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# INDEXED ABSOLUTE CESÀRO SUMMABILITY FOR INFINITE SERIES

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**Abstract:** In the present study, a wider class of sequence is used for a least set of sufficient conditions for absolute Cesàro $\varphi - |C, \alpha, \beta; \delta; \gamma|_k$ summable factor for an infinite series. Many corollaries have been determined by using sutaible conditions in the main theorem. Validation of the theorem done by the previous findings of summablity. In this way, system's stability is improved by finding the conditions for absolute summability.

**Key Words**: Abel's transformation, Absolute Cesàro Summability, Hölder's inequality, Minkowski's inequality

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## 1. Introduction

Let partial sums' sequence of $\sum a_n$ is given by $s_n = \sum_{k=0}^{n} a_k$ and $n^{th}$ sequence to sequence transform of the sequence $\{s_n\}$ is determined by $u_n$, s.t.,

$$u_n = \sum_{k=0}^{\infty} u_{nk} s_k \tag{1.1}$$

An infinite series $\sum a_n$ is absolutely summable, if

$$\lim_{i \to \infty} u_i = s \tag{1.2}$$

and

$$\sum_{i=1}^{\infty} |u_i - u_{i-1}| < \infty. \tag{1.3}$$

**Definition 1 [1]:** Let $\{na_n\}$ be a sequence. The $n^{th}$ Cesàro mean of this sequence is represented by $t_n^{\alpha}$. This mean is of order $\alpha$ $(0 < \alpha \leq 1)$. Then $\sum a_n$ is summable $|C, \alpha; \delta|_k$ for $\delta \geq 0$ and $k \geq 1$, if

$$\sum_{n=1}^{\infty} n^{\delta k - 1} |t_n^{\alpha}|^k < \infty, \tag{1.4}$$

where $t_n^{\alpha}$ is

$$t_n^{\alpha} = \frac{1}{A_n^{\alpha}} \sum_{p=1}^{n} A_{n-p}^{\alpha-1} p a_p, \tag{1.5}$$

and

$$A_n^{\alpha} = \begin{cases} O\left(n^{\alpha}\right), & for\ n > 0, \\ 1, & for\ n = 0, \\ 0, & for\ n < 0. \end{cases} \tag{1.6}$$

**Definition 2:** If sequence of means $\{t_n^{\alpha}\}$ satisfies:

$$\sum_{n=1}^{\infty} \frac{\varphi_n^{k-1}}{n^{k-\delta k}} |t_n^{\alpha}|^k < \infty, \tag{1.7}$$

then $\sum a_n$ is $\varphi - |C, \alpha; \delta|_k$ summable. Where $\{\varphi_n\}$ is a positive real number sequence, $\delta \geq 0$ and $k \geq 1$.

**Definition 3:** Let $t_n^{\alpha,\beta}$ represents the $n^{th}$ Cesàro means of order $(\alpha,\ \beta)$, with $\alpha + \beta > -1$, for a sequence $\{n\, a_n\}$, *i.e.*

$$t_n^{\alpha,\beta} = \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} v a_v, \tag{1.8}$$

where

$$A_n^{\alpha+\beta} = \begin{cases} 0, & n < 0, \\ 1, & n = 0, \\ O(n^{\alpha+\beta}), & n > 0. \end{cases} \tag{1.9}$$

If the sequence $\{t_n^{\alpha,\beta}\}$ satisfies,

$$\sum_{n=1}^{\infty} \frac{\varphi_n^{k-1}}{n^k} |t_n^{\alpha,\beta}|^k < \infty, \tag{1.10}$$

then the infinite series $\sum\limits_{n=0}^{\infty} a_n$ is indexed absolute $\varphi - |C,\ \alpha,\ \beta|_k$ summable.

