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ON CERTAIN LINEAR STRUCTURES OF ORLICZ SPACE $c_0(M,(X,||.||),\bar{a},\bar{\alpha})$ OF VECTOR VALUED DIFFERENCE SEQUENCES

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Abstract: In this article, we introduce and study a new class $c_0(M,(X,||.||),\bar{a},\bar{\alpha})$ of normed space $(X,||.||)$ valued difference sequences with the help of Orlicz function M. This is a generalization of the classical sequence space c_0 . Our primarily interest is to explore some linear structures and investigate the conditions relating to the containment relation of the class $c_0(M,(X,||.||),\bar{a},\bar{\alpha})$ in terms of different $\bar{a}$ and $\bar{\alpha}$.

Key Words: Orlicz Function, Orlicz Space, Sequence Space, Difference Sequence Space

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1. Introduction

The sequence spaces and function spaces have very important position in different branches of mathematics. They occupy prominent position mainly in analysis, for instance, in structure theory of topological linear spaces, summability theory, operator theory, frame theory, Schauder basis theory, approximate theory, etc. This introduces several new concepts in functional analysis and thereby enriching the theory of mathematics.

A sequence space is defined as a linear space of sequences. If ω denotes the set of all functions from the set of positive integers N to the field K, then it becomes a vector space. Sequence space is defined as a linear subspace of ω. A sequence of the form $(x_k)_{k=1}^{\infty}$ is called a single sequence and a sequence of the form $(x_{mn})_{m,n=1}^{\infty}$ is called a double sequence or a matrix.

Let c, c_0, l_∞, and l_p be the linear spaces of convergent, null, bounded, and absolutely p-summable sequences $x = (x_i)$ with complex terms respectively; and norm be given by $||x||_\infty = \sup|x_i|, i \in N$.

Definition 1.1. A non-decreasing, continuous, and convex function $M : [0,\infty) \to [0,\infty)$ is said to be an Orlicz function if M satisfies the following conditions:

(1) $M(0) = 0$
(2) $M(t) > 0$ for $t > 0$
(3) $M(t) \to \infty$ as $t \to \infty$. (see [11])

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It is said to satisfy Δ_2- condition, if $M(2t) \leq QM(t)$, for all $t \geq 0$ and a constant $Q > 0$. It is equivalent to the condition $M(Kt) \leq QKM(t)$, $\forall t$ and $K > 1$.(see [11])

Definition 1.2. Lindenstrauss and Tzafriri [8] had used Orlicz function in order to construct Orlicz sequence space l_M given by

$$l_M = \left\{ \bar{x} = (\xi_k) \in \omega : \sum_{k=1}^{\infty} M\left(\frac{|\xi_k|}{s}\right) < \infty \text{ for some } s > 0 \right\}$$

of scalars (ξ_k). The space is named due to Władysław Orlicz, first defined in 1932 and the first detailed study on Orlicz spaces was given by Krasnosel'skii and Rutickii[11]. The Orlicz sequence space l_M becomes a Banach space when we define the norm as

$$||\bar{x}||_M = \inf\left\{ s > 0 : \sum_{k=1}^{\infty} M\left(\frac{|\xi_k|}{s}\right) \leq 1 \right\}$$

Moreover, l_M is closely related to the space l_p with $M(t) = t^p; 1 \leq p < \infty$.

Definition 1.3. Kizmaz [6] defined the difference sequence spaces by
$c_o(\Delta) = \{\bar{x} = (\xi_k) : \Delta\xi \in c_0\}$
$c(\Delta) = \{\bar{x} = (\xi_k) : \Delta\xi \in c\}$
$l_\infty(\Delta) = \{\bar{x} = (\xi_k) : \Delta\xi \in l_\infty\}$ where, $\Delta\xi = (\Delta\xi_k) = (\xi_k – \xi_{k+1})$ and showed that these spaces are Banach spaces with the norm given by $||\bar{x}|| = |\xi_1| + ||\Delta\xi||_\infty$. A sequence $\bar{x} = (\xi_k)$ is called Δ-convergent if the lim $\Delta\xi_k$ is finite and exists. Every convergent sequence is Δ-convergent but not conversely. If we consider the sequence $\xi_k = 1 + k$ for all natural numbers k, then $(\Delta\xi_k) = (\xi_k – \xi_{k+1}) = -1$ for each natural numbers k. Thus, $\bar{x} = (\xi_k)$ is divergent but it is Δ-convergent.

