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RELATION BETWEEN BMO AND A_2 WEIGHT FUNCTIONS

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Abstract: In this paper, we relate Bounded Mean Oscillation (BMO) function and A_2 weight function. We show that logarithm of any A_2 function is a BMO function and every BMO function is equal to a constant multiple of the logarithm of an A_2 weight function. Moreover, we show that logarithm of any A_p weight function for $1 < p < \infty$ is a BMO function.

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1. Introduction and main result

The space of bounded mean oscillation, abbreviated BMO, is the space of all functions whose deviations from their means over cubes is bounded. The BMO space is frequently used space in analysis. For example the BMO space comes into play in the characterization of L^2 boundedness of noncovolution singular integral operators having standard kernels. Careleson measures have natural relation with BMO functions such that the measures of functions are Carleson measures iff the functions are in BMO. Readers are suggested to refer [1] for more about the BMO space.

On the other hand, the theory of weights play an important role in various fields such as extrapolation theory, vector-valued inequalities and estimates for certain class of non linear differential equation. Moreover, they are very useful in the study of boundary value problems for Laplace's equation in Lipschitz domains. In 1970, Muckenhoupt characterized positive functions w for which the Hardy-Littlewood maximal operator M maps $L^p(\mathbb{R}^n, w(x)dx)$ to itself. Muckenhoupt's characterization actually gave the better understanding of theory of weighted inequalities which then led to the introduction of A_p class and consequently the development of weighted inequalities. For more about the weighted theory, please refer to [1],[2] and [3]. In this article, we relate the BMO function and A_2 weight function. Before this, some definitions are in order:

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Definition: let f be a locally integrable function on $\mathbb{R}^n$ and Q be a measurable set in $\mathbb{R}^n$. Then the mean oscillation of f over Q is

$$\frac{1}{|Q|}\int_Q |f(x) - \mathrm{Avg}_Q f| dx$$

where

$$\mathrm{Avg}_Q f = \frac{1}{|Q|}\int_Q f(x)dx$$

is the mean or average of f over Q.
The BMO norm of a complex valued function f on $\mathbb{R}^n$ is defined as

$$\| f \|_{BMO} = \sup_Q \frac{1}{|Q|}\int_Q |f(x) - \mathrm{Avg}_Q f| dx.$$

In the above definition the supremum is taken over all cubes Q in $\mathbb{R}^n$. Then the function f is said to be of bounded mean oscillation if $\| f \|_{BMO} < \infty$.

Definition: A locally integrable function on $\mathbb{R}^n$ that takes values in the interval $(0, \infty)$ almost everywhere is called a weight. So by definition a weight function can be zero or infinity only on a set whose Lebesgue measure is zero. We use the notation $w(E) = \int_E w(x)dx$ to denote the w-measure of the set E and we reserve the notation $L^n(\mathbb{R}^n, w)$ or $L^p(w)$ for the weighted L^p spaces. We note that $w(E) < \infty$ for all sets E contained in some ball since the weights are locally integrable functions.

Definition: Let $1 < p < \infty$. A weight w is said to be of class A_p if $[w]_{A_p}$ if finite where $[w]_{A_p}$ is defined as

$$[w]_{A_p} = \sup_{Q\text{cubes in}\mathbb{R}^n} \left(\frac{1}{|q|}|w(x)|dx\right)\left(|w(x)|^{\frac{-1}{p-1}}dx\right)^{p-1}.$$

We remark that in the above definition of A_p one can also use set of all balls in $\mathbb{R}^n$ instead of all cubes in $\mathbb{R}^n$. Readers are suggested to read for motivation, properties of A_P weights and much more about the A_P weights.

Theorem 1. *For all functions f in space of BMO defined on $\mathbb{R}^n$, for all cubes Q and $\alpha > 0$, we have*

$$\left|\left\{x : |f(x) - Avg_Q f| > \alpha\right\}\right| \leq e|Q|e^{-A\alpha/\|f\|_{BMO}}$$

with $A = (2^n e)^{-1}$.

