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L^2 SOLUTION OF LINEARIZED CUTOFF BOLTZMANN EQUATION WITH BOUNDARY CONDITION

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Abstract: We are concerned with the existence of L^2 solution for the linearized cutoff Boltzmann equation with the specular reflection condition. And we obtain the weak solution $f \in L^2([0,T];\ L^2_\nu)$ by applying the dual method and the classical functional analysis tool such as Hahn-Banach theorem.

Key Words: Linearized Boltzmann equation, Specular reflection boundary condition, L^2 solution, Dual method.

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1. Introduction

Boundary value problem occurs when the plasma-wall interaction happens. The gas particles are driven by binary collision dynamics following from the intermolecular potential acting on them and by a gas-surface interaction process described by an interaction law. This law gives a balance between the number of the incident particles and the ratio of those

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reflected or captured by the body, see [7, 8]. This phenomenon could be characterized by the Boltzmann equation with boundary condition which could be formulated as follows:

$$(\partial_t + v \cdot \nabla_x)F = Q(F, F),$$

where $F(t, x, v)$ is the distribution function for the gas particles at time $t \geq 0$, position $x \in \Omega$, and velocity $v \in \mathbb{R}^3$. And the collision operator takes the form of

$$\begin{aligned} Q(F_1, F_2) &= \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} |v-u|^\gamma F_1(u^{'})F_2(v^{'})q_0(\theta)d\omega du \\ &\quad - \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} |v-u|^\gamma F_1(u)F_2(v)q_0(\theta)d\omega du \\ &\equiv Q_{gain}(F_1, F_2) - Q_{loss}(F_1, F_2), \end{aligned}$$

where $u^{'} = u + [(v-u)\cdot\omega]\omega, v^{'} = v - [(v-u)\cdot\omega]\omega, \cos\theta = \frac{(u-v)\cdot\omega}{|u-v|}, 0 \leq \gamma \leq 1$ (hard potential) and $0 \leq q_0(\theta) \leq C|\cos\theta|$(angular cutoff), and the boundary condition will be defined shortly.

In terms of the standard perturbation f near Maxwellian such that $F = \mu + \sqrt{\mu}f$, the Boltzmann equation can also be rewritten as

$$\{\partial_t + v\cdot\nabla_x + L\}f = \Gamma(f, f), \qquad f(0, x, v) = f_0(x, v),$$

where the standard linearized Boltzmann operator [5] is given by

$$Lf \equiv \nu f - Kf = -\frac{1}{\sqrt{\mu}}\{Q(\mu, \sqrt{\mu}f) + Q(\sqrt{\mu}f, \mu)\} = \nu f - \int \mathbf{k}(v, v^{'})f(v^{'})dv^{'},$$

with the collision frequency $\nu(v) \equiv \int |v-u|^\gamma \mu(u)q_0(\theta)dud\theta \sim \{1+|v|\}^\gamma$ for $0 \leq \gamma \leq 1$; and

$$\Gamma(f_1, f_2) = \frac{1}{\sqrt{\mu}}\{Q(\sqrt{\mu}f_1, \sqrt{\mu}f_2) \equiv \Gamma_{gain}(f_1, f_2) - \Gamma_{loss}(f_1, f_2).$$

And the coercivity takes the form as following:

$$(Lf, f) \geq c_0\|f\|^2_{L^2_\nu} - c_1\|f\|^2_{L^2},$$

where $\|f\|^2_{L^2_\nu} = \int |f|^2\nu dv$.

As to the boundary conditions for the Boltzmann equation, mainly there are several typical types as below:

1) Bounce-back reflection boundary condition: $\forall x \in \partial\Omega$,

$$M\gamma^+ f(t, x, v) = f(t, x, -v); \tag{1.1}$$

2) Specular reflection boundary condition:

$$M\gamma^+ f(t, x, v) = f(t, x, v - 2(n(x)\cdot v)\cdot n(x)), \quad \forall\ x \in \partial\Omega; \tag{1.2}$$

3) Diffuse boundary condition:

$$\begin{cases} M\gamma^+ f(t, x, v) = c_\mu\sqrt{\mu(v)}\displaystyle\int_{v'\cdot n(x)>0} f(t, x, v')\sqrt{\mu(v')}\{n(x)\cdot v\}dv', \quad \forall (x, v) \in \gamma^- \\ c_\mu \displaystyle\int_{v\cdot n(x)>0} \mu(v)|n(x)\cdot v|dv = 1. \end{cases} \tag{1.3}$$

Now we turn to review some results related to the boundary value problems of Boltzmann equations which have been studied for seveval decades.