Definition 1.4. Let C be the field of complex numbers and X be a normed space over C . Let $\omega(X)$ denote the linear space of all sequences $\bar{x} = (\xi_k), \xi_k \in X, k \geq 1$ with usual coordinate wise addition and scalar multiplication i.e., for all $\bar{x}, \bar{y} \in \omega(X)$ and $\alpha \in C$, $\bar{x} + \bar{y} = (\xi_k + \eta_k)$ and $\alpha\bar{x} = (\alpha\xi_k)$. We shall write $\omega(C)$ by ω. Further, $\bar{\lambda} = (\lambda_k) \in \omega$ and $\bar{x} \in \omega(X)$ we have $\bar{\lambda}\bar{x} = (\lambda_k\xi_k)$. Moreover, a scalar(vector) valued sequence space means a linear subspace of $\omega(X)$.

The various topological and algebraic properties of sequence spaces with the help of Orlicz function have been introduced,studied and investigated as a generalization of various sequence spaces. For instances, we refer a few: Bhardwaj and Bala[19], Maddox [7], Ghosh and Srivastava[3], Kamthan and Gupta[15], Karakaya[18], Khan[17], Kolk[4], Parashar and Choudhary[16], Pahari[14], Rao and Subremanina[10], Savas and Patterson[5], Wilansky[1], Tripathy and Mahanta [2],Srivastava and Pahari[9],Basarir and Altundag[13],and Et et al.[12].

2. The Class $c_0(M, (X, ||.||), \bar{a}, \bar{\alpha})$ of Vector Valued Difference Sequences

Let $\bar{\alpha} = (\alpha_k)$ and $\bar{\gamma} = (\gamma_k)$ be the sequences of complex numbers with non-zero terms and $\bar{a} = (a_k)$ and $\bar{b} = (b_k)$ be sequences of positive real numbers. Let X be a normed space over C, and M be an Orlicz function. Now we introduce a new class

$$c_0(M,(X,||.||),\bar{a},\bar{\alpha}) = \left\{\bar{x}=(\xi_k) : \lim_{k\to\infty} M\left(\frac{||a_k\Delta\xi_k||^{a_k}}{s}\right)=0, \text{where } \xi_k\in X, k\geq 1; \text{for some } s>0\right\}.$$

It is a class of Normed space X-valued sequences. Furthermore, if $a_k = 1\ \forall k\in N$, then $c_0(M,(X,||.||),\bar{a},\bar{\alpha})$ is denoted by $c_0(M,(X,||.||),\bar{\alpha})$ and if $\alpha_k = 1\ \forall k\in N$, then $c_0(M,(X,||.||),\bar{a},\bar{\alpha})$ is denoted by $c_0(M,(X,||.||),\bar{a})$. If $a_k=\alpha_k=1\ \forall k\in N$, then the class $c_0(M,(X,||.||),\bar{a},\bar{\alpha})$ is denoted by $c_0(M,(X,||.||))$.

In this section, we characterize some topological linear structures of $c_0(M,(X,||.||),\bar{a},\bar{\alpha})$ and then investigate some of the inclusion relations between the classes $c_0(M,(X,||.||),\bar{a},\bar{\alpha})$ that arise in terms of $\bar{a}$ and $\bar{\alpha}$. Throughout this paper, we shall denote sup $a_k = S$, $\forall k\in N$. When the sequences a_k and b_k both occur, then we use sup $a_k = S(a)$ and sup $b_k = S(b)$.

