The Nepal Math. Sc. Report
Vol. 34, No. 1 and 2, 2016

ERMAKOV EQUATION AND CAMASSA-HOLM WAVES

HARET C. ROSU[1] AND STEFAN C. MANCAS[2]

[1] *IPICyT, Instituto Potosino de Investigacion Cientifica y Tecnologica, San Luis Potosí, C.P. 78216, S.L.P., Mexico*
[2] *Embry–Riddle Aeronautical University, Daytona Beach, FL 32114-3900, U.S.A.*
mancass@erau.edu

Abstract: Since the works of [1] and [2], it is known that the solution of the Ermakov equation is an important ingredient in the spectral problem of the Camassa-Holm equation. Here, we review this interesting issue and consider in addition more features of the Ermakov equation which have an impact on the behavior of the shallow water waves as described by the Camassa-Holm equation.

Key Words: Ermakov equation, Camassa-Holm equation, nonlinear waves
AMS (MOS) Subject Classification. 34A05, 34A34, 35Q35.

1. Introduction

The Camassa-Holm equation ([3])

$$u_t - u_{xxt} + 2ku_x + 3uu_x = 2u_x u_{xx} + uu_{xxx} \tag{1.1}$$

can be factored ([4]) as

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + 2u_x\right) q(x,t) = 0 \ , \qquad q(x,t) = (u - u_{xx} + k) \ , \tag{1.2}$$

where $q(x,t)$ is known as momentum. There are two conservation laws

$$\frac{\partial}{\partial t}q(x,t) + \frac{\partial}{\partial x}\left[2ku + \frac{3}{2}u^2 - uu_{xx} - \frac{1}{2}(u_x)^2\right] = 0, \tag{1.3}$$

and

$$(\sqrt{q})_t + (u\sqrt{q})_x = 0 \tag{1.4}$$

respectively. Using the momentum, Eq. (1.1) is equivalent to

$$q_t + uq_x + 2u_x q = 0. \tag{1.5}$$

The Lax pairs for CH are (Constantin 2001)

$$\begin{cases} \psi_{xx} = \left(\frac{1}{4} + \lambda q\right)\psi \\ \psi_t = \left(\frac{1}{2\lambda} - u\right)\psi_x + \frac{1}{2}u_x\psi \end{cases} \tag{1.6}$$

and the compatibility condition $\psi_{xxt} = \psi_{txx}$ leads back to Eq. (1.1).

Received June 23, 2016

2. Liouville transformation on CH

The common procedure is to use $q = \left(\frac{dy}{dx}\right)^2$ followed by Liouville's transformation $\phi = q^{\frac{1}{4}}\psi$ in the first equation of the Lax pair system (1.6) to obtain

$$\phi_{yy} - \left[\frac{4q(1+q_{yy}) - 3(q_y)^2}{16q^2}\right]\phi = \lambda\phi. \tag{2.1}$$

In addition, since $q(x,t) = u - u_{xx} + k$ with $\frac{dy}{dx} = \sqrt{q}$, the momentum equation becomes

$$qu_{yy} + \frac{1}{2}q_y u_y - u = k - q \tag{2.2}$$

Therefore, if we know q then one can try to solve CH by finding u in Eq. (2.2). As a scattering Schrödinger problem, we can rewrite Eq. (2.1) as

$$\phi_{yy} - \left[Q + \frac{1}{4k}\right]\phi = \lambda\phi \tag{2.3}$$

with potential Q defined in terms of the momentum by

$$q_{yy} - \frac{3}{4q}(q_y)^2 - 4\left[Q + \frac{1}{4k}\right]q + 1 = 0 \tag{2.4}$$

