The Nepali Mathematical Sciences Report, Vol. 38, No. 1, 2021: 8-15
DOI: https://doi.org/10.3126/nmsr.v38i1.38872

SOME TOPOLOGICAL INDICES OF DOUBLE SQUARE SNAKE GRAPHS

PUSHPALATHA MAHALANK [1] AND BHAIRABA KUMAR MAJHI [2] AND ISMAIL NACI CANGUL [3]

[1] *Department of Mathematics, SDM College of Engineering and Technology, Dharwad-580002, Karnataka, INDIA*
e-mail: mahalankpushpalatha@gmail.com
[2] *Department of Mathematics, School of Applied Sciences, Centurion University of Technology and Management, Bolangir Campus, 767001 Odisha, INDIA*
e-mail: bhairabakumar.majhi@cutm.ac.in
[3] *Department of Mathematics, Faculty of Arts and Science, Bursa Uludag University, 16059 Bursa, TURKEY*
e-mail: cangul@uludag.edu.tr

Abstract: In many areas of science, lattice structures are very useful phenomenons. In network sciences, in chemistry and in social sciences, we face them in the solution of many daily life problems. Several large lattice structures can also be thought as graphs and in that way, are useful in the study of large networks. A very recently defined and studied class of such networks are snake graphs which has close relations with number theory due to the use of continued fractions. Following the works on square snake graphs, in this work, we study some non-Zagreb type topological graph indices of some interesting lattice structures called as double square snake graphs. We first obtain the vertex and edge partitions of these graphs and calculate their indices by means of these partitions.

Key Words: Zagreb index, vertex degrees, graph index, square snake graphs, double square snake graphs
AMS (MOS) Subject Classification. 05C07, 05C30, 05C38.

Received: April 17, 2021 Accepted: May 17, 2021 Published Online: June 30, 2021

1. Introduction

Special lattice structures called as the snake graphs are studied in different contexts in mathematics and other sciences. They have finite or infinite one or two dimensional repetitions of some geometric shape. They are, at the same time, planar bipartite graphs. In [2, 3, 4], Canakci et al. studied snake graphs in relation with cluster algebras. In [8], Shiffler constructed snake graphs consisting of square tiles and used them to establish relations with perfect matchings and positive continued fractions which are used in estimating real numbers by some infinite sequences of rational numbers. As a result, the idea of continued fractions have been a popular and useful tool in number theory. They are used in the solutions of some Diophantine equations. Bradshaw et al. continued the above work in [1] and established their relations with linear algebra by studying their characteristic polynomials in relation with the Chebycheff polynomials of the first and second type. In [5], snake graphs are studied in relation with strongly *-graphs.

The Zagreb indices of square snake graphs, see Fig. 1, are recently studied in [6].

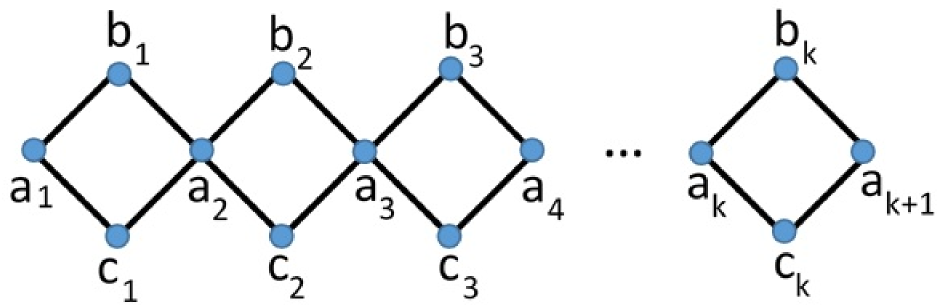

Figure 1 The square snake graph $C^1_{4,k}$

In this work, we consider double square snake graphs given in Fig. 2. Some Zagreb type topological indices of double square snake graphs have been determined in [7]. In this work, we calculate some additive and multiplicative topological graph indices of double square snake graphs.

