k=1}^n d_k^2(x)}\right]_{Q_{nj}} \left[\int_{Q_{nj}^{'}} e^{d_{n+1}(x)-\frac{1}{2}d_{n+1}^2(x)}dx + \int_{Q_{nj}^{''}} e^{d_{n+1}(x)-\frac{1}{2}d_{n+1}^2(x)}dx\right] \\
&= \sum_{j=0}^{2^n} \left[e^{\sum_{k=1}^n d_k(x)-\frac{1}{2}\sum_{k=1}^n d_k^2(x)}\right]_{Q_{nj}} \left[\int_{Q_{nj}^{'}} e^{a_{n+1}-\frac{1}{2}a_{n+1}^2}dx + \int_{Q_{nj}^{''}} e^{-a_{n+1}-\frac{1}{2}a_{n+1}^2}dx\right] \\
&= \sum_{j=0}^{2^n} \left[e^{\sum_{k=1}^n d_k(x)-\frac{1}{2}\sum_{k=1}^n d_k^2(x)}\right]_{Q_{nj}} \left[e^{a_{n+1}-\frac{1}{2}a_{n+1}^2}\frac{1}{2^{n+1}} + e^{-a_{n+1}-\frac{1}{2}a_{n+1}^2}\frac{1}{2^{n+1}}\right] \\
&= \sum_{j=0}^{2^n} 2\left[e^{\sum_{k=1}^n d_k(x)-\frac{1}{2}\sum_{k=1}^n d_k^2(x)}\right]_{Q_{nj}} \left(\frac{e^{a_{n+1}}+e^{-a_{n+1}}}{2}\right) e^{-\frac{1}{2}a_{n+1}^2}\frac{1}{2^{n+1}} \\
&= \sum_{j=0}^{2^n} 2\left[e^{\sum_{k=1}^n d_k(x)-\frac{1}{2}\sum_{k=1}^n d_k^2(x)}\right]_{Q_{nj}} \cosh(a_{n+1})e^{-\frac{1}{2}a_{n+1}^2}\frac{1}{2^{n+1}} \\
&\leq \sum_{j=0}^{2^n} \left[e^{\sum_{k=1}^n d_k(x)-\frac{1}{2}\sum_{k=1}^n d_k^2(x)}\right]_{Q_{nj}} e^{\frac{1}{2}a_{n+1}^2}e^{-\frac{1}{2}a_{n+1}^2}\frac{1}{2^n}
\end{aligned}$$

$$
\begin{aligned}
g(n+1) &\leq \sum_{j=0}^{2^n} \left[ e^{\sum_{k=1}^n d_k(x) - \frac{1}{2}\sum_{k=1}^n d_k^2(x)} \right]_{Q_{nj}} |Q_{nj}| \\
&= \sum_{j=0}^{2^n} \int_{Q_{nj}} e^{\sum_{k=1}^n d_k(x) - \frac{1}{2}\sum_{k=1}^n d_k^2(x)} dx \\
&= \int_0^1 e^{\sum_{k=1}^n d_k(x) - \frac{1}{2}\sum_{k=1}^n d_k^2(x)} dx \\
&= g(n).
\end{aligned}
$$

So, $g(n+1) \leq g(n)$. This with $g(1) \leq 1$ gives $g(n) \leq 1$. Consequently,

$$\int_0^1 \exp\left( \sum_{k=1}^n d_k(x) - \frac{1}{2}\sum_{k=1}^n d_k^2(x) \right) dx \leq 1.$$

$$\int_0^1 \exp\left( \sum_{k=1}^n a_k r_k(x) - \frac{1}{2}\sum_{k=1}^n a_k^2 \right) dx \leq 1.$$

Now if we rescale the function $f_n(x) = \sum_{k=1}^n a_k r_k(x)$ as $\alpha f_n(x) = \alpha \sum_{k=1}^n a_k r_k(x)$ where $\alpha$ is any real number, we get

$$\int_0^1 \exp\left( \alpha \sum_{k=1}^n a_k r_k(x) - \frac{\alpha^2}{2}\sum_{k=1}^n a_k^2 \right) dx \leq 1.$$

□

**Lemma 2.4** (Estimate 2)**.** *Suppose $\{r_k\}$ is a sequence of Rademacher functions and $\{a_k\}_{k=1}^{\infty}$ is a sequence of real numbers. Then for any $\lambda > 0$, we have*

$$\left| \left\{ x \in [0,1) : \sup_{m \geq 1} \left| \sum_{k=1}^m a_k r_k(x) \right| > \lambda \right\} \right| \leq 6 \exp\left( \frac{-\lambda^2}{2\sum_{k=1}^{\infty} a_k^2} \right).$$