Let $q = E^4$, then we obtain the Ermakov equation

$$E_{yy} - \left[Q + \frac{1}{4k}\right]E + \frac{1}{4}E^{-3} = 0 \tag{2.5}$$

The solution of Eq. (2.5) is given by Pinney (1950)

$$E = \sqrt{F_1^2 - \frac{1}{4}\left(\frac{F_2}{W}\right)^2} \tag{2.6}$$

where F_1 and F_2 are the two independent solutions of the linear ODE

$$F_{yy} - \left[Q + \frac{1}{4k}\right]F = 0 \tag{2.7}$$

The Ermakov-Lewis invariant is

$$\mathcal{I} = \frac{1}{2}\left[(EF_y - FE_y)^2 - \frac{1}{4}\left(\frac{E}{F}\right)^{-2}\right]. \tag{2.8}$$

By using E from Eq. (2.5) and F as a superposition of homogenous solutions $F = aF_1 + bF_2$, the invariant is

$$\mathcal{I} = \frac{1}{2}\left(-\frac{a^2}{4} + b^2W^2\right) = const. \tag{2.9}$$

It is easy to show that Eq. (2.7) and (2.5) are related to the linear third order ODE

$$\phi_{yyy} - 4\left[Q + \frac{1}{4k}\right]\phi_y - 2Q_y\phi = 0\ , \tag{2.10}$$

which is of maximal symmetry algebra $sp(5)$ ([5]) and has ϕ itself as an integrating factor; thus Eq. (2.10) becomes

$$\phi\phi_{yy} - \frac{1}{2}(\phi_y)^2 - 2\left[Q + \frac{1}{4k}\right]\phi^2 + \frac{1}{2} = 0\ . \tag{2.11}$$

3. Solutions

Let us use the solitonic potential

$$Q = -2K^2 \operatorname{sech}^2(\theta) \ , \qquad \theta = Ky + \Omega t - \alpha \tag{3.1}$$

where $K = \frac{\mu}{\sqrt{k}}$, $\Omega = \frac{\mu}{2\lambda}$, $\mu = \frac{1}{2}\sqrt{1+4\lambda k}$, and α, an arbitrary phase. Thus, Eq. (2.10) becomes

$$\phi_{\theta\theta\theta} + 4\left[2\ \operatorname{sech}^2\theta - \frac{1}{1-\frac{2k}{c}}\right]\phi_\theta - 8\ \operatorname{sech}^2\theta\ \tanh\theta\ \phi = 0 \tag{3.2}$$

and has solution

$$\phi(\theta) = \frac{c}{2\sqrt{k}}\left(\operatorname{sech}^2\theta + \frac{2k}{c}\ \tanh^2\theta\right)\ , \tag{3.3}$$

where $c = -\frac{1}{2\lambda}$. Since $q = \phi^2$, we get

$$q(\theta) = \frac{c^2}{4k}\left(\operatorname{sech}^2\theta + \frac{2k}{c}\ \tanh^2\theta\right)^2 \tag{3.4}$$

In the θ variable Eq. (2.2) becomes

$$\frac{1}{4k}\left(1-\frac{2k}{c}\right)\left(qu_{\theta\theta} + \frac{1}{2}q_\theta u_\theta\right) - u = k - q \tag{3.5}$$

and after we substitute q, q_θ in Eq. (3.5) it becomes

$$\left[c-k+k\ \cosh^2(2\theta)\right]^2 u_{\theta\theta} - 2(c-2k)\left[c-k+k\ \cosh^2(2\theta)\right]\ \tanh\theta\ u_\theta - 16ck^2u = f(\theta) \tag{3.6}$$

with nonhomogenous function given by $f(\theta) = -(c-2k)\ \operatorname{sech}^4\theta\,[4ck\,(c+2k\ \cosh(2\theta)]$. For the particular case of $k \to 0$ Eq. (3.6) simplifies to

$$u_{\theta\theta} - 2\ \tanh\theta\ u_\theta = 0 \tag{3.7}$$

with solution

$$u(\theta) = \mathcal{C}_1 + \mathcal{C}_2\left(\frac{\theta}{2} + \frac{1}{4}\ \sinh 2\theta\right). \tag{3.8}$$