3. Some Topological Linear Structures on $c_0(M,(X,||.||),\bar{a},\bar{\alpha})$

In this section, we will study the linear structure of $c_0(M,(X,||.||),\bar{a},\bar{\alpha})$ of vector valued difference sequences defined by using Orlicz function M. It is a generalization of sequence space c_0. Also, we will investigate the conditions pertaining to the containment relations of $c_0(M,(X,||.||),\bar{a},\bar{\alpha})$ in terms of $\bar{a}$ and $\bar{\alpha}$. In this article, the following inequality will be used: $|x+y|^{u_k}\leq H\{|x|^{u_k}+|y|^{u_k}\}$, Where; $x,y\in C, 0<a_k\leq \sup_k a_k = S$, and $H=\max(1,2^{S-1})$. Throughout the article we shall denote $c_k=\frac{b_k}{a_k}$ and $\delta_k = |\frac{\alpha_k}{\gamma_k}|^{a_k}$.

Theorem 3.1. *The class $c_0(M,(X,||.||),\bar{a},\bar{\alpha})$ of difference sequences is a linear space over C if and only if $\sup_k a_k = S<\infty$.*

Proof. Necessary part: Let $c_0(M,(X,||.||),\bar{a},\bar{\alpha})$ be a linear space over C but $\sup_k a_k=\infty$. Then there exists a sequence of positive integers $(k(n))$ satisfying the condition $1\leq k(n)<k(n+1); n\geq 1$, and $a_{k(n)}>n; n\geq 1$. Let $z\in X$ with $||z||=1$. We now define a sequence $\bar{x}=(\xi_k)$ as

$$\Delta\xi_k = \begin{cases} {\alpha_{k(n)}}^{-1} n^{-\frac{2}{a_{k(n)}}} z & \text{if } k=k(n);\ n\geq 1 \\ 0 & \text{otherwise} \end{cases} \tag{3.1}$$

Let $s>0$ be given. Then from (3.1), using convexity of Orlicz function M, we can write

$$M\left(\frac{||\alpha_k\Delta\xi_k||^{a_k}}{s}\right) = M\left(\frac{||n^{-\frac{2}{a_{k(n)}}}z||^{a_{k(n)}}}{s}\right) = M\left(\frac{1}{n^2 s}\right)\leq M\left(\frac{1}{s}\right)\frac{1}{n^2}$$

and $M\left(\frac{||\alpha_k\Delta\xi_k||^{a_k}}{s}\right)=0$, otherwise. This shows that $\lim_{k\to\infty} M\left(\frac{||\alpha_k\Delta\xi_k||^{a_k}}{s}\right)=0$ and therefore $\bar{x}\in c_0(X,||.||,\bar{a},\bar{\alpha})$. But on the other hand, for any $s>0$ and taking $v=4$, we find that for $k=k(n); n\geq 1$

$$M\left(\frac{||\alpha_k v\Delta\xi_k||^{a_k}}{s}\right) = M\left(\frac{||4n^{-\frac{2}{a_{k(n)}}}z||^{a_{k(n)}}}{s}\right) = M\left(\frac{4^n}{n^2 s}\right)\geq M\left(\frac{1}{s}\right)$$

This shows that $\lim_{k\to\infty} M\left(\frac{||\alpha_k v\Delta\xi_k||^{a_k}}{s}\right) \neq 0$ and $v\bar{x}$ does not belong $c_0(M,(X,||.||),\bar{a},\bar{\alpha})$, which contradicts our assumption.

For sufficiency part, assume that $\sup_k a_k = S < \infty$. Let $\bar{x} = (\xi_k), \bar{y} = (\eta_k) \in c_0(X,||.||,\bar{a},\bar{\alpha})$ and $\beta, v \in C$. Then there exists positive real numbers s_1 and s_2 such that $\lim_{k\to\infty} M\left(\frac{||\alpha_k\Delta\xi_k||^{a_k}}{s_1}\right) = 0$ and $\lim_{k\to\infty} M\left(\frac{||\alpha_k\Delta\eta_k||^{a_k}}{s_2}\right) = 0$. We now choose $s_3 > 0$ such that $\max(1,|\beta|^s) \leq \frac{s_3}{2Hs_1}$ and $\max(1,|v|^s) \leq \frac{s_3}{2Hs_2}$. Now applying the convex and non decreasing properties of Orlicz function, we have