In the L^1 framework, Mischler [9] demonstrated the existence of DiPerna-Lions renormalized solutions [2] to the Boltzmann equation and the Vlasov-Poisson-Boltzmann system for the initial boundary value problem. More precisely, Mischler considered the following problem:

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + \nabla_x \phi \cdot \nabla_v f = Q(f,f) \\ \Delta\phi(t,x) = \int f(t,x,v)dv, \quad (t,x) \in (0,\infty) \times \Omega \\ \phi(t,x) = 0, \quad (t,x) \in (0,\infty) \times \partial\Omega \\ \gamma^- f = M\gamma^+ f \\ \iint_{\Omega\times\mathbb{R}^3} f_0(1+|v|^2+|\log f_0|)dvdx < \infty, \end{cases} \tag{1.4}$$

and he obtained a solution f to (1.4) such that

$$\sup_{[0,T]} \iint_{\Omega\times\mathbb{R}^3} f(1+|v|^2+|\log f|)dvdx \le C_T.$$

For the L^∞-framework, Guo [5] obtained an L^∞ solution for the Boltzmann equation in the bounded domains with four basic types of boundary conditions: in-flow, bounce-back reflection, specular reflection and diffuse reflection. For instance,

Proposition 1.1. *([5]) Assume $w(v) = (1+\rho^2|v|^2)^\beta, \rho > 0, \beta \in \mathbb{R}, w^{-2}(1+|v|)^3 \in L^1$, then $\exists\ \delta > 0$, such that if $F_0(x,v) = \mu + \sqrt{\mu} f_0$ and $\|wf_0\|_\infty \le \delta$. There exists a unique solution*

$$F(t,x,v) = \mu + \sqrt{\mu} f$$

to the specular boundary value problem

$$\begin{cases} (\partial_t + v\cdot\nabla_x + L)f = \Gamma(f,f) \\ M\gamma^+ f(t,x,v) = f(t,x,v-2(n(x)\cdot v)\cdot n(x)) \\ f(0,x,v) = f_0(x,v). \end{cases} \tag{1.5}$$

Moreover,

$$\sup_{0\le t\le\infty} e^{\lambda t}\|wf(t)\|_\infty \le C\|wf_0\|_\infty, \quad \textit{for some } \lambda > 0.$$

Further, Guo, Kim, Tonon and Trescases [6] considered the regularity of solution for Boltzmann equation in the convex domain based on the existence of L^∞ solution.

For the L^2 framework, Esposito ect. [3] considered the linearized Boltzmann equation for diffuse reflection boundary conditions and established an L^2-solution for the following system of equation:

$$\begin{cases} \partial_t f + v\cdot\nabla_x f + Lf = g \\ \gamma^- f = M\gamma^+ f \\ f(0) = f_0, \end{cases} \tag{1.6}$$

where

$$M\gamma^+ f(t,x,v) = c_\mu \sqrt{\mu(v)} \int_{v'\cdot n(x)>0} f(t,x,v')\sqrt{\mu(v')}\{n(x)\cdot v\}dv', \quad \forall (x,v)\in\gamma^-$$

is the diffuse reflection boundary condition and $\iint_{\Omega\times\mathbb{R}^3} g(t,x,v)\sqrt{\mu}dv = 0$.

To study the linearized Boltzmann equation with specular boundary conditions is interesting. In this context, we consider the existence of L^2 solution for the linearized cutoff Boltzmann equation with the specular reflection boundary condition, i.e.,

$$\begin{cases} \partial_t f + v\cdot\nabla_x f + Lf = H \\ \gamma^- f = M\gamma^+ f \\ f(0) = f_0, \end{cases} \tag{1.7}$$

where M defined by (1.2), i.e. the specular reflection boundary condition, and $H\in L^2([0,T]\times\Omega\times\mathbb{R}^3)$.