**Definition 4:** For a series $\sum a_n$, if the condition given below is satisfied,

$$\sum_{n=1}^{\infty} \frac{\varphi_n^{(\delta k + k - 1)}}{n^k} |t_n^{\alpha,\beta}|^k < \infty, \tag{1.11}$$

then it is said to be $\varphi - |C,\ \alpha,\ \beta;\ \delta|_k$ summable, where $\delta \geq 0$, $k \geq 1$ and $\{\varphi_n\}$ is a sequence containing (+)ve real numbers.

**Definition 5:** The series $\sum a_n$ is indexed $\varphi - |C,\, \alpha,\, \beta;\, \delta;\, \gamma|_k$ summable for $k \geq 1$, $\delta \geq 0$ and $\gamma$ is a real number, if

$$\sum_{n=1}^{\infty} \frac{\varphi_n^{\gamma(\delta k+k-1)}}{n^k} |t_n^{\alpha,\,\beta}|^k < \infty. \tag{1.12}$$

Bor [2-6] has determined various important results by using absolute summability factors for infinite series with the application of diffferent classes of sequences. Özarslan derived the theorems on absolute matrix summable factors [7, 8] and $(K,\, 1,\, \alpha)$ summable factor has been used by Parashar in [9]. Absolute product summability has been used by Chandra and Jain [10] for Fourier series. Various theorems on absolute Cesàro summability have been established by Sonker and Munjal in [11, 12] and they used triangle matrices for infinite series in [13].

## 2. Known results

A positive sequence $B = \{B_n\}$ is quasi-$f$-power increasing sequence [14] with $K = K(B, f) \geq 1$ for all $1 \leq m \leq n$ s.t.

$$K f_n B_n \geq f_m B_m \tag{2.1}$$

$$f = [f_n\,(\varsigma,\ \eta)] = \{n^{\varsigma}(\log n\ )^{\eta},\ 0 < \varsigma < 1, \eta \geq 0\}\,. \tag{2.2}$$

A wider class has been used in [15] and Bor [16] used absolute summable factor of order $\alpha$ for the result.

**Theorem 2.1.** *Let $\{B_n\}$ be a wider class (a quasi-f-power sequence), which is an increasing sequence for $\varsigma$ (0 <$\varsigma$ <1). Assume $\exists$ a sequence $\{D_n\}$ s.t. it is $\xi$-quasi-monotone with conditions:*

$$\sum n\xi_n B_n = O\,(1)\,, \tag{2.3}$$

$$\Delta D_n \leq \xi_n, \tag{2.4}$$

$$|\Delta\lambda_n| \leq |D_n|\,, \tag{2.5}$$

$$\sum D_n B_n \ is\ convergent\ for\ all\ n. \tag{2.6}$$

*If the following two conditions*

$$\sum_{i=1}^{p} \frac{(w_i^{\alpha})^k}{i} = O\,(B_p) \qquad as \quad p \to \infty, \tag{2.7}$$

$$|\lambda_i|\, B_i = O\,(1) \qquad as \qquad i \to \infty, \tag{2.8}$$

*are satisfied, then $|C, \alpha|_k$ summable factor has been followed by infinite series $\sum a_n \lambda_n$ with $0 < \alpha \leq 1$ and $k \geq 1$.*

## 3. Main results

Increasing sequences are very useful for establishing a number of results on absolute summable factor. In the present study, quasi-f-power sequence is playing an important role for summable factor of a generalized series. Conditions are determined on absolute summable factor which are sufficient for an infinite series to make it absolute summable.

**Theorem 3.1.** *Let $\{B_n\}$ be a wider class of sequence (A quasi-f-power sequence), which is a increasing sequence for $\varsigma$ ($0<\varsigma<1$). Assume $\exists$ a sequence $\{D_n\}$ s.t. it is $\xi$-quasi-monotone with conditions:*