Solving Eq. (3.6) yields to general solution $u(\theta) = u_p(\theta) + \mathcal{C}_1 u_1 + i\mathcal{C}_2 u_2$. Denoting $k_c = \frac{2k}{c}$ the particular solution is

$$u_p(\theta) = \frac{c(1-k_c)}{1-k_c+k_c\ \cosh^2\theta} = \frac{c(1-k_c)}{1+k_c\ \sinh^2\theta} \tag{3.9}$$

while the general solutions are

$$\begin{cases} u_1(\theta) = \cosh\left\{\frac{2}{\sqrt{1-k_c}}\theta - 2\operatorname{arctanh}\left[\sqrt{1-k_c}\ \tanh\ \theta\right]\right\}, & 0 < k_c < 1 \\ u_2(\theta) = \sinh\left\{\frac{2}{\sqrt{1-k_c}}\theta - 2\operatorname{arctanh}\left[\sqrt{1-k_c}\ \tanh\ \theta\right]\right\}, & 0 < k_c < 1 \end{cases} \tag{3.10}$$

and

$$\begin{cases} u_1(\theta) = \cos\left\{\frac{2}{\sqrt{k_c-1}}\theta + 2\arctan\left[\sqrt{k_c-1}\ \tanh\ \theta\right]\right\}, & k_c > 1 \\ u_2(\theta) = -i\ \sin\left\{\frac{2}{\sqrt{k_c-1}}\theta + 2\arctan\left[\sqrt{k_c-1}\ \tanh\ \theta\right]\right\}, & k_c > 1. \end{cases} \tag{3.11}$$

See Fig. 1 (top) for the particular solution u_p if $k_c < 1$ which shows that the soliton profiles are tending to a compacton when k_c is small, and on (bottom) when $k_c > 1$ which shows that soliton profiles are more and more peakon-like. In Figs. 2 and 3 we show the

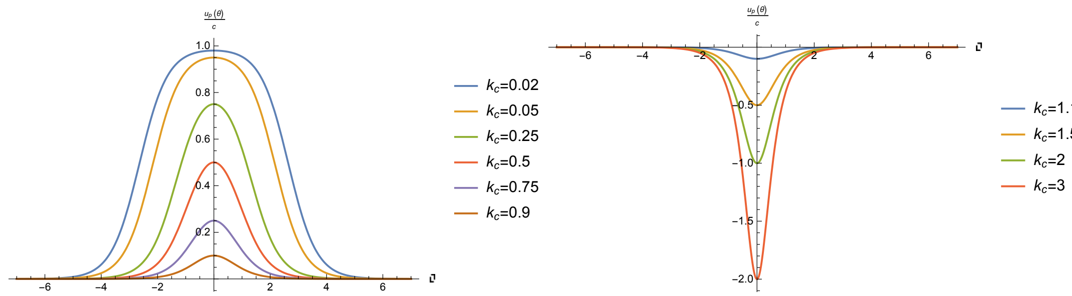

FIGURE 1. The particular solution $\frac{u_p}{c}$ for $k_c = \frac{2k}{c} < 1$ (top) and $k_c = \frac{2k}{c} > 1$ (bottom).

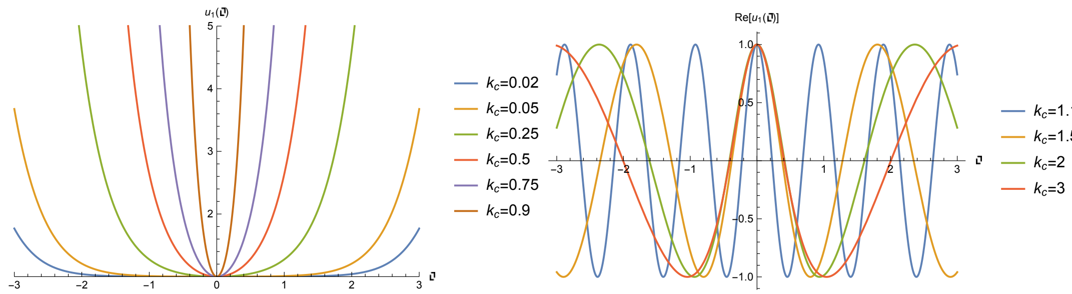

FIGURE 2. The particular solution u_1 for $k_c = \frac{2k}{c} < 1$ (top) and $k_c = \frac{2k}{c} > 1$ (bottom).