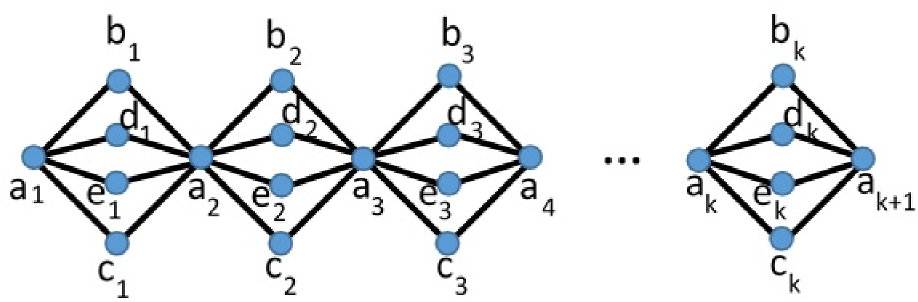

Figure 2 The double square snake graph $C^2_{4,k}$

Modeling by graphs is a very useful tool that allows to use graphs in solving many real life problems. Many daily problems can indeed be modeled by a graph. As the most frequent example, a chemical molecule can be modeled by a graph in the following way: A vertex corresponds to each atom in the molecule and an edge corresponds to each chemical bond between two atoms. A graph obtained in that way is called a chemical graph. In Chemistry, QSPR and QSAR studies are realized by means of some simple mathematical formulae. A topological index or a molecular descriptor is a mathematical formula calculated by means of either vertex degrees, distances, graph matrices or some graph parameters. They are used to obtain a real number corresponding to a given graph and commenting on such a number, one can obtain physico-chemical properties of the corresponding molecules. Modeling by graphs can also be used in other areas of science, especially in social network studies and other social sciences where there are relations between things, persons, firms, countries etc. There are several classes of topological graph indices including Zagreb indices, atom-bond-connectivity indices, geometric-arithmetic indices, Randic indices etc. In this paper, we determine some non-Zagreb-type topological graph indices of the double square snake graphs $C_{4,k}^2$.

2. Some additive topological indices of double square snake graphs

In this section, we will determine some additive non-Zagreb-type topological graph indices of the double square snake graphs $C_{4,k}^2$. The following indices are used in this work:

The harmonic index of a graph G denoted by $H(G)$ is defined by

$$H(G) = \sum_{uv \in E(G)} \frac{2}{d(u)+d(v)}.$$

In a similar way, for a fixed real number α, the generalized harmonic index is defined by

$$H_\alpha^*(G) = \sum_{uv \in E(G)} \left(\frac{2}{d(u)+d(v)}\right)^\alpha.$$

Another topological graph index is the sum-connectivity index given by

$$\chi(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{d(u)+d(v)}}.$$

For a fixed real number α, the general sum connectivity index is defined by

$$H_\alpha(G) = \sum_{uv \in E(G)} (d(u)+d(v))^\alpha.$$

Inverse sum indeg index is denoted by $ISI(G)$ and defined by

$$ISI(G) = \sum_{uv \in E(G)} \frac{d(u)d(v)}{d(u)+d(v)}.$$

One of the most important topological graph indices is called the Randic index which is defined by

$$R(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{d(u)d(v)}}.$$

There are some variants of the Randic index as below. The reverse Randic index is defined by

$$RR(G) = \sum_{uv \in E(G)} \sqrt{d(u)d(v)}$$

and the generalized Randic index is similarly defined by

$$R_\alpha(G) = \sum_{uv \in E(G)} \frac{1}{(d(u)d(v))^\alpha}.$$

Another group of frequently used topological graph indices are the atom-bond-connectivity indices. The classical atom-bond-connectivity index of G is defined by

$$ABC(G) = \sum_{uv \in E(G)} \sqrt{\frac{d(u)+d(v)-2}{d(u)d(v)}}.$$

The geometric-arithmetic index of G is defined by

$$GA(G) = \sum_{uv \in E(G)} \frac{2\sqrt{d(u)d(v)}}{d(u)+d(v)}.$$

A graph G is called regular if all vertices have the same vertex degree. Regularity is an important property which makes calculations easier. Naturally, most of the graphs are not regular and called irregular. It is sometimes important to decide which one of the two given graphs is more regular. To this aim, several irregularity indices are defined. Albertson index is one of these irregularity indices defined by

$$Alb(G) = \sum_{uv\in E(G)} |d(u) - d(v)|.$$

Another useful irregularity measure is the sigma index denoted by $\sigma(G)$ defined by

$$\sigma(G) = \sum_{uv\in E(G)} (d(u) - d(v))^2.$$

Another frequently used irregularity index is the Bell index defined by

$$B(G) = \sum_{u\in V(G)} \left(d(u) - \tfrac{2m}{n}\right).$$

The final irregularity index we shall be calculating in this work is the ireegularity index denoted by $Irr(G)$ and defined by

$$Irr(G) = \sum_{u\in V(G)} \left|d(u) - \tfrac{2m}{n}\right|.$$

Here, note that the fraction $2m/n$ is the ratio of the sum of the vertex degrees and the number of vertices in a graph G. Therefore it is the average vertex degree.