$$\sum n\xi_n B_n = O(1), \tag{3.1}$$

$$\Delta D_n \leq \xi_n, \tag{3.2}$$

$$|\Delta\lambda_n| \leq |D_n|, \tag{3.3}$$

$$|\lambda_n| B_n = O(1) \quad as \quad n\to\infty. \tag{3.4}$$

*If the following two conditions*

$$\sum D_n B_n < \infty \ for\ all\ n, \tag{3.5}$$

$$\sum_{n=1}^{m} \frac{(w_n^{\alpha,\beta})^k \varphi_n^{\gamma(\delta k+k-1)}}{n^k} = O(B_m)\ as\ m\to\infty, \tag{3.6}$$

$$\sum_{n=v}^{m} \frac{\varphi_n^{\gamma(\delta k+k-1)}}{n^{k(1+\alpha+\beta)}} = \frac{\varphi_v^{\gamma(\delta k+k-1)}}{v^{k(1+\alpha+\beta)-1}}, \tag{3.7}$$

*are satisfied, then generalized summable factor $\varphi-|C,\alpha,\beta;\delta;\gamma|_k$ has been followed by infinite series $\sum a_n\lambda_n$, where $k\geq 1$, $\alpha+\beta>0$, $0<\alpha\leq 1$, $\beta>-1, \delta\geq 0$, $\gamma$ (real number). The term of sequence $\{w_n^{\alpha,\beta}\}$ is*

$$w_n^{\alpha,\beta} = \begin{cases} \max\limits_{1\leq v\leq n} |t_v^{\alpha,\beta}|, & \beta>-1,\ 0<\alpha<1, \\ |t_n^{\alpha,\beta}|, & \beta>-1,\ \alpha=1. \end{cases} \tag{3.8}$$

## 4. Proof of the Theorem

The series $\sum a_n\lambda_n$ will follow $\varphi-|C,\alpha,\beta;\delta;\gamma|_k$ summable factor, if the $n^{th}$ mean satisfies the condition,

$$\sum_{n=1}^{\infty} \frac{\varphi_n^{\gamma(\delta k+k-1)}}{n^k} |t_n^{\alpha,\beta}|^k < \infty. \tag{4.1}$$

The $n^{th}$ sequence to sequence transform $(C,\alpha,\beta)$ of $\{na_n\lambda_n\}$ is

$$T_n^{\alpha,\beta} = \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} v a_v \lambda_v$$

$$= \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n-1} \Delta\lambda_v \sum_{p=1}^{v} A_{n-p}^{\alpha-1} A_p^{\beta} p a_p + \frac{\lambda_n}{A_n^{\alpha+\beta}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} v a_v. \tag{4.2}$$

By taking modulus value of both sides and using the concept of modulus,

$$\begin{aligned} |T_n^{\alpha,\beta}| &\leq \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n-1} |\Delta\lambda_v| \left| \sum_{p=1}^{v} A_{n-p}^{\beta-1} A_p^{\alpha} p a_p \right| + \frac{|\lambda_n|}{A_n^{\alpha+\beta}} \left| \sum_{v=1}^{n} A_{n-v}^{\beta-1} A_v^{\alpha} v a_v \right| \\ &\leq \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n-1} A_v^{\alpha,\beta} w_v^{\alpha,\beta} |\Delta\lambda_v| + |\lambda_n| w_n^{\alpha,\beta} \\ &= T_{n,1}^{\alpha,\beta} + T_{n,2}^{\alpha,\beta}. \end{aligned} \tag{4.3}$$

With the use of Minkowski's inequality's concept,

$$|T_n^{\alpha,\beta}|^k = |T_{n,1}^{\alpha,\beta} + T_{n,2}^{\alpha,\beta}|^k \leq 2^k \Big( |T_{n,1}^{\alpha,\beta}|^k + |T_{n,2}^{\alpha,\beta}|^k \Big). \tag{4.4}$$

It is enough to prove that

$$\sum_{n=1}^{\infty} \frac{\varphi_n^{\gamma(\delta k+k-1)}}{n^k} |T_{n,r}^{\alpha,\beta}|^k < \infty, \quad for\ r = 1, 2. \tag{4.5}$$