$$\lim_{k\to\infty} M\left(\frac{||\alpha_k(\beta\Delta\xi_k+v\Delta\eta_k)||^{a_k}}{s_3}\right)$$
$$\leq \lim_{k\to\infty} M\left(\frac{H||\beta\alpha_k\Delta\xi_k||^{a_k}+H||v\alpha_k\Delta\eta_k)||^{a_k}}{s_3}\right)$$
$$= \lim_{k\to\infty} M\left(\frac{H|\beta|^{a_k}||\alpha_k\Delta\xi_k||^{a_k}}{s_3} + \frac{H|v|^{a_k}||\alpha_k\Delta\eta_k)||^{a_k}}{s_3}\right)$$
$$\leq \lim_{k\to\infty} M\left(\frac{H\max(1,|\beta|^s)||\alpha_k\Delta\xi_k||^{a_k}}{s_3} + \frac{H\max(1,|v|^s)||\alpha_k\Delta\eta_k)||^{a_k}}{s_3}\right)$$
$$\leq \lim_{k\to\infty} M\left(\frac{1}{2s_1}||\alpha_k\Delta\xi_k||^{a_k} + \frac{1}{2s_2}||\alpha_k\Delta\eta_k)||^{a_k}\right)$$
$$\leq \frac{1}{2}\lim_{k\to\infty} M\left(\frac{||\alpha_k\Delta\xi_k||^{a_k}}{s_1}\right) + \frac{1}{2}\lim_{k\to\infty} M\left(\frac{||\alpha_k\Delta\eta_k||^{a_k}}{s_2}\right)$$
$$= 0.$$

This implies that $c_0(M,(X,||.||),\bar{a},\bar{\alpha})$ forms a linear space over C. □

Lemma 3.2. *For any sequence* $\bar{a} = (a_k), c_0(M,(X,||.||),\bar{a},\bar{\alpha}) \subset c_0(M,(X,||.||),\bar{a},\bar{\gamma})$ *if lim inf* $_k$ $\delta_k > 0$.

Proof. Assume that lim $\inf_k$ $\delta_k > 0$ i.e. lim $\inf_k|\frac{\alpha_k}{\gamma_k}|^{a_k} > 0$. Then there exists $q > 0$ such that

$q|\gamma_k|^{a_k} < |\alpha_k|^{a_k}$ $\forall k$ sufficiently large. Let $\bar{x} = (\xi_k) \in c_0(M,(X,||.||),\bar{a},\bar{\alpha})$, then for some $s > 0$, we have

$\lim_{k\to\infty} M\left(\frac{||\alpha_k\Delta\xi_k||^{a_k}}{s}\right) = 0$. Now we choose $s_1 > 0$ such that $s \leq qs_1$. Using the non decreasing property of Orlicz function, we have $M\left(\frac{||\gamma_k\Delta\xi_k||^{a_k}}{s_1}\right) \leq M\left(\frac{||\alpha_k\Delta\xi_k||^{a_k}}{s}\right)$. This implies that $\bar{x} \in c_0(M,(X,||.||),\bar{a},\bar{\gamma})$ and hence $c_0(M,(X,||.||),\bar{a},\bar{\alpha}) \subset c_0(M,(X,||.||),\bar{a},\bar{\gamma})$.
□

Lemma 3.3. *Let* $\bar{a} = (a_k)$. *If* $c_0(M,(X,||.||),\bar{a},\bar{\alpha}) \subset c_0(M,(X,||.||),\bar{a},\bar{\gamma})$ *then lim* $\inf_k$ $\delta_k > 0$.

Proof. Assume that $c_0(M,(X,||.||),\bar{a},\bar{\alpha}) \subset c_0(M,(X,||.||),\bar{a},\bar{\gamma})$ holds but lim inf $_k|\frac{\alpha_k}{\gamma_k}|^{a_k} = 0$. Then there exists a sequence of positive integers $(k(n))$ satisfying the condition
$1 \leq k(n) < k(n+1), n \geq 1$ and

$$n^2|\alpha_{k(n)}|^{a_{k(n)}} < |\gamma_{k(n)}|^{a_{k(n)}}, \forall n > 1 \tag{3.2}$$

. Let $z \in X$, with $||z|| = 1$. We now define a sequence $\bar{x} = (\xi_k)$ as

$$\Delta\xi_k = \begin{cases} {\alpha_{k(n)}}^{-1} n^{-\frac{2}{a_{k(n)}}} z & \text{if } k = k(n);\ n \geq 1 \\ 0 & \text{otherwise} \end{cases} \tag{3.3}$$