This theorem is popularly known as John-Nirenberg theorem and for the proof, please refer to [4].

We first establish the following result:
Let ν be a real-valued locally integrable function on R^n and let $1 < p < \infty$. Then $e^\nu \in A_p$ if and only if the following two conditions are satisfied for some constant $c < \infty$:

(a) $\displaystyle\sup_{Q\text{cubes}} \frac{1}{|Q|}\int_Q e^{\nu(t)-\nu_Q}dt \leq C$

(b) $\displaystyle\sup_{Q\text{cubes}} \frac{1}{|Q|}\int_Q e^{-(\nu(t)-\nu_Q)\frac{1}{p-1}}dt \leq C$

Suppose $e^{\nu} \in A_p, 1 < p < \infty$..Then,

$$\begin{aligned}\frac{1}{|Q|}\int_Q e^{\nu(t)-\nu_Q}dt &= e^{-\nu_Q}\frac{1}{|Q|}\int_Q e^{\nu(t)}dt \\ &= \left(e^{\frac{-\nu_Q}{p-1}}\right)^{p-1}\frac{1}{|Q|}\int_Q e^{\nu(t)}dt. \\ &\leq \left(\frac{1}{|Q|}\int_Q e^{\frac{-\nu(t)}{p-1}}dt\right)^{p-1}\left(\frac{1}{|Q|}\int_Q e^{\nu(t)}dt\right) \\ &\leq [e^{\nu(t)}]_{A_p} < \infty.\end{aligned}$$

This proves (a). Again,

$$\begin{aligned}\frac{1}{|Q|}\int_Q e^{-(\nu(t)-\nu_Q)\frac{1}{p-1}}dt &= \frac{1}{|Q|}\int_Q e^{\frac{-\nu(t)}{p-1}}.e^{\frac{\nu_Q}{p-1}}dt. \\ &\leq \left(\frac{1}{|Q|}\int_Q e^{\frac{\nu(t)}{p-1}}dt\right)\left(\frac{1}{|Q|}\int_Q e^{\nu(t)}dt\right)^{\frac{1}{p-1}} \\ &\leq [e^{\nu}]_{A_p}\end{aligned}$$

This proves (b). Conversely, suppose that the conditions (a) and (b) hold. We need to show that $e^{\nu} \in A_p$. This follows because,

$$\begin{aligned}&\left(\frac{1}{|Q|}\int_Q e^{\nu(t)}dt\right)\left(\frac{1}{|Q|}\int_Q \left(e^{\nu(t)}\right)^{\frac{-1}{p-1}}dt\right)^{p-1} \\ &= \left(\frac{1}{|Q|}\int_Q e^{\nu(t)-\nu_Q}dt\right)\left(\frac{1}{|Q|}\int_Q e^{\frac{-(\nu(t)-\nu_Q)}{p-1}}dt\right)^{p-1} \\ &< C\end{aligned}$$

by (a) and (b). In particular for p=2, we have: $e^{v} \in A_2 \Longleftrightarrow$ for some constant$C < \infty$

$$\sup_Q \frac{1}{|Q|}\int_Q e^{V(t)-V_Q}dt \leq C$$

$$\sup_Q \frac{1}{|Q|}\int_Q e^{-(V(t)-V_Q)}dt \leq C$$

$$\Longleftrightarrow \sup_Q \frac{1}{|Q|}\int_Q e^{|V(t)-V_Q|}dt \leq C$$

if $\varphi \in A_2$, then

$$\left(\frac{1}{|Q|}\int_Q \varphi\right)\left(1/|Q|\int_Q \varphi^{-1}\right) \leq C$$

equivalently

$$\left(\frac{1}{|Q|}\int_Q \varphi/\varphi_Q\right)\left(1/|Q|\int_Q \varphi_Q/\varphi\right) \leq C.$$

By Jensen's inequality, each function is at least 1 and at most C, therefore

$$\frac{1}{|Q|}\int_Q e^{|\log\varphi-\log\varphi_Q|} \leq 2C$$

and so

$$\frac{1}{|Q|}\int_Q |\log\varphi - \log\varphi_Q| \le 2C.$$

This proves that $log\varphi \in BMO$.