**For $\mathbf{T_{n,1}^{\alpha,\beta}}$:** By applying Abel's transformation and Hölder's inequality, we have

$$\begin{aligned} &\sum_{n=2}^{m+1} \frac{\varphi_n^{\gamma(\delta k+k-1)}}{n^k} |T_{n,1}^{\alpha,\beta}|^k \\ &\leq \sum_{n=2}^{m+1} \frac{\varphi_n^{\gamma(\delta k+k-1)}}{n^k} \frac{1}{(A_n^{\alpha+\beta})^k} \left( \sum_{v=1}^{n-1} A_v^{\alpha,\beta} w_v^{\alpha,\beta} |\Delta\lambda_v| \right)^k \\ &\leq \sum_{n=2}^{m+1} \frac{\varphi_n^{\gamma(\delta k+k-1)}}{n^{k(1+\alpha+\beta)}} \sum_{v=1}^{n-1} v^{(\alpha+\beta)k} (w_v^{\alpha,\beta})^k |D_v| \left( \sum_{v=1}^{n-1} |D_v| \right)^{k-1} \\ &= O(1) \sum_{v=1}^{m} v^{(\alpha+\beta)k} (w_v^{\alpha,\beta})^k |D_v| \sum_{n=v+1}^{m+1} \frac{\varphi_n^{\gamma(\delta k+k-1)}}{n^{k(1+\alpha+\beta)}} \\ &= O(1) \sum_{v=1}^{m} v^{(\alpha+\beta)k} (w_v^{\alpha,\beta})^k |D_v| \frac{\varphi_v^{\gamma(\delta k+k-1)}}{v^{k(1+\alpha+\beta)-1}} \\ &= O(1) \sum_{v=1}^{m} v |D_v| (w_v^{\alpha,\beta})^k \frac{\varphi_v^{\gamma(\delta k+k-1)}}{v^k} \\ &= O(1) \sum_{v=1}^{m-1} \Delta(v|D_v|) \sum_{r=1}^{v} (w_r^{\alpha,\beta})^k \frac{\varphi_r^{\gamma(\delta k+k-1)}}{r^k} \\ &\quad + O(1) m |D_m| \sum_{v=1}^{m} (w_v^{\alpha,\beta})^k \frac{\varphi_v^{\gamma(\delta k+k-1)}}{v^k} \\ &= O(1) \sum_{v=1}^{m-1} \Big| (v+1)\Delta|D_v| - |D_v| \Big| B_v + O(1) m |D_m| B_m \\ &= O(1) \sum_{v=1}^{m-1} v |\Delta D_v| B_v + O(1) \sum_{v=1}^{m-1} |D_v| B_v + O(1) m |D_m| B_m \end{aligned}$$

$$
\begin{aligned}
&= O(1)\sum_{v=1}^{m-1} v\xi_v B_v + O(1)\sum_{v=1}^{m-1} |D_v| B_v + O(1) m |D_m| B_m \\
&= O(1) \ as \ \ m \to \infty, \qquad (4.6)
\end{aligned}
$$

**For $\mathbf{T^{\alpha,\beta}_{n,2}}$:** By applying Abel's transformation and Hölder's inequality, we have

$$
\begin{aligned}
\sum_{n=2}^{m} \frac{\varphi_n^{\gamma(\delta k+k-1)}}{n^k} |T^{\alpha,\beta}_{n,2}|^k &= O(1)\sum_{n=1}^{m} |\lambda_n| (w_n^{\alpha,\beta})^k \frac{\varphi_n^{\gamma(\delta k+k-1)}}{n^k} \\
&= O(1)\sum_{n=1}^{m-1} \Delta|\lambda_n| \sum_{v=1}^{n} (w_v^{\alpha,\beta})^k \frac{\varphi_v^{\gamma(\delta k+k-1)}}{v^k} \\
&\quad + O(1)|\lambda_m| \sum_{n=1}^{m} (w_n^{\alpha,\beta})^k \frac{\varphi_n^{\gamma(\delta k+k-1)}}{n^k} \\
&= O(1)\sum_{n=1}^{m-1} |\Delta\lambda_n| B_n + O(1)|\lambda_m| B_m \\
&= O(1)\sum_{n=1}^{m-1} |D_n| B_n + O(1)|\lambda_m| B_m \\
&= O(1) \ as \ m \to \infty. \qquad (4.7)
\end{aligned}
$$