Let $s > 0$ be given. Then for $k = k(n), n \geq 1$, using convexity of Orlicz function, we have

$$M\left(\frac{||\alpha_k \Delta\xi_k||^{a_k}}{s}\right) = M\left(\frac{||n^{-\frac{2}{a_{k(n)}}} z||^{a_{k(n)}}}{s}\right) = M\left(\frac{1}{n^2 s}\right) \leq M\left(\frac{1}{s}\right)\frac{1}{n^2}$$

and $M\left(\frac{||\alpha_k \Delta\xi_k||^{a_k}}{s}\right) = 0$, otherwise. This shows that $\lim_{k\to\infty} M\left(\frac{||\alpha_k \Delta\xi_k||^{a_k}}{s}\right) = 0$ and therefore $\bar{x} \in c_0(X, ||.||, \bar{a}, \bar{\alpha})$.

But on the other hand, for any $s > 0$ and $k = k(n), n \geq 1$ and from (3.2) and (3.3), we obtain

$M\left(\frac{||\gamma_k \Delta\xi_k||^{a_k}}{s}\right) = M\left(\frac{||\frac{\gamma_{k(n)}}{\alpha_{k(n)}} n^{-\frac{2}{a_{k(n)}}} z||^{a_{k(n)}}}{s}\right) = M\left(|\frac{\gamma_{k(n)}}{\alpha_{k(n)}}|^{a_{k(n)}} \frac{1}{n^2 s}\right) \geq M\left(\frac{1}{s}\right)$. This shows that $\lim_{k\to\infty} M\left(\frac{||\gamma_k \Delta\xi_k||^{a_k}}{s}\right) \neq 0$ and hence $\bar{x} \notin c_0(M, (X, ||.||), \bar{a}, \bar{\gamma})$, a contradiction. This completes the proof. □

Next, combining Lemma (3.2) and Lemma(3.3), we obtain the theorem given below.

Theorem 3.4. *For any* $\bar{a} = (a_k), c_0(M, (X, ||.||), \bar{a}, \bar{\alpha}) \subset c_0(M, (X, ||.||), \bar{a}, \bar{\gamma})$ *if and only if lim inf*$_k$ $\delta_k > 0$.

Theorem 3.5. *For any* $\bar{a} = (a_k), c_0(M, (X, ||.||), \bar{a}, \bar{\gamma}) \subset c_0(M, (X, ||.||), \bar{a}, \bar{\alpha})$ *if and only if lim sup*$_k$ $\delta_k > 0$.

Proof. Let lim sup $_k |\frac{\alpha_k}{\gamma_k}|^{a_k} < \infty$. Then there exists $Q > 0$ such that
$Q|\gamma_k|^{a_k} > |\alpha_k|^{a_k}$ $\forall k$ sufficiently large. Then analogous to the proof of Lemma (3.2),the sufficient part follows.
For the necessity part of the theorem, suppose $c_0(M, (X, ||.||), \bar{a}, \bar{\gamma}) \subset c_0(M, (X, ||.||), \bar{a}, \bar{\alpha})$ holds. Suppose lim sup$_k$ $\delta_k = \infty$. Then there exists a sequence of positive integers $k(n)$ satisfying $1 \leq k(n) < k(n+1); n \geq 1$, for which

$$|\alpha_{k(n)}|^{a_{k(n)}} > n^2 |\gamma_{k(n)}|^{a_{k(n)}}, \forall n > 1 \tag{3.4}$$

Now as proved in Lemma (3.3), corresponding to $z \in X$ with $||z|| = 1$ we can construct a sequence $\bar{x} = (\xi_k)$ by

$$\Delta\xi_k = \begin{cases} {\gamma_{k(n)}}^{-1} n^{-\frac{2}{a_{k(n)}}} z & \text{if } k = k(n);\ n \geq 1 \\ 0 & \text{otherwise} \end{cases} \tag{3.5}$$

Now in view of (3.4) and (3.5), we can show that $\bar{x} \in c_0(M, (X, ||.||), \bar{a}, \bar{\gamma})$ but $\bar{x} \notin c_0(M, (X, ||.||), \bar{a}, \bar{\alpha})$ which contradicts our assumption. This completes the proof. □

On combining Theorem (3.4) and Theorem (3.5), one can obtain the following theorem.