*Proof.* Define $f_n(x) = \sum_{k=1}^n a_k r_k(x)$. Then for all $m \leq n$ with $n$ being fixed and $x \in Q_m, |Q_m| = \frac{1}{2^m}$ we have

$$f_m(x) = \frac{1}{|Q_m|} \int_{Q_m} f_n(y) dy.$$

Fix $x$. Let $Mf_n$ denote the Hardy-Littlewood Maximal function associated to the function $f_n$. Then, for any real number $\alpha$ employing Jensen's inequality, we get

$$
\begin{aligned}
e^{\alpha|f_m(x)|} &= \exp\left( \alpha \left| \int_{Q_m} f_n(y) d\left( \frac{y}{|Q_m|} \right) \right| \right) \\
&\leq \frac{1}{|Q_m|} \int_{Q_m} e^{(\alpha|f_n(y)|)} dy \\
&\leq M\left( e^{\alpha|f_m(x)|} \right)(x).
\end{aligned}
$$

Using this inequality and H-L Maximal theorem, we have

$$\begin{aligned}\left|\left\{x\in[0,1):\sup_{1\le m\le n}|f_m(x)|>\lambda\right\}\right| &= \left|\left\{x\in[0,1):\sup_{1\le m\le n}e^{\alpha|f_m(x)|}>e^{\alpha\lambda}\right\}\right|\\ &\le \left|\left\{x\in[0,1):M\left(e^{\alpha|f_m|}\right)(x)>e^{\alpha\lambda}\right\}\right|\\ &\le \frac{3}{e^{\alpha\lambda}}\int_0^1 e^{(\alpha|f_n(y)|)}dy\\ &= \frac{3}{e^{\alpha\lambda}}\exp\left(\frac{\alpha^2}{2}\sum_{k=1}^n a_k^2\right)\int_0^1\exp\left(\alpha|f_n(y)|-\frac{\alpha^2}{2}\sum_{k=1}^n a_k^2\right)dy.\end{aligned}$$

Using Estimate 1, we have

$$\begin{aligned}&\int_0^1\exp\left(\alpha|f_n(y)|-\frac{\alpha^2}{2}\sum_{k=1}^n a_k^2\right)dy\\ &=\int_{\{y:f_n(y)\ge 0\}}\exp\left(\alpha f_n(y)-\frac{\alpha^2}{2}\sum_{k=1}^n a_k^2\right)dy+\int_{\{y:f_n(y)<0\}}\exp\left(-\alpha f_n(y)-\frac{\alpha^2}{2}\sum_{k=1}^n a_k^2\right)dy\\ &=\int_{\{y:f_n(y)\ge 0\}}\exp(\left(\alpha f_n(y)-\frac{\alpha^2}{2}\sum_{k=1}^n a_k^2\right)dy+\int_{\{y:f_n(y)<0\}}\exp\left(-\alpha f_n(y)-\frac{(-\alpha)^2}{2}\sum_{k=1}^n a_k^2\right)dy\\ &\le 1+1\\ &=2.\end{aligned}$$

With this inequality, we get

$$\begin{aligned}\left|\left\{x\in[0,1):\sup_{1\le m\le n}|f_m(x)|>\lambda\right\}\right| &= \frac{3}{e^{\alpha\lambda}}\exp\left(\frac{\alpha^2}{2}\sum_{k=1}^n a_k^2\right)2\\ &= \frac{6}{e^{\alpha\lambda}}\exp\left(\frac{\alpha^2}{2}\sum_{k=1}^n a_k^2\right).\end{aligned}$$

We choose $\alpha=\frac{\lambda}{\sum_{k=1}^n a_k^2}$. Then

$$\left|\left\{x\in[0,1):\sup_{1\le m\le n}|f_m(x)|>\lambda\right\}\right|\le 6\exp\left(\frac{-\lambda^2}{2\sum_{k=1}^n a_k^2}\right).$$

Here, we note that $\sum_{k=1}^n a_k^2\nearrow\sum_{k=1}^\infty a_k^2$. This gives

$$\frac{-1}{\sum_{k=1}^n a_k^2}\le\frac{-1}{\sum_{k=1}^\infty a_k^2}.$$

So,

$$\left|\left\{x\in[0,1):\sup_{1\le m\le n}|f_m(x)|>\lambda\right\}\right|\le 6\exp\left(\frac{-\lambda^2}{2\sum_{k=1}^\infty a_k^2}\right).$$

Then, using the continuity of measure, we get

$$\left|\left\{x\in[0,1):\sup_{m\ge 1}|f_m(x)|>\lambda\right\}\right|\le 6\exp\left(\frac{-\lambda^2}{2\sum_{k=1}^\infty a_k^2}\right).$$

□

Now we prove our main theorem.