Collecting (4.2) - (4.7), we have

$$
\sum_{n=1}^{\infty} \frac{\varphi_n^{\gamma(\delta k+k-1)}}{n^k} |T_n^{\alpha,\beta}|^k < \infty. \qquad (4.8)
$$

Hence the proof of the theorem is completed.

## 5. Corollaries

**Corollary 5.1.** *Let $\{B_n\}$ be a wider class of sequence (quasi-f-power sequence), which is an increasing sequence for $\varsigma$ ($0 < \varsigma < 1$). Assume $\exists$ a sequence $\{D_n\}$ s.t. it is $\xi$-quasi-monotone with conditions (3.1-3.5) and*

$$
\sum_{i=1}^{p} \frac{\varphi_i^{(k-1)} (w_i^{\alpha})^k}{i^k} = O(B_p) \qquad as \qquad p \to \infty, \qquad (5.1)
$$

$$
\sum_{i=v}^{p} \frac{\varphi_i^{(k-1)}}{i^{(\alpha+1)k}} = O\left( \frac{\varphi_v^{(k-1)}}{v^{(\alpha+1)k-1}} \right), \qquad (5.2)
$$

*then $\varphi - |C,\alpha|_k$ summable factor followed by the series $\sum a_n\lambda_n$ with $k \geq 1, 0 < \alpha \leq 1$ and $w_n^{\alpha}$ is*

$$
w_n^{\alpha} = \begin{cases} \max\limits_{1\leq v\leq n} |t_v^{\alpha}|, & 0 < \alpha < 1, \\ |t_n^{\alpha}|, & \alpha = 1. \end{cases} \qquad (5.3)
$$

**Proof:** Use $\gamma = 1$, $\delta = 0$ and $\beta = 0$ in the present result.

**Corollary 5.2.** *Let $\{B_n\}$ be a wider class of sequence (quasi-f-power sequence), which is an increasing sequence for $\varsigma$ $(0 <\varsigma <1)$. Assume $\exists$ a sequence $\{D_n\}$ s.t. it is $\xi$-quasi-monotone with conditions (3.1-3.5) and*

$$\sum_{i=1}^{p} \frac{(w_i^{\alpha})^k}{i} = O\left(B_p\right) \qquad as \qquad p \to \infty, \tag{5.4}$$

*then $|C,\alpha|_k$ summable factor followed by $\sum a_n \lambda_n$ series with $k \geq 1, 0 < \alpha \leq 1$ and $w_n^{\alpha}$ is*

$$w_n^{\alpha} = \begin{cases} \max\limits_{1 \leq v \leq n} |t_v^{\alpha}|, & 0 < \alpha < 1, \\ |t_n^{\alpha}|, & \alpha = 1. \end{cases} \tag{5.5}$$

**Proof:** By using $\varphi_n = n$, $\gamma = 1$, $\delta = 0$ and $\beta = 0$ in the present resent.

## 6. Conclusion

Present work is on absolute summability factor which makes the system stable. If an impulse response be absolutely summable, then the system is BIBO stable, i.e.,

$$BIBO \ stable \iff \sum_{n=-\infty}^{\infty} |h\left(n\right)| < \infty. \tag{6.1}$$

With the help of summable factor, error is minimized and the output is made stable. Absolute summable factor is used to predict the input data and the complete changes in the process.

Present work is applicable in rectification of signals in Filter. By finding the corollary, we can be concluded that present result is very important. This is a generalized research on absolute summability which is used for find various previous results. Validation of present work is done by corollary 5.2, which is establised by Bor [16] for infinite series to be absolute summable.

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## Ethical Approval

This article does not contain any studies with human participants or animals performed by any of the authors.

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