Theorem 3.6. *For any* $\bar{a} = (a_k), c_0(M, (X, ||.||), \bar{a}, \bar{\alpha}) = c_0(M, (X, ||.||), \bar{a}, \bar{\gamma})$ *if and only if* $0 <$ *lim inf*$_k$ $\delta_k \leq$ *lim sup*$_k$ $\delta_k < \infty$.

Corollary 3.7. *Let* $\bar{a} = (a_k)$. *Then we have*

(1) $c_0(M, (X, ||.||), \bar{a}, \bar{\alpha}) \subset c_0(M, (X, ||.||), \bar{a})$ *if and only if lim inf* $_k|\alpha_k|^{a_k} > 0$.

(2) $c_0(M,(X,||.||),\bar{a}) \subset c_0(M,(X,||.||),\bar{a},\bar{\alpha})$ *if and only if lim sup*$_k |\alpha_k|^{a_k} < \infty$.
(3) $c_0(M,(X,||.||),\bar{a},\bar{\alpha}) = c_0(M,(X,||.||),\bar{a})$ *if and only if* $0 < \text{lim inf}_k\, |\alpha_k|^{a_k} \le \text{lim sup}_k\, |\alpha_k|^{a_k} < \infty$.

Proof. The statements (1), (2), (3) follow by taking $\gamma_k = 1, \forall k \in N$ in Theorem (3.4), (3.5), (3.6). □

Lemma 3.8. *For any* $\bar{\alpha} = (\alpha_k)$,*if* $c_0(M,(X,||.||),\bar{a},\bar{\alpha}) \subset c_0(M,(X,||.||),\bar{b},\bar{\alpha})$ *then* $\text{lim inf}_k c_k > 0$.

Proof. Suppose that $c_0(M,(X,||.||),\bar{a},\bar{\alpha}) \subset c_0(M,(X,||.||),\bar{b},\bar{\alpha})$ holds but $\lim\inf_k c_k = 0$ i.e $\lim\inf_k \frac{b_k}{a_k} = 0$. Then there exists a sequence of positive integers $(k(n))$ satisfying $1 \le k(n) < k(n+1)$, for which

$$nb_{k(n)} < a_{k(n)}, \forall n \ge 1 \tag{3.6}$$

Now, let $z \in X$ with $||z|| = 1$. We can construct a sequence $\bar{x} = (\xi_k)$ by

$$\Delta\xi_k = \begin{cases} \alpha_{k(n)}^{-1} n^{-\frac{1}{a_{k(n)}}} z & \text{if } k = k(n);\ n \ge 1 \\ 0 & \text{otherwise} \end{cases} \tag{3.7}$$

Let $s > 0$.Then for $k = k(n), n \ge 1$ and using properties of Orlicz function, we have

$$M\left(\frac{||\alpha_k \Delta\xi_k||^{a_k}}{s}\right) = M\left(\frac{||n^{-\frac{1}{a_{k(n)}}} z||^{a_{k(n)}}}{s}\right) = M\left(\frac{||z||^{a_{k(n)}}}{ns}\right) \le \frac{1}{n} M\left(\frac{1}{s}\right)$$

and $M\left(\frac{||\alpha_k\Delta\xi_k||^{a_k}}{s}\right) = 0$ for $k \ne k(n), n \ge 1$.

Thus $\lim_{k\to\infty} M\left(\frac{||\alpha_k\Delta\xi_k||^{a_k}}{s}\right) = 0$ and hence $\bar{x} \in c_0(M,(X,||.||),\bar{a},\bar{\alpha})$. But, for each $k = k(n), n \ge 1$ and from (3.6) and (3.7), we have

$$M\left(\frac{||\alpha_k \Delta\xi_k||^{b_k}}{s}\right) = M\left(\frac{||n^{-\frac{1}{a_{k(n)}}} z||^{b_k(n)}}{s}\right) \ge M\left(\frac{1}{sn^{\frac{1}{n}}}\right) \ge M\left(\frac{1}{s\sqrt{e}}\right).$$