## 3. Proof of the main theorem

*Proof.* Let us take $\theta > 1$, $\beta > 0$ and $f_n(x) = \sum_{k=1}^{n} a_k r_k(x)$, $d_k = f_k - f_{k-1}$. Define $\eta_k(x)$ as follows

$$\eta_k(x) = \begin{cases} min\left(n : \sqrt{\sum_{k=1}^{n+1} d_k^2(x)} > \theta^k\right) & \\ \infty, & \text{if } \sqrt{\sum_{k=1}^{\infty} d_k^2(x)} \le \theta^k. \end{cases}$$

By the definition, $\eta_k$ is the smallest index and so $\sqrt{\sum_{k=1}^{\eta_k+1} d_k^2(x)} > \theta^k$. Moreover, we have $\sqrt{\sum_{k=1}^{\eta_k} d_k^2(x)} \le \theta^k$. Let $p \wedge q$ denote the minimum of $p$ and $q$. Next we define

$$\tilde{f}_n(x) = f_{n \wedge \eta_k(x)} = \begin{cases} f_1(x), f_2(x), \cdots, f_{\eta_k}(x), f_{\eta_k}(x), \cdots & \text{for } \eta_k \ne \infty \\ f_1(x), f_2(x), \cdots & \text{for } \eta_k = \infty. \end{cases}$$

We first show that $\sqrt{\sum_{k=1}^{\infty} \tilde{d_k}^2(x)} \le \theta^k$. For $n < \eta_k(x)$,

$$\sqrt{\sum_{k=1}^{n} \tilde{d_k}^2(x)} = \sqrt{\sum_{k=1}^{n} d_k^2(x)} \le \sqrt{\sum_{k=1}^{\eta_k} d_k^2(x)} \le \theta^k.$$

For $n \ge \eta_k(x)$, $\sqrt{\sum_{k=1}^{n} \tilde{d_k}^2(x)} = \sqrt{\sum_{k=1}^{\eta_k} d_k^2(x)} \le \theta^k$. So for all $n$ we have $\sqrt{\sum_{k=1}^{n} \tilde{d_k}^2(x)} \le \theta^k$. This gives $\sqrt{\sum_{k=1}^{\infty} \tilde{d_k}^2(x)} \le \theta^k$. Take $\lambda = \sqrt{2(1+\beta)^2\theta^{2k}\ln\ln\theta^k}$. Using Estimate 2, we have

$$\begin{aligned} \left|\left\{x \in [0,1) : \sup_{n\ge1} |\tilde{f}_n(x)| > \sqrt{2(1+\beta)^2\theta^{2k}\ln\ln\theta^k}\right\}\right| &\le 6\exp\left(\frac{-2(1+\beta)^2\theta^{2k}\ln\ln\theta^k}{2\sum_{k=1}^{\infty}\tilde{d_k}^2(x)}\right) \\ &\le 6\exp\left(\frac{-2(1+\beta)^2\theta^{2k}\ln\ln\theta^k}{2\theta^{2k}}\right) \\ &= \frac{6}{(k\ln\theta)^{(1+\beta)^2}}. \end{aligned}$$

Taking summation,

$$\sum_{k=1}^{\infty}\left|\left\{x \in [0,1) : \sup_{n\ge1} |\tilde{f}_n(x)| > \sqrt{2(1+\beta)^2\theta^{2k}\ln\ln\theta^k}\right\}\right| \le \sum_{k=1}^{\infty} \frac{6}{(\ln\theta)^{(1+\beta)^2}}\frac{1}{k^{(1+\beta)^2}} < \infty.$$

Then, by Borel-Cantelli Lemma (Lemma 2.2), we have for a.e. $x$,

$$\sup_{n\ge1} |\tilde{f}_n(x)| \le \sqrt{2(1+\beta)^2\theta^{2k}\ln\ln\theta^k}$$

for sufficiently large $k$, $k \ge N$ for some $N$ which depends on $x$. Take $x$ so that $f_n(x) = \sum_{k=1}^{n} a_k r_k(x)$ is unbounded. We note that set where $f_n(x) = \sum_{k=1}^{n} a_k r_k(x)$ converges is almost everywhere equal to the set where $\sqrt{\sum_{k=1}^{\infty} d_k^2(x)}$ is finite (see [2]). Consequently, we have $\sqrt{\sum_{k=1}^{\infty} d_k^2(x)} = \infty$. So, we have for all $i$, $\eta_i(x) < \infty$. Let $n \ge \eta_N$. Let us choose $k$ satisfying $\eta_k(x) < n \le \eta_{k+1}(x)$. We have $\eta_k(x) < n$ so that $\eta_k(x) \le n-1$. Then we have $\sqrt{\sum_{k=1}^{n} d_k^2(x)} = \sqrt{\sum_{k=1}^{(n-1)+1} d_k^2(x)} > \theta^k$.