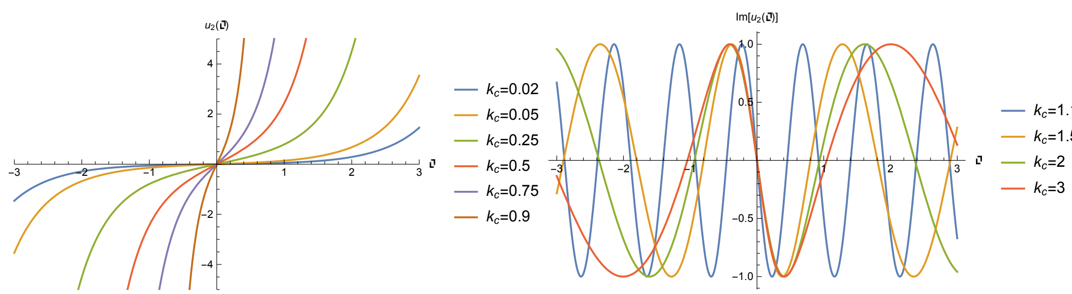

FIGURE 3. Same as in the previous figure for particular solution u_2.

particular solution u_1 and u_2 for both cases.

To find the solution in terms of x and t, we need to use the relation between θ and x. Since $\sqrt{q(\theta)} = \frac{dy}{dx}$, we have

$$\int \frac{d\theta}{\operatorname{sech}^2\theta + k_c \tanh^2\theta} = \frac{\sqrt{1-k_c}}{2k_c}(x - ct - x_0). \tag{3.12}$$

To find this integral we notice that if we differentiate the argument of the homogenous solutions u_1 and u_2, we have

$$\frac{d}{d\theta}\left\{\frac{2}{\sqrt{1-k_c}}\theta - 2\operatorname{arctanh}\left[\sqrt{1-k_c}\,\tanh\theta\right]\right\} = \frac{2k_c}{\sqrt{1-k_c}}\frac{1}{\operatorname{sech}^2\theta + k_c \tanh^2\theta} \tag{3.13}$$

Using Eqs. (3.12) and (3.13) we conclude that

$$x - ct - x_0 = \begin{cases} \frac{2}{\sqrt{1-k_c}}\theta - 2\operatorname{arctanh}\left[\sqrt{1-k_c}\,\tanh\theta\right], & 0 < k_c < 1 \\ \frac{2}{\sqrt{k_c-1}}\theta + 2\arctan\left[\sqrt{k_c-1}\,\tanh\theta\right], & k_c > 1 \end{cases} \tag{3.14}$$

References

[1] Constantin, A., On the scattering problem for the Camassa-Holm equation, *Proc. R. Soc. Lond. A*, **457**, 953–970, 2001.

[2] Johnson, R.S., On solutions of the Camassa-Holm equation, *Proc. R. Soc. Lond. A*, **459**, 1687-1708, 2003.

[3] Camassa, R., Holm, D.D., An integrable shallow water equation with peaked solitons, *Phys. Rev. Lett.*, **71**, 1661-1664, 1993.

[4] Gilson, C., Pickering, A., Factorization and Painlevé analysis of a class of nonlinear 3d-order PDEs, *J. Phys. A: Math. Gen.*, **28**, 2871-2888, 1995.

[5] Abraham-Shrauner, B. & al., Hidden and contact symmetries of ordinary differential equations, *J. Phys. A: Math. Gen.*, **28**, 6707-6716, 1993.

[6] Pinney, E., The nonlinear differential equation $y' + p(x)y' + cy^{-3} = 0$, *Proc. Amer. Math. Soc.* **1**, 681, 1950.