The Nepali Mathematical Sciences Report, Vol. 38, No.1, 2021: 39-50
DOI:https://doi.org/10.3126/nmsr.v38i1.38910

THE PILLAI–CHOWLA METHOD FOR AN ERROR TERM IN THE MEAN SQUARE OF $\delta_k(n)$

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Abstract: Let $k \geq 2$ be a fixed square-free integer and $\delta_k(n)$ be the greatest divisor of n which is coprime to k. We consider the error term $E_k^{(2)}(x)$ in the mean square of $\delta_k(n)$, $E_k^{(2)}(x) := \sum_{n\leq x}\delta_k^2(n) - \frac{k^2}{3\sigma(k^2)}x^3$, where $\sigma(n) = \sum_{d|n} d$. Using the Pillai–Chowla method we show $\sum_{n\leq x} E_k^{(2)}(n) \sim \frac{k^2}{6\sigma(k^2)}x^3$ (as $x\to\infty$) and $\int_1^\infty \frac{E_k^{(2)}(t)}{t^3}dt = \frac{k^2}{3\sigma(k^2)}$. To prove them, we make a framework for the Pillai–Chowla method.

Key Words: The greatest divisor, Mean square of an arithmetical function, Mean values of error terms
AMS (MOS) Subject Classification. 11A25, 11N37

Received: December 29,2020 Accepted: May 17, 2021 Published Online: June 30, 2021

1. Introduction

Let $\varphi(n)$ be the Euler function, that is, $\varphi(n) = \sum_{\substack{1\leq m\leq n\\(m,n)=1}} 1$. Mertens [13] showed that for any $x \geq 2$

$$\sum_{n\leq x}\varphi(n) = \frac{3}{\pi^2}x^2 + O\left(x\log x\right). \tag{1.1}$$

In [16], Pillai and Chowla studied the error term in (1.1)

$$E(x) := \sum_{n\leq x}\varphi(n) - \frac{3}{\pi^2}x^2 \quad (x\geq 1) \tag{1.2}$$

and showed that

$$\sum_{n\leq x} E(n) = \frac{3}{2\pi^2}x^2 + o\left(x^2\right), \tag{A}$$

$$E(x) \neq o(x\log\log\log x). \tag{B}$$

This work is supported by JSPS KAKENHI: 19K03449 .

The former is improved by Suryanarayana and Sitaramachandra Rao [21] and the latter is developed by Erdős and Shapiro [7] and Montgomery [14]. In these notes, we shall apply the method to obtain (A) due to Pillai and Chowla [16] to the mean square of $\delta_k(n)$, where $\delta_k(n)$ denotes the greatest divisor of n which is coprime to a fixed square-free integer $k = p_1 p_2 \cdots p_{\nu(k)}$ (p_i are distinct primes). As for the average $\sum_{n\le x} \delta_k(n)$, Joshi and Vaidya [12] showed that

$$\sum_{n\le x} \delta_k(n) = \frac{k}{2\sigma(k)} x^2 + O_k(x), \quad (\text{as } x \to \infty),$$

where $\sigma(n) = \sum_{d|n} d$ is the divisor function. Let

$$E_k(x) := \sum_{n\le x} \delta_k(n) - \frac{k}{2\sigma(k)} x^2 \quad (x \ge 1). \tag{1.3}$$

On $E_k(x)$, many results were shown in the literature, Suryanarayana [20], Joshi and Vaidya [12], Ramachandran [17], Herzog and Maxsein [9], Adhikari, Balasubramanian and Sankaranarayanan [3], Adhikari and Balasubramanian [2], Pétermann [15], Adhikari [1], Adhikari and Soundararajan [5], Adhikari and Chakraborty [4], Adhikari and Thangadurai [6], lately Singh [19], Igawa and Sharma [11].

As a generalisation of an Ω-result in Joshi and Vaidya [12], Herzog and Maxsein [9] obtained $E_k(x) = \Omega_\pm(x)$, moreover, deduced that

$$\left.\begin{aligned} &\sum_{n\le x} E_k(n) = \frac{k}{4\sigma(k)} x^2 + O_k\left(x(\log x)^{\nu(k)}\right), \\ &\int_1^x E_k(t)dt = O_k\left(x(\log x)^{\nu(k)}\right), \end{aligned}\right\} \quad (\text{as } x \to \infty). \tag{C}$$

In addition they obtained an estimate for $\int_1^x E_k(t)^2 dt$.