To realize our goal, we mainly adopt the dual method. Comparatively, in [3], the authors adopted the iteration method. To be more precise, we establish the functional inequality $|l(P^*g)|\le c\|P^*g\|_{L^2([0,T];\ L^2_{\nu^{-1}})}$, $\forall g\in W$, and Hahn-Banach theorem, Riesz representation theorem are applied to establish this estimate.

Lastly, we introduce some definitions and notations for later use.

i) ([10]) The trace operator $\gamma^\pm f = f|_{\Sigma^\pm}$ are defined primarily on $f\in C_0^1(\bar{D})$, which could be extended to $W_2 = \{f\in L^2(D) \mid (\partial_t + v\cdot\nabla_x)f\in L^2(D)\}$, where

$$\Sigma^+ = \{(x,v)\in\partial\Omega\times\mathbb{R}^3 : \quad n(x)\cdot v > 0\},$$

$$\Sigma^- = \{(x,v)\in\partial\Omega\times\mathbb{R}^3 : \quad n(x)\cdot v < 0\},$$

$$D = (0,T)\times\Omega\times\mathbb{R}^3,\ \Omega \text{ is a domain in } \mathbb{R}^3.$$

ii) Denote the boundary inner product

$$\langle g_1, g_2\rangle_\pm = \iint_{\pm v\cdot n(x)>0} g_1(x,v)g_2(x,v)|v\cdot n(x)|dvdS_x,$$

and

$$\|\gamma^\pm g\|^2_{L^{2,\pm}} = \int_{\Sigma^\pm} |g|^2\ |v\cdot n(x)|dvdS_x,$$

where dS_x is the standard surface measure on $\partial\Omega$, $n(x)$ is outward normal vector, and the L^2 inner product is defined as $(f,g) = \int_{[0,T)\times\Omega\times\mathbb{R}^3} fgdsdxdv$.

iii) We call $f\in L^2([0,T];L^2_\nu)$ is weak solution if

$$(f, (-\partial_t - v\cdot\nabla_x + L)g) = \int_0^T (H,g)_{L^2}ds + (f_0, g(0))_{L^2}, \quad \forall g\in W,$$

where

$$W =: \{g\in L^2_\nu \mid \ (\partial_t + v\cdot\nabla_x)g\in L^2,\ \gamma^\pm g\in L^{2,\pm},\ \gamma^+ g = M\gamma^- g,\ g(T) = 0\}.$$

Now we are in the position to state the main theorem and give its proof as well.

2. The main theorem and its proof

Theorem 2.1. *Assume $f_0 \in L^2(\Omega \times \mathbb{R}^3), H \in L^2([0,T] \times \Omega \times \mathbb{R}^3)$, then there exists a weak solution $f \in L^2([0,T];\ L^2_\nu)$ to (1.7).*

Proof. Our proof relies on the dual method. Firstly, we introduce a formal adjoint operator defined as follows:

$$P^* g = -\partial_t g - v \cdot \nabla_x g + Lg, \quad g \in W. \tag{2.1}$$

Then we are going to prove the following claim.

Claim: $\|g(t,\cdot)\|^2 + \int_t^T \|g(s)\|^2_{L^2_\nu} ds \lesssim \int_t^T \|P^* g\|^2_{L^2_{\nu^{-1}}} ds.$