This shows that $\lim_{k\to\infty} M\left(\frac{||\alpha_k\Delta\xi_k||^{b_k}}{s}\right) \ne 0$ and so $\bar{x}$ does not belong to $c_0(M,(X,||.||),\bar{b},\bar{\alpha})$, a contradiction. □

Lemma 3.9. *For any* $\bar{\alpha} = (\alpha_k), c_0(M,(X,||.||),\bar{a},\bar{\alpha}) \subset c_0(M,(X,||.||),\bar{b},\bar{\alpha})$ *if* $\text{lim inf}_k c_k > 0$

Proof. Assume that $\lim\inf_k c_k > 0$ i.e. $\lim\inf_k \frac{b_k}{a_k} > 0$. Then there exists $q > 0$ such that $\frac{b_k}{a_k} > q, \forall k$ sufficiently large. Let $\bar{x} = (\xi_k) \in c_0(M,(X,||.||),\bar{a},\bar{\alpha})$.Then for some $s > 0$, $\lim_{k\to\infty} M\left(\frac{||\alpha_k\Delta\xi_k||^{a_k}}{s}\right) = 0$. Hence for a given $\epsilon > 0$, if we choose a positive number η such that $\eta < 1$ satisfying $n^q M\left(\frac{1}{s}\right) < \epsilon$, then we have $M\left(\frac{||\alpha_k\Delta\xi_k||^{a_k}}{s}\right) < M\left(\frac{\eta}{s}\right), \forall k$ sufficiently large. Since M is non decreasing, therefore $\forall k$ sufficiently large, we have $||\alpha_k\xi_k||^{a_k} < \eta < 1$ and hence $||\alpha_k\xi_k|| < 1$. Since Orlicz function is convex, we can write,

$$M\left(\frac{||\alpha_k\Delta\xi_k||^{b_k}}{s}\right) \le M\left(\frac{[||\alpha_k\Delta\xi_k||^{a_k}]^q}{s}\right) \le M\left(\frac{\eta^q}{s}\right) \le \eta^q M\left(\frac{1}{s}\right) < \epsilon, \forall k,$$

sufficiently large. This implies $\lim_{k\to\infty} M\left(\frac{||\alpha_k \Delta\xi_k||^{b_k}}{s}\right) = 0$ and so $\bar{x} \in c_0(M,(X,||.||),\bar{b},\bar{\alpha})$ and hence $c_0(M,(X,||.||),\bar{a},\bar{\alpha}) \subset c_0(M,(X,||.||),\bar{b},\bar{\alpha})$. □

Next combining lemma (3.8) and lemma (3.9), one can obtain the following theorem.

Theorem 3.10. *For any $\bar{\alpha} = (\alpha_k), c_0(M,(X,||.||),\bar{a},\bar{\alpha}) \subset c_0(M,(X,||.||),\bar{b},\bar{\alpha})$ if and only if lim $\inf_k c_k > 0$*

Lemma 3.11. *For any sequence $\bar{\alpha} = (\alpha_k)$, if $c_0(M,(X,||.||),\bar{b},\bar{\alpha}) \subset c_0(M,(X,||.||),\bar{a},\bar{\alpha})$ then lim $\sup_k c_k < \infty$.*

Proof. Let the inclusion holds but lim $\sup_k c_k = \infty$. That is, lim $\sup_k \frac{b_k}{a_k} = \infty$. Then there exists a sequence of positive integers $(k(n))$ satisfying $1 \leq k(n) < k(n+1), n \geq 1$, for which

$$b_{k(n)} > n a_{k(n)}; \forall n \geq 1. \tag{3.8}$$

Let $z \in X$ with $||z|| = 1$. Now, We can define a sequence $\bar{x} = (\xi_k)$ by

$$\Delta\xi_k = \begin{cases} \alpha_{k(n)}^{-1} n^{-\frac{1}{b_{k(n)}}} z & \text{if } k = k(n);\ n \geq 1 \\ 0 & \text{otherwise} \end{cases} \tag{3.9}$$

Suppose $s > 0$.Then using properties of Orlicz function, for each $k = k(n), n \geq 1$; we obtain

$$M\left(\frac{||\alpha_k \Delta\xi_k||^{b_k}}{s}\right) = M\left(\frac{||n^{-\frac{1}{b_{k(n)}}} z||^{b_{k(n)}}}{s}\right) = M\left(\frac{||z||^{b_{k(n)}}}{ns}\right) \leq \frac{1}{n} M\left(\frac{1}{s}\right)$$

and $M\left(\frac{||\alpha_k \Delta\xi_k||^{b_k}}{s}\right) = 0$ for each $k \neq k(n), n \geq 1$.