Next we prove that if $\varphi \in BMO$, then $e^{C_\varphi \varphi} \in A_2$ for some constant C_φ. From the part (a), to show $e^{C_\varphi\varphi} \in A_2$ it suffices to prove $\sup_Q \frac{1}{|Q|}\int_Q e^{|f-f_Q|}dt \le C$ where $f = e^{C_\varphi \varphi}$. John-Nirenberg Theorem gives,

$$|x \in Q : |f(x) - avgf| > \alpha| \le C|Q|e^{-A\alpha/\|f\|_{BMO}}.$$

Let $Q_i = x \in Q : |f| \ge |f(x) - Avgf| > i$. Then

$$\frac{1}{|Q|}\int_Q e^{|f-f_Q|}dt = \frac{1}{|Q|}\sum_{i=1}^{\infty}\int_{Q_i} e^{|f-f_Q|}dt \le \frac{1}{|Q|}\sum_{i=1}^{\infty} e^{i+1}C|Q|e^{-Ai/\|f\|_{BMO}}$$

We take $C_\varphi, s.t.\|f\|_{BMO} = A/2$ so that

$$\frac{1}{|Q|}\int_Q e^{|f-f_Q|}dt \le Ce\sum_{i=1}^{\infty} e^{-i} < \infty.$$

This shows that $e^{C_\varphi\varphi} \in A_2$. This proves that every BMO function is equal to a constant multiple of the logarithm of an A_2 weight function.
Finally we show that logarithm of any A_p weight function for $1 < p < \infty$ is a BMO function. We already proved that $\|\log\varphi\|_{BMO} \le [\varphi]_{A_2}$ and when $1 < p \le 2$

$$\varphi_{A_Q} = \sup_Q (\frac{1}{|Q|}\int_Q \varphi(x)dx)(\frac{1}{|Q|}\int_Q \varphi(x)^{-1/p-1}dx)^{p-1}$$

$$\ge \sup_Q (\frac{1}{|Q|}\int_Q \varphi(x)dx)(\frac{1}{|Q|}\int_Q \varphi^{-1}dx) = [\varphi]_{A_2}.$$

Therefore $\|\log\varphi\|_{BMO} \le [\varphi]_{A_p}$ when $1 < p \le 2$

$$\|log\varphi^{-1/p-1}\|_{BMO} \le [\varphi^{-1/p-1}]_{A_{p'}}$$

when $p > \infty$. That is:

$$\sup_Q\left(\frac{1}{|Q|}\int_Q |\log\varphi^{-1/p-1} - \log\varphi_Q^{-1/p-1}|dt\right) \le \sup_Q\left(\frac{1}{|Q|}\int_Q p^{-1/p-1}\right)\left(\frac{1}{|Q|}\int_Q \varphi^{1/(p-1)(p'-1)}\right)^{p'-1}$$

$$\frac{1}{(p-1)}\sup_Q\left(\frac{1}{|Q|}\int |\log\varphi - \log\varphi_Q|dt\right) \le \sup\left(\frac{1}{|Q|}\int_Q \varphi^{-1/p-1}\right)\left(\frac{1}{|Q|}\int_Q \varphi\right)^{p-1}$$

$$\sup_Q\left(\frac{1}{|Q|}\int |\log\varphi - \log\varphi_Q|dt\right) \le (p-1)\sup\left(\frac{1}{|Q|}(1/|Q|\int_Q \varphi^{-1/p-1})^{p-1}\right)^{1/p-1}.$$

Consequently, when $p > 2$.

$$\|\log\varphi\|_{BMO} \le (p-1)[\varphi]_{A_p}^{1/p-1}.$$

This shows that the $\log\varphi$ is a BMO function.

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