We now analyse the results corresponding to (C) for an error term in the mean square $\sum_{n\le x} \delta_k^2(n)$. In our previous paper [8], we had for any $x \ge 1$

$$\sum_{n\le x} \delta_k^2(n) = \frac{k^2}{3\sigma(k^2)} x^3 + O_k\left(x^2\right), \tag{1.4}$$

and we defined an error term in formula (1.4) as

$$E_k^{(2)}(x) := \sum_{n\le x} \delta_k^2(n) - \frac{k^2}{3\sigma(k^2)} x^3 \quad (x \ge 1). \tag{1.5}$$

Moreover, in [8], we introduced another error term $R_k^{(2)}(x)$ as

$$R_k^{(2)}(x) := E_k^{(2)}(x) + \frac{k}{6} x \quad (x \ge 1) \tag{1.6}$$

and showed that for any prime p

$$R_p^{(2)}(x) = \Omega_\pm\left(x^2\right)$$

which is a corresponding result to [12]. However, in these present notes we would like to change the definition (1.6) of $R_k^{(2)}(x)$ as

$$R_k^{(2)}(x) := E_k^{(2)}(x) + (-1)^{\nu(k)+1} \frac{k}{6} x \quad (x \ge 1), \tag{1.7}$$

where k is a fixed square-free integer and $\nu(k)$ is the number of distinct prime divisors.[1]

Using the method in [16] and some ideas from [17], [9], [3] and [11], we obtain the following theorem.

Theorem 1.1. *Let $k \geq 2$ be a fixed square-free integer. As for $E_k^{(2)}(x)$ and $R_k^{(2)}(x)$ (defined by (1.5) and (1.7), respectively), for any $x \geq 1$ we have*

$$\text{(a)} \quad \sum_{n\leq x} E_k^{(2)}(n) = \frac{k^2}{6\sigma(k^2)}x^3 + O_k\left(x^2\right),$$
$$\sum_{n\leq x} R_k^{(2)}(n) = \frac{k^2}{6\sigma(k^2)}x^3 + O_k\left(x^2\right),$$
$$\text{(b)} \quad \int_1^x E_k^{(2)}(t)dt = O_k\left(x^2\right),$$
$$\int_1^x R_k^{(2)}(t)dt = O_k\left(x^2\right).$$

It is trivial that assertion (a) of Theorem 1.1 implies $R_k^{(2)}(n)$, $E_k^{(2)}(n) \neq o(n^2)$. On the other hand, we also obtain the following estimations.

Theorem 1.2. *We have*

$$\text{(a)} \quad \int_1^\infty \frac{E_k^{(2)}(t)}{t^3}dt = \frac{k^2}{3\sigma(k^2)},$$
$$\text{(b)} \quad \int_1^\infty \frac{R_k^{(2)}(t)}{t^3}dt = \frac{k^2}{3\sigma(k^2)} + \frac{(-1)^{\nu(k)+1}k}{6}.$$

Theorem 1.3. *We have*

$$\text{(a)} \quad \int_1^x \frac{E_k^{(2)}(t)}{t^2}dt = O_k(1),$$
$$\text{(b)} \quad \int_1^x \frac{R_k^{(2)}(t)}{t^2}dt = \frac{(-1)^{\nu(k)+1}k}{6}\log x + O_k(1).$$

To deduce Theorems 1.2 and 1.3 we find a framework for the Pillai–Chowla method in [16] for the result (A). Here we note that as for $E(x)$ in the above (1.2) Suryanarayana [22, p. 184] has shown that

$$\int_1^\infty \frac{E(t)}{t^2}dt = \frac{3}{\pi^2}. \tag{1.8}$$

This impies the framework for the Pillai–Chowla method.

Notation. Throughout this paper, we use $[x]$ to express the integer part of a positive number x, and $\{x\} = x - [x]$ to denote a fractional part of $x \geq 0$. For a positive integer n we write $n = dd'$ as a product of positive integers d and d'.

[1]In [8], even if we use the new definition (1.7) of $R_k^{(2)}(x)$ we can obtain the same assertions for theorems.

2. Proof of Theorem 1.1

To prove Theorem 1.1, we shall apply ideas of Pillai and Chowla [16], Ramachandran [17], Herzog and Maxsein [9], Adhikari, Balasubramanian and Sankaranarayanan [3], Igawa and Sharma [11] and Segal [18]. First, we collect some lemmas and then we prove Theorem 1.1.

As we mentioned $k \geq 2$ is a fixed square-free integer, moreover, $\nu(k)$ denotes the number of distinct prime divisors of k, by p a prime number.

Lemma 2.1 (cf. [17], [9], [3] and [11]). *For any positive integer n we have a Dirichlet convolution for $\delta_k^2(n)$ as*

$$\delta_k^2(n) = \sum_{d|n} g_{(k,2)}(d) \left(\frac{n}{d}\right)^2, \tag{2.1}$$

where $g_{(k,2)}(n)$ is a multiplicative function which is defined by $g_{(k,2)}(1) = 1$ and

$$g_{(k,2)}\left(p^m\right) = \begin{cases} 1 - p^2 & (\textit{if } p|k) \\ 0 & (\textit{if } p \nmid k) \end{cases}, \tag{2.2}$$

for any prime powers p^m $(m = 1, 2, \ldots)$.