This shows that $\lim_{k\to\infty} M\left(\frac{||\alpha_k \Delta\xi_k||^{b_k}}{s}\right) = 0$ and so $\bar{x} \in c_0(M,(X,||.||),\bar{b},\bar{\alpha})$. But, for each $k = k(n), n \geq 1$ and from (3.8) and (3.9), we obtain

$$M\left(\frac{||\alpha_k \Delta\xi_k||^{a_k}}{s}\right) = M\left(\frac{||n^{-\frac{1}{b_{k(n)}}} z||^{a_k(n)}}{s}\right) \geq M\left(\frac{1}{s n^{\frac{1}{n}}}\right) \geq M\left(\frac{1}{s\sqrt{e}}\right)$$

. This implies that $\lim_{k\to\infty} M\left(\frac{||\alpha_k \Delta\xi_k||^{a_k}}{s}\right) \neq 0$ and so $\bar{x}$ does not belong to $c_0(M,(X,||.||),\bar{a},\bar{\alpha})$, which is a contradiction. □

Lemma 3.12. *For any sequence $\bar{\alpha} = (\alpha_k), c_0(M,(X,||.||),\bar{b},\bar{\alpha}) \subset c_0(M,(X,||.||),\bar{a},\bar{\alpha})$ if lim $\sup_k c_k < \infty$.*

Proof. Assume that lim $\sup_k c_k$i.e. lim $\sup_k \frac{b_k}{a_k} < \infty$. Then $\exists Q > 0$ such that $\frac{b_k}{a_k} < Q, \forall k$ sufficiently large. Then analogous to Lemma (3.8), we can easily show that $c_0(M,(X,||.||),\bar{b},\bar{\alpha}) \subset c_0(M,(X,||.||),\bar{a},\bar{\alpha})$. □

Next, combining the Lemma (3.11) and Lemma (3.12), one can obtain the following theorem.

Theorem 3.13. *For any sequence $\bar{\alpha} = (\alpha_k), c_0(M,(X,||.||),\bar{b},\bar{\alpha}) \subset c_0(M,(X,||.||),\bar{a},\bar{\alpha})$ if and only if lim $\sup_k c_k < \infty$.*

On combining the Theorem (3.10) and Theorem (3.13), one can obtain the following theorem.

Theorem 3.14. *For any sequence* $\bar{\alpha} = (\alpha_k), c_0(M,(X,||.||),\bar{a},\bar{\alpha}) = c_0(M,(X,||.||),\bar{b},\bar{\alpha})$ *if and only if* $0 < lim\ inf_k c_k \leq lim\ sup_k c_k < \infty$.

Corollary 3.15. *For the sequence* $\bar{\alpha} = (\alpha_k)$*, following statements holds:*

(1) $c_0(M,(X,||.||),\bar{\alpha}) = c_0(M,(X,||.||),\bar{a},\bar{\alpha})$ *if and only if* $lim\ inf_k a_k > 0$.
(2) $c_0(M,(X,||.||),\bar{a},\bar{\alpha}) \subset c_0(M,(X,||.||),\bar{\alpha})$ *if and only if* $lim\ sup_k a_k < \infty$.
(3) $c_0(M,(X,||.||),\bar{a},\bar{\alpha}) = c_0(M,(X,||.||),\bar{\alpha})$ *if and only if* $0 < lim\ inf_k a_k \leq lim\ sup_k a_k < \infty$.

Proof. The statements $(1),(2),(3)$ follow by taking $a_k = 1; \forall k$ and replacing $\bar{b}$ by $\bar{a}$ in the theorems $(3.12),(3.13),(3.14)$ respectively. □

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