Before determining the above additive topological graph indices of non-Zagreb-type for the double square snake graphs, we must first determine the vertex and edge partitions of the double square snake graph $C_{4,k}^2$. It has $5k+1$ vertices and $8k$ edges. The vertex degrees are 2, 4 or 8 and the vertex partition of $C_{4,k}^2$ together with the parity of each number is therefore appears as in Table 1:

TABLE 1. Vertex partition of $C_{4,k}^2$

d_i	$\sharp\ d_i$
2	$4k$
4	2
8	$k-1$

Similarly, the edge partition of $C_{4,k}^2$ together with the parity of each edge type is shown in Table 2:

TABLE 2. Edge partition of $C_{4,k}^2$

(d_i, d_j)	$\sharp\ (d_i, d_j)$
(2,4)	8
(2,8)	$8(k-1)$

Now, we are ready to determine the additive non-Zagreb-type topological graph indices of the double square snake graph $C_{4,k}^2$ listed above. We shall be using the vertex and edge partitions of these graphs given in Tables 1 and 2. The results obtained here can be commented to obtain information on several chemical properties of the corresponding molecular structures.

Theorem 2.1. *Some non-Zagreb-type additive indices of double square snake graph $C_{4,k}^2$ are as follows:*

$$\begin{array}{ll}
H(C_{4,k}^2) & = \frac{8}{5}(3k+2), \\
H_\alpha^*(C_{4,k}^2) & = 8 \cdot \left(\frac{2}{2+4}\right)^\alpha + 8(k-1) \cdot \left(\frac{2}{2+8}\right)^\alpha, \\
\chi(C_{4,k}^2) & = \frac{4}{15}(5\sqrt{6} + 3\sqrt{10}(k-1)), \\
H_\alpha(C_{4,k}^2) & = 2^{\alpha+3}(3^\alpha + 5^\alpha(k-1)) = 8(3^{-\alpha} + 5^{-\alpha}(k-1)), \\
ISI(C_{4,k}^2) & = \frac{32}{5}(2k-1), \\
R(C_{4,k}^2) & = 2(k-1+\sqrt{2}), \\
RR(C_{4,k}^2) & = 16(2(k-1)+\sqrt{2}), \\
R_\alpha(C_{4,k}^2) & = 8(2^{-3\alpha} + 2^{-4\alpha}(k-1)), \\
ABC(C_{4,k}^2) & = 4\sqrt{k}, \\
GA(C_{4,k}^2) & = \frac{16}{15}(6(k-1) + 5\sqrt{2}), \\
Alb(C_{4,k}^2) & = 16(3k-2), \\
\sigma(C_{4,k}^2) & = 32(9k-8), \\
B(C_{4,k}^2) & = \frac{16k(45k^2-64k+33)-32}{(5k+1)^2}, \\
Irr(C_{4,k}^2) & = \frac{4k(17k-4)}{5k+1}.
\end{array}$$

Proof. Let us start with the harmonic index. By the definition, using the edge partition of $C_{4,k}^2$ given in Table 2, we obtain

$$\begin{aligned}
H(C_{4,k}^2) & = \textstyle\sum_{v \in V(G)} \frac{2}{d(u)+d(v)} \\
& = 8 \cdot \frac{2}{6} + 8(k-1) \cdot \frac{2}{10} \\
& = \frac{8}{5}(3k+2).
\end{aligned}$$

Secondly, the generalized harmonic index is

$$\begin{aligned}
H_\alpha^*(C_{4,k}^2) & = 8 \cdot \left(\frac{2}{2+4}\right)^\alpha + 8(k-1) \cdot \left(\frac{2}{2+8}\right)^\alpha \\
& = 8(3^{-\alpha} + 5^{-\alpha}(k-1)).
\end{aligned}$$