Proof. We consider the Dirichlet series $\sum_{n=1}^{\infty} \delta_k^2(n) n^{-s}$ for $\operatorname{Re} s > 2$, which is convergent absolutely by the trivial estimate $|\delta_k(n)| \leq n$. Note that by the definition (2.2) of $g_{(k,2)}(n)$ we observe that

$$\prod_{p|k} \frac{1 - \frac{p^2}{p^s}}{1 - \frac{1}{p^s}} = \sum_{n=1}^{\infty} \frac{g_{(k,2)}(n)}{n^s} \quad (\operatorname{Re} s > 0). \tag{2.3}$$

Since $\delta_k(n)$ is a multiplicative function satisfying for any prime powers p^m $(m \geq 1)$,

$$\delta_k(p^m) = \begin{cases} p^m & (\text{if } p \nmid k) \\ 1 & (\text{if } p|k) \end{cases},$$

we observe that

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\delta_k^2(n)}{n^s} &= \prod_p \left(1 + \frac{\delta_k^2(p)}{p^s} + \frac{\delta_k^2(p^2)}{p^{2s}} + \cdots\right) \\ &= \prod_{p|k} \left(1 - \frac{1}{p^s}\right)^{-1} \prod_{p \nmid k} \left(1 - \frac{p^2}{p^s}\right)^{-1} \\ &= \frac{1}{\prod_p \left(1 - \frac{p^2}{p^s}\right)} \prod_{p|k} \frac{1 - \frac{p^2}{p^s}}{1 - \frac{1}{p^s}} \\ &= \sum_{n=1}^{\infty} \frac{n^2}{n^s} \sum_{n=1}^{\infty} \frac{g_{(k,2)}(n)}{n^s} \\ &= \sum_{n=1}^{\infty} \frac{\sum_{d|n} g_{(k,2)}(d) \left(\frac{n}{d}\right)^2}{n^s}. \end{aligned}$$

Then we have the assertion of Lemma 2.1. $\square$

Remark 2.2. By (2.2) and identity (2.3) we observe that for $\operatorname{Re} s > 0$

$$\sum_{n=1}^{\infty} \frac{|g_{(k,2)}(n)|}{n^s} = \prod_{p|k} \left(1 + \frac{p^2-1}{p^s-1}\right). \tag{2.4}$$

Lemma 2.3 (cf. [17, pp. 337–338], [9, p. 147], [3, p. 832])**.** *Let $k = p_1 p_2 \cdots p_{\nu(k)}$ be a square-free integer ≥ 2. For any integer $n \geq 1$ and real $x \geq 1$ we have*

(a) $|g_{(k,2)}(n)| \leq \prod_{p|k} (p^2-1)$

(b) $\sum_{n \leq x} |g_{(k,2)}(n)| \leq k^2 \left(\frac{\log 2x}{\log 2}\right)^{\nu(k)}$

(c) $\sum_{n=1}^{\infty} \frac{g_{(k,2)}(n)}{n^3} = \frac{k^2}{\sigma(k^2)}$

(d) $\sum_{n \leq x} \frac{g_{(k,2)}(n)}{n^3} = \frac{k^2}{\sigma(k^2)} + O_k\left(\frac{(\log 2x)^{\nu(k)}}{x^3}\right)$

(e) $\sum_{n=1}^{\infty} \frac{g_{(k,2)}(n)}{n^2} = 0$

(f) $\sum_{n \leq x} \frac{g_{(k,2)}(n)}{n^2} = O_k\left(\frac{(\log 2x)^{\nu(k)}}{x^2}\right)$

(g) $\sum_{n=1}^{\infty} \frac{|g_{(k,2)}(n)|}{n^2} = 2^{\nu(k)}$

(h) $\sum_{n=1}^{\infty} \frac{g_{(k,2)}(n)}{n} = (-1)^{\nu(k)} k$

(i) $\sum_{n \leq x} \frac{g_{(k,2)}(n)}{n} = (-1)^{\nu(k)} k + O_k\left(\frac{(\log 2x)^{\nu(k)}}{x}\right)$

(j) $\sum_{n=1}^{\infty} \frac{|g_{(k,2)}(n)|}{n} = \prod_{p|k} (p+2) \leq 2^{\nu(k)} k$