In fact, on the one hand, by the Green formula [4], we have

(2.2)
$$\begin{aligned}
\iint_{\Omega\times\mathbb{R}^3} (v\cdot\nabla_x g) g dv dx &= \frac{1}{2}\iint_{\Omega\times\mathbb{R}^3} v\cdot\nabla_x g^2 dv dx \\
&= \frac{1}{2}\iint_{\partial\Omega\times\mathbb{R}^3} g^2\cdot(n(x)\cdot v) dv dS_x \qquad \text{(Green's formula)} \\
&= \frac{1}{2}\Big(\iint_{n(x)\cdot v>0} |g|^2\cdot(n(x)\cdot v) dv dS_x + \iint_{n(x)\cdot v<0} |g|^2\cdot(n(x)\cdot v) dv dS_x \\
&= \frac{1}{2}\Big(\iint_{n(x)\cdot v>0} |g|^2\cdot|(n(x)\cdot v)| dv dS_x - \iint_{n(x)\cdot v<0} |g|^2\cdot|(n(x)\cdot v)| dv dS_x\Big) \\
&= \frac{1}{2}\Big(\|\gamma^+ g\|^2_{L^2} - \|\gamma^- g\|^2_{L^2}\Big).
\end{aligned}$$

Thus,

$$(P^* g, g) = -\frac{1}{2}\Big(\frac{d}{dt}\|g\|^2_{L^2} + \|\gamma^+ g\|^2_{L^{2,+}} - \|\gamma^- g\|^2_{L^{2,-}}\Big) + (Lg, g)_{L^2}, \tag{2.3}$$

multiplying e^{2cs} with c to be determined, and integrating with respect to s in $[t,T]$, we have

$$\begin{aligned}
\int_t^T e^{2cs}(P^* g, g) ds \quad &= \quad -\frac{1}{2}\int_t^T e^{2cs}\frac{d}{ds}\|g\|^2_{L^2} ds + \frac{1}{2}\int_t^T e^{2cs}(\|\gamma^- g\|^2_{L^{2,-}} - \|\gamma^+ g\|^2_{L^{2,+}}) ds \\
&\quad + \int_t^T e^{2cs}(Lg, g) ds \\
&=: \quad I_1 + I_2 + I_3
\end{aligned}$$

For I_1,

$$\begin{aligned}
I_1 \quad &= \quad -\frac{1}{2}\int_t^T \frac{d}{ds}(e^{2cs}\|g\|^2_{L^2}) ds + \frac{1}{2}\int_t^T 2c e^{2cs}\|g\|^2_{L^2} ds \\
&= \quad -\frac{1}{2}(e^{2cT}\|g(T)\|^2 - e^{2ct}\|g(t)\|^2_{L^2}) + \frac{1}{2}\int_t^T 2c e^{2cs}\|g(s)\|^2_{L^2} ds,
\end{aligned}$$

note that $g(T) = 0$, we have $I_1 \geq \frac{1}{2}\|g(t)\|^2_{L^2} + c\int_t^T e^{2cs}\|g(s)\|^2_{L^2} ds$.

For I_2, it is easy to get $I_2 = 0$ since $\|\gamma^+ g\|^2_{L^{2,+}} = \|\gamma^- g\|^2_{L^{2,-}}$.
For I_3, note that

$$(Lg, g) \geq c_0 \|g\|^2_{L^2_\nu} - c_1 \|g\|^2_{L^2}$$

and

$$I_3 \geq c_0 \int_t^T e^{2cs} \|g\|^2_{L^2_\nu} ds - c_1 \int_t^T e^{2cs} \|g\|^2_{L^2} ds.$$

It follows that ,

$$\begin{aligned} \int_t^T e^{2cs} |(P^* g, g)| ds \geq \frac{1}{2} \|g(t)\|^2_{L^2} + c \int_t^T e^{2cs} \|g(s)\|^2_{L^2} ds \\ + c_0 \int_t^T e^{2cs} \|g\|^2_{L^2_\nu} ds - c_1 \int_t^T e^{2cs} \|g\|^2_{L^2} ds. \end{aligned} \tag{2.4}$$

On the other hand, by Young's inequality with ϵ [4], taking ϵ small enough, and the Cauchy-Schwartz inequality, we derive

$$\begin{aligned} \int_t^T e^{2cs} |(P^* g, g)| ds &\lesssim \int_t^T e^{2cs} (c_\epsilon \|P^* g\|_{L^2_{\nu-1}} + \epsilon \|g\|^2_{L^2_\nu}) ds \\ &\lesssim \int_t^T e^{2cs} c_\epsilon \|P^* g\|_{L^2_{\nu-1}} ds + \epsilon \int_t^T e^{2cs} \|g\|^2_{L^2_\nu} ds. \end{aligned}$$