The sum-connectivity index of $C_{4,k}^2$ is

$$\begin{aligned}
\chi(C_{4,k}^2) & = 8 \cdot \frac{1}{\sqrt{6}} + 8(k-1) \cdot \frac{1}{\sqrt{10}} \\
& = \frac{4}{15}(5\sqrt{6} + 3\sqrt{10}(k-1))
\end{aligned}$$

and the generalized sum-connectivity index would be

$$\begin{aligned}
H_\alpha(C_{4,k}^2) & = 8 \cdot (2+4)^\alpha + 8(k-1) \cdot (2+8)^\alpha \\
& = 2^{\alpha+3}(3^\alpha + 5^\alpha(k-1)).
\end{aligned}$$

Next, the inverse sum indeg index is found to be

$$\begin{aligned} ISI(C_{4,k}^2) &= \textstyle\sum_{uv\in E(C_{4,k}^2)} \frac{d(u)d(v)}{d(u)+d(v)} \\ &= 8\cdot\tfrac{2\cdot 4}{2+4} + 8(k-1)\cdot\tfrac{2\cdot 8}{2+8} \\ &= \tfrac{64}{5}k - \tfrac{32}{5}. \end{aligned}$$

The Randic index of double square snake graph $C_{4,k}^2$ is

$$\begin{aligned} R(C_{4,k}^2) &= 8\cdot\tfrac{1}{\sqrt{8}} + 8(k-1)\tfrac{1}{\sqrt{16}} \\ &= 2(k-1+\sqrt{2}), \end{aligned}$$

its reverse Randic index would be

$$\begin{aligned} RR(C_{4,k}^2) &= 8\sqrt{2\cdot 4} + 8(k-1)\sqrt{2\cdot 8} \\ &= 16(2(k-1)+\sqrt{2}), \end{aligned}$$

and also the generalized Randic index of $C_{4,k}^2$ will be

$$\begin{aligned} R_\alpha(C_{4,k}^2) &= 8\cdot\tfrac{1}{(2\cdot 4)^\alpha} + 8(k-1)\cdot\tfrac{1}{(2\cdot 8)^\alpha} \\ &= 8(2^{-3\alpha} + 2^{-4\alpha}(k-1)). \end{aligned}$$

The atom bond connectivity index of $C_{4,k}^2$ is

$$\begin{aligned} ABC(C_{4,k}^2) &= 8\cdot\sqrt{\tfrac{2+4-2}{2\cdot 4}} + 8(k-1)\sqrt{\tfrac{2+8-2}{2\cdot 8}} \\ &= 4\sqrt{2}k, \end{aligned}$$

and the geometric-arithmetic index will be

$$\begin{aligned} GA(C_{4,k}^2) &= 8\tfrac{2\sqrt{2\cdot 4}}{2+4} + 8(k-1)\tfrac{2\sqrt{2\cdot 8}}{2+8} \\ &= \tfrac{16}{15}(6(k-1)+5\sqrt{2}). \end{aligned}$$

Finally we calculate the irregularity indices of $C_{4,k}^2$ as follows:

$$\begin{aligned} Alb(C_{4,k}^2) &= 8\cdot|2-4| + 8(k-1)\,|2-8| \\ &= 16(3k-2), \end{aligned}$$

$$\begin{aligned} \sigma(C_{4,k}^2) &= 8(2-4)^2 + 8(k-1)(2-8)^2 \\ &= 32(9k-8), \end{aligned}$$

$$\begin{aligned} B(C_{4,k}^2) &= 4k\left(2-\tfrac{16k}{5k+1}\right)^2 + 2\left(4-\tfrac{16k}{5k+1}\right)^2 + (k-1)\left(8-\tfrac{16k}{5k+1}\right)^2 \\ &= \tfrac{16k(45k^2-64k+33)-32}{(5k+1)^2} \end{aligned}$$

and

$$\begin{aligned} Irr(C_{4,k}^2) &= 4k\left|2-\tfrac{16k}{5k+1}\right| + 2\left|4-\tfrac{16k}{5k+1}\right| + (k-1)\left|8-\tfrac{16k}{5k+1}\right| \\ &= \tfrac{4k(17k-4)}{5k+1}. \end{aligned}$$