Proof. By the definition (2.2) of $g_{(k,2)}(n)$, the first assertion (a) is trivial. Using prime factors of $k = p_1 p_2 \cdots p_{\nu(k)}$ we have

$$\sum_{n \leq x} |g_{(k,2)}(n)| \leq \prod_{i=1}^{\nu(k)} \left(p_i^2 \sum_{p_i^m \leq x} 1\right) \leq k^2 \prod_{i=1}^{\nu(k)} \left(1 + \frac{\log x}{\log 2}\right) = k^2 \left(\frac{\log 2x}{\log 2}\right)^{\nu(k)}$$

that is, the second assertion (b). Assertions (c), (e) and (h) are trivial by (2.3). As for the assertion (d), using partial summation with (b) and (c) we have

$$\sum_{n \leq x} \frac{g_{(k,2)}(n)}{n^3} = \sum_{n=1}^{\infty} \frac{g_{(k,2)}(n)}{n^3} - \lim_{T \to \infty} \sum_{x < n \leq T} \frac{g_{(k,2)}(n)}{n^3} = \frac{k^2}{\sigma(k^2)} + O_k\left(\frac{(\log 2x)^{\nu(k)}}{x^3}\right).$$

We obtain an assertion (f) by estimation as in the above equation. As for the assertions (g) and (j), we put $s = 2$ and $s = 1$ in the identity (2.4), respectively. By (h) and partial summation we have assertion (i). $\square$

Remark 2.4. A certain generalisation of Lemma 2.3 is discussed in Igawa and Sharma [11].

Lemma 2.5 (cf. [9, Theorem 1], [3, Lemma 3.1], [16])**.** *For any $x \geq 1$ we have*

(a) $\displaystyle\sum_{n\leq x} \delta_k^2(n) = \frac{k^2}{3\sigma(k^2)}x^3 - x^2 \sum_{d'\leq x} \frac{g_{(k,2)}(d')}{d'^2}\left\{\frac{x}{d'}\right\} + O_k(x),$

(b) $\displaystyle\sum_{n\leq x} \frac{\delta_k^2(n)}{n^2} = \frac{k^2}{\sigma(k^2)}x - \sum_{d'\leq x} \frac{g_{(k,2)}(d')}{d'^2}\left\{\frac{x}{d'}\right\} + O_k\left(\frac{(\log 2x)^{\nu(k)}}{x^2}\right),$

(c) $\displaystyle\sum_{n\leq x} \frac{\delta_k^2(n)}{n} = \frac{k^2}{2\sigma(k^2)}x^2 - x \sum_{d'\leq x} \frac{g_{(k,2)}(d')}{d'^2}\left\{\frac{x}{d'}\right\} + O_k(1).$

Proof. Using the Dirichlet convolution (2.1) of Lemma 2.1 we have

$$\begin{aligned}
\sum_{n\leq x} \delta_k^2(n) &= \sum_{dd'\leq x} d^2 g_{(k,2)}(d') \\
&= \sum_{d'\leq x} g_{(k,2)}(d') \sum_{d\leq x/d'} d^2 \\
&= \frac{1}{3}\sum_{d'\leq x} g_{(k,2)}(d')\left[\frac{x}{d'}\right]^3 + \frac{1}{2}\sum_{d'\leq x} g_{(k,2)}(d')\left[\frac{x}{d'}\right]^2 + \frac{1}{6}\sum_{d'\leq x} g_{(k,2)}(d')\left[\frac{x}{d'}\right] \\
&=: S_1 + S_2 + S_3 \quad \text{(say)}
\end{aligned} \tag{2.5}$$

estimate S_1, S_2, and S_3.

By (d), (b) and (j) of Lemma 2.3 we observe that

$$\begin{aligned}
S_1 &= \frac{x^3}{3}\sum_{d'\leq x} \frac{g_{(k,2)}(d')}{d'^3} - x^2 \sum_{d'\leq x} \frac{g_{(k,2)}(d')}{d'^2}\left\{\frac{x}{d'}\right\} + x \sum_{d'\leq x} \frac{g_{(k,2)}(d')}{d'}\left\{\frac{x}{d'}\right\}^2 \\
&\quad - \frac{1}{3}\sum_{d'\leq x} g_{(k,2)}(d')\left\{\frac{x}{d'}\right\}^3 \\
&= \frac{k^2}{3\sigma(k^2)}x^3 - x^2 \sum_{d'\leq x} \frac{g_{(k,2)}(d')}{d'^2}\left\{\frac{x}{d'}\right\} + O_k(x) + O_k\left((\log 2x)^{\nu(k)}\right).
\end{aligned} \tag{2.6}$$

By (f), (b) and (j) of Lemma 2.3 we have

$$\begin{aligned}
S_2 &= \frac{x^2}{2}\sum_{d'\leq x} \frac{g_{(k,2)}(d')}{d'^2} - x \sum_{d'\leq x} \frac{g_{k,2}(d')}{d'}\left\{\frac{x}{d'}\right\} + \frac{1}{2}\sum_{d'\leq x} g_{k,2}(d')\left\{\frac{x}{d'}\right\}^2 \\
&= O_k(x) + O_k\left((\log 2x)^{\nu(k)}\right)
\end{aligned} \tag{2.7}$$

By (i) and (b) of Lemma 2.3 we see

$$S_3 = \frac{x}{6}\sum_{d'\leq x} \frac{g_{(k,2)}(d')}{d'} - \frac{1}{6}\sum_{d'\leq x} g_{(k,2)}(d')\left\{\frac{x}{d'}\right\} = (-1)^{\nu(k)}\frac{k}{6}x + O_k\left((\log 2x)^{\nu(k)}\right). \tag{2.8}$$

Collecting these results (2.5)–(2.8) we reach assertion (a).