$$\begin{aligned}|f_n(x)| &\leq \sup_{1\leq m\leq \eta_{k+1}} |f_{m\wedge\eta_{k+1}}(x)| \\ &\leq \sup_{m\geq 1} |f_{m\wedge\eta_{k+1}}(x)| \\ &\leq \sqrt{2(1+\beta)^2\theta^{2(k+1)}\ln\ln\theta^{k+1}} \\ &= (1+\beta)\theta\theta^k\sqrt{2\ln(\ln\theta^k+\ln\theta)}.\end{aligned}$$

Using $\theta^k < \sqrt{\sum_{k=1}^{n} d_k^2(x)}$, we get

$$|f_n(x)| < (1+\beta)\theta\sqrt{2\sum_{k=1}^{n} d_k^2(x)\ln\left(\ln\sqrt{\sum_{k=1}^{n} d_k^2(x)}+\ln\theta\right)}.$$

This gives

$$\begin{aligned}\limsup_{n\to\infty}\frac{|f_n(x)|}{\sqrt{2\sum_{k=1}^{n} d_k^2(x)\ln\ln\sqrt{\sum_{k=1}^{n} d_k^2(x)}}} &\leq \limsup_{n\to\infty}(1+\beta)\theta\frac{\sqrt{\ln\left(\ln\sqrt{\sum_{k=1}^{n} d_k^2(x)}+\ln\theta\right)}}{\sqrt{\ln\ln\sqrt{\sum_{k=1}^{n} d_k^2(x)}}} \\ &= (1+\beta)\theta\sqrt{\limsup_{n\to\infty}\frac{\ln\left(\ln\sqrt{\sum_{k=1}^{n} d_k^2(x)}+\ln\theta\right)}{\ln\ln\sqrt{\sum_{k=1}^{n} d_k^2(x)}}}.\end{aligned}$$

We show

$$\limsup_{n\to\infty}\sqrt{\frac{\ln\left(\ln\sqrt{\sum_{k=1}^{n} d_k^2(x)}+\ln\theta\right)}{\ln\ln\sqrt{\sum_{k=1}^{n} d_k^2(x)}}} = 1.$$

Let $X = \ln(\sqrt{\sum_{k=1}^{n} d_k^2(x)})$. This gives

$$\limsup_{n\to\infty}\sqrt{\frac{\ln\left(\ln(\sqrt{\sum_{k=1}^{n} d_k^2(x)})+\ln\theta\right)}{\ln(\ln(\sqrt{\sum_{k=1}^{n} d_k^2(x)}))}} = \sqrt{\limsup_{n\to\infty}\frac{\ln(X+\ln\theta)}{\ln X}} = 1.$$

Using this, we have

$$\limsup_{n\to\infty}\frac{|f_n(x)|}{\sqrt{2\sum_{k=1}^{n} d_k^2(x)\ln\ln\sqrt{\sum_{k=1}^{n} d_k^2(x)}}} \leq (1+\beta)\theta.$$

We now let $\theta \searrow 1$. This gives

$$\limsup_{n\to\infty}\frac{|f_n(x)|}{\sqrt{2\sum_{k=1}^{n} d_k^2(x)\ln\ln\sqrt{\sum_{k=1}^{n} d_k^2(x)}}} \leq (1+\beta).$$

This is true for all $\beta > 0$. Hence for a.e. $x$,

$$\limsup_{n\to\infty}\frac{|f_n(x)|}{\sqrt{2\sum_{k=1}^{n} d_k^2(x)\ln\ln\sqrt{\sum_{k=1}^{n} d_k^2(x)}}} \leq 1.$$

Hence

$$\limsup_{n\to\infty} \frac{|\sum_{k=1}^{n} a_k r_k(x)|}{\sqrt{2\sum_{k=1}^{n} a_k^2 \ln\ln\sqrt{\sum_{k=1}^{n} a_k^2}}} \leq 1$$

for a.e. $x$ in the set where $\{\sum_{k=1}^{n} a_k r_k(x)\}$ is unbounded. □

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