□

3. Some multiplicative topological indices of $C_{4,k}^2$

In this section, we shall determine some of the multiplicative topological graph indices of the double square snake graphs. Again, we shall be using the vertex and edge partitions of these graphs given in Tables 1 and 2. The multiplicative indices we shall calculate are as follows:

$$\begin{aligned}
\Pi_1(G) &= \textstyle\prod_{u\in V(G)} d(u)^2,\\
\Pi_2(G) &= \textstyle\prod_{uv\in E(G)} d(u)d(v),\\
\Pi_3(G) &= \textstyle\prod_{u\in V(G)} d(u)^3,\\
NK(G) &= \textstyle\prod_{u\in V(G)} d(u),\\
\Pi_1^*(G) &= \textstyle\prod_{uv\in E(G)}(d(u)+d(v)),\\
GA\Pi(G) &= \textstyle\prod_{uv\in E(G)} \frac{2\sqrt{d(u)d(v)}}{d(u)+d(v)},\\
H\Pi_1(G) &= \textstyle\prod_{uv\in E(G)}(d(u)+d(v))^2,\\
H\Pi_2(G) &= \textstyle\prod_{uv\in E(G)}(d(u)\cdot d(v))^2,\\
\\
R\Pi(G) &= \textstyle\prod_{uv\in E(G)} \frac{1}{\sqrt{d(u)d(v))}},\\
\chi\Pi(G) &= \textstyle\prod_{uv\in E(G)} \frac{1}{\sqrt{d(u)+d(v))}},\\
ABC\Pi(G) &= \textstyle\prod_{uv\in E(G)} \sqrt{\frac{d(u)+d(v)-2}{d(u)d(v)}}.
\end{aligned}$$

Theorem 3.1. *Some multiplicative indices of double square snake graph $C_{4,k}^2$ are as follows:*

$$\begin{aligned}
\Pi_1(C_{4,k}^2) &= 2^{14k+2},\\
\Pi_2(C_{4,k}^2) &= 2^{32k-8},\\
\Pi_3(C_{4,k}^2) &= 2^{21k+3},\\
NK(C_{4,k}^2) &= 2^{7k+1},\\
\Pi_1^*(C_{4,k}^2) &= (2^k\cdot 3\cdot 5^{k-1})^8,\\
GA\Pi(C_{4,k}^2) &= 2^{16k+11}\cdot 5^{16(1-k)}/3,\\
H\Pi_1(C_{4,k}^2) &= 2^{16k}\cdot 3^{16}\cdot 5^{16(k-1)},\\
H\Pi_2(C_{4,k}^2) &= 2^{16(4k-1)},\\
R\Pi(C_{4,k}^2) &= 2^{4(1-4k)},\\
\chi\Pi(C_{4,k}^2) &= [2^k\cdot 3^4\cdot 5^{4(k-1)}]^{-1},\\
ABC\Pi(C_{4,k}^2) &= 2^{-4k}.
\end{aligned}$$

Proof. The proof uses the vertex and edge partitions of $C_{4,k}^2$ in Tables 1 and 2. As the calculations are similar to Theorem 1, we prefer to omit the details. □

4. Summary and conclusions

In this work, double square snake graphs are considered as special network structures and some of their indices are calculated. Similar studies can be done for other lattice types and for other topological graph indices.

References

[1] J.P. Bradshaw, P. Lampe, D. Ziga, Snake graphs and their characteristic polynomials, 2019, arXiv.org>math>arXiv:1910.11823v1.

[2] I. Çanakçi, R. Schiffler, Snake graph calculus and cluster algebras from surfaces, *Journal of Algebra*, Vol. 382 pp 240-281, 2013.

[3] I. Çanakçi, R. Schiffler, Snake graph calculus and cluster algebras from surfaces II: Selfcrossing snake graphs, *Mathematische Zeitschrift*, Vol. 281 (1) pp 55-102, 2015.

[4] I. Çanakçi, R. Schiffler, Snake graph calculus and cluster algebras from surfaces III: Band graphs and snake rings, *International Mathematics Research Notices*, rnx157, 2017.

[5] I.I. Jadav, G.V. Ghodasara, Snakes related strongly*-graphs, *International Journal of Advanced Engineering Research and Science*, Vol. 3 (9) pp 240-245, 2016.

[6] P. Mahalank, B.K. Majhi, I. N. Cangul, Zagreb indices of square snake graphs, Montes Taurus Journal of Mathematics (preprint)

[7] B.K. Majhi, P. Mahalank, I. N. Cangul, Several Zagreb indices of double square snake graphs, (preprint)

[8] R. Schiffler, Snake Graphs, Perfect Matchings and Continued Fractions, *Snapshots of Modern Mathematics form Oberwolfach*, No: 1, Mathematisches Forschungsinstitut Oberwolfach, 2019 DOI: 10.14760/SNAP-2019-001-EN.