As demomstrated above we obtain assertions (b) and (c) as follows.

$$\begin{aligned}
\sum_{n\leq x} \frac{\delta_k^2(n)}{n^2} &= \sum_{d'\leq x} \frac{g_{(k,2)}(d')}{d'^2}\left[\frac{x}{d'}\right] = x\sum_{d'\leq x} \frac{g_{(k,2)}(d')}{d'^3} - \sum_{d'\leq x} \frac{g_{(k,2)}(d')}{d'^2}\left\{\frac{x}{d'}\right\} \\
&= \frac{k^2}{\sigma(k^2)}x - \sum_{d'\leq x} \frac{g_{(k,2)}(d')}{d'^2}\left\{\frac{x}{d'}\right\} + O_k\left(\frac{(\log 2x)^{\nu(k)}}{x^2}\right)
\end{aligned}$$

and

$$\sum_{n\le x}\frac{\delta_k^2(n)}{n}=\frac{x^2}{2}\sum_{d'\le x}\frac{g_{(k,2)}(d')}{d'^3}-x\sum_{d'\le x}\frac{g_{(k,2)}(d')}{d'^2}\left\{\frac{x}{d'}\right\}+\frac{1}{2}\sum_{d'\le x}\frac{g_{(k,2)}(d')}{d'}\left\{\frac{x}{d'}\right\}^2$$
$$+\frac{x}{2}\sum_{d'\le x}\frac{g_{(k,2)}(d')}{d'}-\frac{1}{2}\sum_{d'\le x}\frac{g_{(k,2)}(d')}{d'}\left\{\frac{x}{d'}\right\}$$
$$=\frac{k^2}{2\sigma(k^2)}x^2-x\sum_{d'\le x}\frac{g_{(k,2)}(d')}{d'^2}\left\{\frac{x}{d'}\right\}+O_k(1).$$

□

Remark 2.6. In (b) and (c) of Lemma 2.5 we use (g) Lemma 2.1 (g) to get

$$\sum_{n\le x}\frac{\delta_k^2(n)}{n^2}=\frac{k^2}{\sigma(k^2)}x+O_k(1),$$
$$\sum_{n\le x}\frac{\delta_k^2(n)}{n}=\frac{k^2}{2\sigma(k^2)}x^2+O_k(x),$$

as $x\to\infty$.

Definition 2.7 (cf. [3, p. 831], [16])**.** For any $x\ge 1$ we shall define $H_k^{(2)}(x)$ and $G_k^{(1)}(x)$ by

$$H_k^{(2)}(x):=\sum_{n\le x}\frac{\delta_k^2(n)}{n^2}-\frac{k^2}{\sigma(k^2)}x, \tag{2.9}$$

$$G_k^{(1)}(x):=\sum_{n\le x}\frac{\delta_k^2(n)}{n}-\frac{k^2}{2\sigma(k^2)}x^2. \tag{2.10}$$

Lemma 2.8 (cf. [16, Theorem III], [3, Lemma 3.3])**.** *For any $x\ge 1$ we have*

$$\sum_{n\le x}H_k^{(2)}(n)=\frac{k^2}{2\sigma(k^2)}x+O_k(1). \tag{2.11}$$

Proof. First of all, let $x\ge 1$ be any positive integer. Using the definition (2.9) we observe that

$$\sum_{n\le x}H_k^{(2)}(n)=\sum_{n\le x}\left(\sum_{m\le n}\frac{\delta_k^2(m)}{m^2}-\frac{k^2}{\sigma(k^2)}n\right)$$
$$=(x+1)\sum_{n\le x}\frac{\delta_k^2(n)}{n^2}-\sum_{n\le x}\frac{\delta_k^2(n)}{n}-\frac{k^2}{2\sigma(k^2)}x^2-\frac{k^2}{2\sigma(k^2)}x. \tag{2.12}$$

Here, we use (b) and (c) of Lemma 2.5, to obtain assertion (2.11) for any positive integer $x\ge 1$. If $x\ge 1$ is real, then

$$\sum_{n\le x}H_k^{(2)}(n)=\sum_{n\le [x]}H_k^{(2)}(n)=\frac{k^2}{2\sigma(k^2)}[x]+O_k(1) \tag{2.13}$$

which implies (2.11). □

We give another proof of Lemma 2.8 to elucidate the Pillai–Chowla method in [16]. We prepare the next lemma.