Consequently,

$$\begin{aligned} \int_t^T e^{2cs} c_\epsilon \|P^* g\|_{L^2_{\nu-1}} ds &\geq \frac{1}{2} \|g(t)\|^2 + \int_t^T (c - c_1) e^{2cs} \|g\|^2_{L^2} ds \\ &\quad + \int_t^T (c_0 - \epsilon) e^{2cs} \|g\|^2_{L^2_\nu} ds. \end{aligned}$$

Note ϵ is small and taking $c > c_1$, then the claim holds.
Now we define a functional l on $P^* W$ as follows:

$$P^* g \longrightarrow l(P^* g) = \int_0^T (H, g)_{L^2} ds + (f_0, g(0))_{L^2} \tag{2.5}$$

By the claim proved above , we have

$$\begin{aligned} |l(P^* g)| &\leq \int_0^T \|H\|_{L^2} \|g\|_{L^2} ds + \|f_0\|_{L^2} \|g_0\|_{L^2} \\ &\leq \Big(\int_0^T \|H\|_{L^2} ds + \|f_0\|_{L^2} \Big) \Big(\int_0^T \|P^* g\|^2_{L^2_{\nu-1}} ds \Big)^{\frac{1}{2}}, \end{aligned}$$

i.e.

$$|l(P^* g)| \leq c \Big(\int_0^T \|P^* g\|^2_{L^2_{\nu-1}} ds \Big)^{\frac{1}{2}} = c \|P^* g\|_{L^2([0,T];\ L^2_{\nu-1})}.$$

Thanks to the Hahn - Banach Theorem [1], l has a bounded extension $\tilde{l}$ to $L^2([0,T], L^2_{\nu-1})$. Finally, the Riesz representation theorem guarantees there exists $f \in L^2([0,T], L^2_\nu)$, such that

$$\tilde{l}(h) = (f, h), \quad \forall\ h \in L^2([0,T], L^2_{\nu-1}). \tag{2.6}$$

Restricting to $P^*W \subset L^2([0,T], L^2_{\nu^{-1}})$, we have

$$l(P^*g) = (f, P^*g), \quad \forall\, g \in W, \tag{2.7}$$

i.e.

$$(f, (-\partial_t - v\cdot\nabla_x + L)g) = \int_0^T (H,g)_{L^2}ds + (f_0, g(0))_{L^2} \tag{2.8}$$

which ends the proof of Theorem 2.1. □

References

[1] Conway, J.B., A course in functional analysis, Springer, 2007.

[2] DiPerna, R.J., Lions, P.L., On the Cauchy problem for Boltzmann equation: global existence and weak stability, *Ann. Math.* 130 (1989) 321-366.

[3] Esposito, R., Guo, Y., Kim, C., Marra, R., Non-isothermal boundary in the Boltzmann theory and Fourier law, *Comm. Math. Phys.* 323 (2013) 177–239.

[4] Evans, L.C., Partial differential equations, American Mathematical Society, Vol. 19, 1997.

[5] Guo, Y., Decay and Continuity of the Boltzmann Equation in Bounded Domains, *Arch. Ration. Mech. Anal.* 197 (2010) 713-809.

[6] Guo, Y., Kim, C., Tonon, D., Trescases, A., Regularity of the Boltzmann Equation in Convex Domains, *Invent. Math.* (2017) 115–290.

[7] Grad, H., Asymptotic theory of the Boltzmann equation, II. in Rarefied Gas Dynamics, Vol. 1, *Ed. LAURMANN, J., Acad. Press*, (1963) 26-59.

[8] Grad, H., Asymptotic equivalence of the Navier-Stokes and nonlinear Boltzmann equation, *Proc. Symp. Appl. Math.* 17 (1965) 154-183.

[9] Mischler, S., On the Initial Boundary Value Problem for the Vlasov-Poisson-Boltzmann System, *Comm. Math. Phys.* 210 (2000) 447-466.

[10] Ukai, S., Solutions of the Boltzmann equation. Pattern and Waves, *Stud. Math. Appl.* 18 (1986) 37-96.