Lemma 2.9 (cf. [16, Theorem II], [3, Lemmas 3.2]). *For any $x \geq 1$ we have*

(a) $E_k^{(2)}(x) - x^2 H_k^{(2)}(x) = O_k(x)$,
(b) $E_k^{(2)}(x) - xG_k^{(1)}(x) = O_k(x)$.

Proof. By the definitions (1.5), (2.9) and Lemma 2.5 (a) and (b) we see the assertion (a). Moreover, by the definitions (1.5), (2.10) and Lemma 2.5 (a), (c) we obtain the assertion (b). □

Using Lemma 2.9 we will prove Lemma 2.8.

The second proof of Lemma 2.8.

Let $x \geq 1$ be any positive integer. In the identity (2.12), we use definitions (2.9) and (2.10) to show

$$\begin{aligned}\sum_{n\leq x} H_k^{(2)}(n) &= \frac{k^2}{2\sigma(k^2)}x + (x+1)H_k^{(2)}(x) - G_k^{(1)}(x)\\ &= \frac{k^2}{2\sigma(k^2)}x + (x+1)\frac{x^2}{x^2}H_k^{(2)}(x) - \frac{x}{x}G_k^{(1)}(x)\\ &\quad - \frac{x+1}{x^2}E_k^{(2)}(x) + \frac{x+1}{x^2}E_k^{(2)}(x)\\ &= \frac{k^2}{2\sigma(k^2)}x - \frac{x+1}{x^2}\Big(E_k^{(2)}(x) - x^2H_k^{(2)}(x)\Big)\\ &\quad + \frac{1}{x}\Big(E_k^{(2)}(x) - xG_k^{(1)}(x)\Big) + \frac{E_k^{(2)}(x)}{x^2}. \end{aligned} \tag{2.14}$$

Here we use Lemma 2.9 and formula (1.4) to get the assertion (2.11) for any positive integer $x \geq 1$. Moreover, using (2.13) we complete the proof for Lemma 2.8. □

To complete a proof of Theorem 1.1 we shall recall Segal's lemma in [18].

Lemma 2.10 ([18, p. 279, p. 765]). *Let $f(n)$ be a real valued arithmetical function and $g(x)$, $R(x)$ certain real valued functions for $x \geq 1$. If*

(a) $\sum_{n\leq x} f(n) = g(x) + R(x)$,
(b) *$g(x)$ is twice continuously differentiable,*
(c) *$g''(x) > 0$ or $g''(x) < 0$,*

then we have

$$\sum_{n\leq x} R(n) = \frac{1}{2}g(x) + (1 - \{x\})R(x) + \int_1^x R(t)dt + O(|g'(x)|) + O(1).$$

Now we shall prove Theorem 1.1.

Proof of Theorem 1.1.
By Lemma 2.9 (a) and Lemma 2.8 we have the first assertion of (a) as follows.

$$\begin{aligned}\sum_{n\le x} E_k^{(2)}(n) &= \sum_{n\le x} n^2 H_k^{(2)}(n) + O_k(x^2)\\ &= x^2 \sum_{n\le x} H_k^{(2)}(n) - \int_1^x 2t \sum_{n\le t} H_k^{(2)}(n)dt + O_k(x^2)\\ &= \frac{k^2}{6\sigma(k^2)}x^3 + O_k(x^2). \qquad (2.15)\end{aligned}$$

From this we have the second assertion of (a). Here, we use Lemma 2.10 with formula (2.15), to get the assertion (b) of Theorem 1.1. □

3. Pillai-Chowla method and Proof of Theorems 1.2 and 1.3

Finally, we shall prove Theorems 1.2 and 1.3. We make reference to a similar results on the error terms (1.2) and (1.3). To this end, we prepare the following lemma which is a framework in [16].

Lemma 3.1. *As for a real valued arithmetical function $f(n)$, a constant $\alpha \neq 0$, a positive integer l, and a real parameter $x \geq 1$, we set*

$$\begin{aligned}E(x) &:= \sum_{n\le x} f(n) - \alpha x^{1+l},\\ H^{(l)}(x) &:= \sum_{n\le x} \frac{f(n)}{n^l} - \alpha(1+l)x,\\ G^{(l-1)}(x) &:= \sum_{n\le x} \frac{f(n)}{n^{l-1}} - \frac{\alpha(1+l)}{2}x^2 \quad (\textit{if } l \geq 2).\end{aligned}$$

Then we obtain the following assertions.

(a) $E(x) - x^l H^{(l)}(x) = -l x^l \left(\int_1^x \frac{E(t)}{t^{1+l}}dt - \alpha\right)$,

(b) *When $l \geq 2$ we have* $E(x) - x^{l-1}G^{(l-1)}(x) = -(l-1)x^{l-1}\left(\int_1^x \frac{E(t)}{t^l}dt - \frac{\alpha}{2}\right)$,

(c) *The assumption $E(x) - x^l H^{(l)} = o(x^l)$ (as $x \to \infty$) is equivalent to*
$$\int_1^\infty \frac{E(t)}{t^{1+l}}dt = \alpha.$$

(d) *When $l \geq 2$, the assumption $E(x) - x^{l-1}G^{(l-1)}(x) = O\left(x^{l-1}\right)$ is equivalent to*
$$\int_1^x \frac{E(t)}{t^l}dt = O(1).$$

(e) *When $l = 1$, for any positive integer $x \geq 1$ we have*
$$\begin{aligned}\sum_{n\le x} H^{(1)}(n) &= \alpha x + H^{(1)}(x) - \left(E(x) - xH^{(1)}(x)\right)\\ &= \alpha x + \frac{E(x)}{x} - \frac{x+1}{x}\left(E(x) - xH^{(1)}(x)\right).\end{aligned}$$

(f) *When $l \geq 2$, for any positive integer $x \geq 1$ we have*

$$\begin{aligned}\sum_{n\leq x} H^{(l)}(n) &= \frac{\alpha(1+l)}{2}x + (x+1)H^{(l)}(x) - G^{(l-1)}(x)\\ &= \frac{\alpha(1+l)}{2}x + \frac{E(x)}{x^l} - \frac{x+1}{x^l}\left(E(x) - x^l H^{(l)}(x)\right)\\ &\quad + \frac{1}{x^{l-1}}\left(E(x) - x^{l-1}G^{(l-1)}(x)\right).\end{aligned}$$

Proof. By partial summation we see that

$$\sum_{n\leq x}\frac{f(n)}{n^l} = \alpha(1+l)x + \frac{E(x)}{x^l} + l\left(\int_1^x \frac{E(t)}{t^{1+l}}dt - \alpha\right). \tag{3.1}$$

To get assertions (a) and (c), assume that $l \geq 2$. The identity

$$\sum_{n\leq x}\frac{f(n)}{n^{l-1}} = \frac{\alpha(1+l)}{2}x^2 + \frac{E(x)}{x^{l-1}} + (l-1)\left(\int_1^x \frac{E(t)}{t^l}dt - \frac{\alpha}{2}\right) \tag{3.2}$$

yields assertions (b) and (d). Note that (3.2) and (b) hold for $l = 1$.

For any integer $x \geq 1$

$$\begin{aligned}\sum_{n\leq x} H^{(l)}(n) &= \sum_{n\leq x}\left(\sum_{m\leq n}\frac{f(m)}{m^l} - \alpha(1+l)n\right)\\ &= (x+1)\sum_{n\leq x}\frac{f(n)}{n^l} - \sum_{n\leq x}\frac{f(n)}{n^{l-1}} - \frac{\alpha(1+l)}{2}x^2 - \frac{\alpha(1+l)}{2}x.\end{aligned} \tag{3.3}$$

When $l = 1$, by the definitions $E(x)$ and $H^{(1)}(x)$ we see assertion (e) as follows.

$$\begin{aligned}\sum_{n\leq x} H^{(1)}(n) &= \alpha x + H^{(1)}(x) - \left(E(x) - xH^{(1)}(x)\right)\\ &= \alpha x + H^{(1)}(x) + \frac{E(x)}{x} - \frac{E(x)}{x} - \left(E(x) - xH^{(1)}(x)\right)\\ &= \alpha x + \frac{E(x)}{x} - \frac{x+1}{x}\left(E(x) - xH^{(1)}(x)\right).\end{aligned}$$

When $l \geq 2$, by the definitions $H^{(l)}(x)$ and $G^{(l-1)}(x)$ we have assertion (f) as

$$\begin{aligned}\sum_{n\leq x} H^{(l)}(n) &= \frac{\alpha(1+l)}{2}x + (x+1)H^{(l)}(x) - G^{(l-1)}(x)\\ &= \frac{\alpha(1+l)}{2}x + (x+1)\frac{x^l}{x^l}H^{(l)}(x) - \frac{x^{l-1}}{x^{l-1}}G^{(l-1)}(x)\\ &\quad + \frac{x+1}{x^{l-1}}E(x) - \frac{x+1}{x^l}E(x)\\ &= \frac{\alpha(1+l)}{2}x + \frac{E(x)}{x^l} - \frac{x+1}{x^l}\left(E(x) - x^l H^{(l)}(x)\right)\\ &\quad + \frac{1}{x^{l-1}}\left(E(x) - x^{l-1}G^{(l-1)}(x)\right).\end{aligned}$$

Note that it is true for $l = 1$. □

Remark 3.2. We also obtain the second equalities in Lemma 3.1 (e) and (f) as follows. In (3.3), we use (3.1) and (3.2), then

$$\sum_{n\le x} H^{(l)}(n) = \alpha(1+l)x + \frac{E(x)}{x^l} + (x+1)l\left(\int_1^x \frac{E(t)}{t^{1+l}}dt - \alpha\right) - (l-1)\left(\int_1^x \frac{E(t)}{t^l}dt - \frac{\alpha}{2}\right).$$

Here, we use the assertions (a) and (b) in Lemma 3.1, hence

$$\sum_{n\le x} H^{(l)}(n) = \alpha(1+l)x + \frac{E(x)}{x^l} - \frac{x+1}{x^l}\left(E(x) - x^l H^{(l)}(x)\right) + \frac{1}{x^{l-1}}\left(E(x) - x^{l-1}G^{(l-1)}(x)\right).$$

We obtain Theorems 1.2 and 1.3 as follows.

Proof of Theorems 1.2 and 1.3.
In Lemma 3.1, we set $f(n) = \delta_k^2(n)$, $\alpha = \frac{k^2}{3\sigma(k^2)}$, and $l = 2$. By Lemma 2.9 (a) we have known $E(x) - x^2H(x) = o\left(x^2\right)$, hence by Lemma 3.1 (c) we obtain the first assertion of Theorem 1.2, and the second assertion. Moreover, by Lemma 2.9 (b) and Lemma 3.1 (d) we obtain the assertions of Theorem 1.3. □

In this occasion, we shall apply Lemma 3.1 to error terms in two averages $\sum_{n\le x}\delta_k(n)$ and $\sum_{n\le x}\varphi(n)$. In [3], Adhikari, Balasubramanian and Sankaranarayanan showed the formula

$$\sum_{n\le x}\frac{\delta_k(n)}{n} = \frac{k}{\sigma(k)}x + O_k(\log 2x).$$

Moreover, as for $E_k(x)$ defined in (1.3), and

$$H_k(x) := \sum_{n\le x}\frac{\delta_k(n)}{n} - \frac{k}{\sigma(k)}x,$$

they deduced a relation between $E_k(x)$ and $H_k(x)$ in Lemma 3.2 in [3, p. 383] as

$$E_k(x) = xH_k(x) + o(x). \tag{3.4}$$

Therefore by Lemma 3.1 (c) we obtain an analogue of Theorem 1.2 as

Theorem 3.3.

$$\int_1^\infty \frac{E_k(t)}{t^2}dt = \frac{k}{2\sigma(k)}.$$

Remark 3.4. We would like to remark that lemmas in [3] leads the first assertion of (C). Actually, using (II) $\sum_{n\le x}|g(n)| = O\left((\log 2x)^{\nu(k)}\right)$ in [3, p. 832] we can replace $o(x)$ in the above (3.4) by $O\left((\log 2x)^{\nu(k)}\right)$ and obtain

$$\sum_{n\le x} H_k(n) = \frac{k}{2\sigma(k)}x + O_k\left((\log 2x)^{\nu(k)}\right)$$

which is a sharp form of Lemma 3.3 in [3, p. 384]. Therefore we have (C), as the proof of Theorem 1.1.

We shall go back to (1.1) the average of the Euler function $\varphi(n)$. Using Lemma 3.1 we explain (1.8). By [16] we know that

$$\sum_{n\le x}\varphi(n)=\frac{3}{\pi^2}x^2-x\sum_{d\le x}\frac{\mu(d)}{d}\left\{\frac{x}{d}\right\}+o(x),$$
$$\sum_{n\le x}\frac{\varphi(n)}{n}=\frac{6}{\pi^2}x-\sum_{d\le x}\frac{\mu(d)}{d}\left\{\frac{x}{d}\right\}+o(1),$$

and Theorem III in [16] : $E(x)=xH(x)+o(x)$, where $E(x)$ is the function defined in (1.1) and $H(x)$ is defined as

$$H(x):=\sum_{n\le x}\frac{\varphi(n)}{n}-\frac{6}{\pi^2}x\quad(x\ge 1).$$

Hence, by Lemma 3.1 (c) we obtain (1.8). The idea of Lemma 3.1 (a) and (c) is same as [22, p. 183, (3.9)].

Note that Lemma 3.1 (f) with (1.4) (i.e. $E_k^{(2)}(x)=O_k\left(x^2\right)$) and Lemma 2.9 give the assertion (2.11) of Lemma 2.8. Then we reach (a) of Theorem 1.1. Also, Lemma 3.1 (d) leads the first assertion of (C) and (A). We may state that Lemma 3.1 is a framework of the Pillai–Chowla method in [16].

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