The Nepali Math. Sc. Report
Vol. 35, No.1 and 2, 2018

ESCAPING SETS OF HYPERBOLIC SEMIGROUPS

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Abstract: *In this paper, we mainly study hyperbolic semigroups from which we get non-empty escaping sets and Eremenko's conjecture remains valid. We prove that if each generator of bounded type transcendental semigroups is hyperbolic, then the semigroups are themselves hyperbolic and all components of escaping sets are unbounded.*

Key Words: Escaping sets, Eremenko's conjecture, transcendental semigroups, post singularly bounded (finite) semigroups, hyperbolic semigroups.
AMS (MOS) [2010] Subject Classification. 37F10, 30D05

1. Introduction

Throughout this paper, we denote the *complex plane* by $\mathbb{C}$ and the set of integers greater than zero by $\mathbb{N}$. We assume the function $f : \mathbb{C} \to \mathbb{C}$ is *transcendental entire function* unless otherwise stated. For any $n \in \mathbb{N}$, f^n always denotes the nth *iterates* of f. The *escaping set* of f is defined by

$$I(f) = \{z \in \mathbb{C} : f^n(z) \to \infty \text{ as } n \to \infty\}$$

and any point $z \in I(S)$ is called an *escaping point*. For a transcendental entire function f, the escaping set $I(f)$ was first studied by A. Eremenko [2]. He showed that

(1) $I(f) \neq \emptyset$,
(2) the boundary of this set is a Julia set $J(f)$ (that is, $J(f) = \partial I(f)$),
(3) $I(f) \cap J(f) \neq \emptyset$, and
(4) $\overline{I(f)}$ has no bounded component.

Furthermore,

5. $I(f^n) = I(f)$ for all $n \in \mathbb{N}$.
6. $I(f)$ is completely invariant under f.

This research work of the first author is supported by a PhD faculty fellowship of the University Grants Commission, Nepal.

In view of the statement (4), he posed a question:

Is every component of $I(f)$ unbounded?

This question is considered as an important open problem of transcendental dynamics, and nowadays, it is famous as *Eremenko's conjecture.* Note that the complement of the Julia set $J(f)$ in $\mathbb{C}$ is the *Fatou set* $F(f)$. A connected maximal open subset of $F(f)$ is called *Fatou component.*

For any holomorphic function f, we call

$$C(f) = \{z \in \mathbb{C} : f'(z) = 0\}$$

(where f' represents derivative of f with respect to z) by the set of *critical points* and

$$CV(f) = \{w \in \mathbb{C} : w = f(z) \text{ such that } f'(z) = 0 \text{ for some } z\}$$

by the set of *critical values.* The set $AV(f)$ consisting of all $w \in \mathbb{C}$ such that there exists a curve $\Gamma : [0, \infty) \to \mathbb{C}$ so that $\Gamma(t) \to \infty$ and $f(\Gamma(t)) \to w$ as $t \to \infty$ is the set of *asymptotic values* of f and

$$SV(f) = \overline{(CV(f) \cup AV(f))}$$

is the set of *singular values* of f. If $SV(f)$ is finite, then f is said to be of *finite type.* If $SV(f)$ is bounded, then f is said to be of *bounded type.* The sets

$$\mathscr{S} = \{f : f \text{ is of finite type}\}$$

and

$$\mathscr{B} = \{f : f \text{ is of bounded type}\}$$

are respectively known as *Speiser class* and *Eremenko-Lyubich class.*

The main concern of this paper is to study of escaping sets of transcendental semigroups. So, we start our formal study from the notion of transcendental semigroups. The set $\text{Hol}(\mathbb{C})$ denotes a set of all holomorphic functions of $\mathbb{C}$. If $f \in \text{Hol}(\mathbb{C})$, then f is either a polynomial or a transcendental entire function. The composite of two entire functions is an entire function. So, this fact makes the set $\text{Hol}(\mathbb{C})$ a semigroup with semigroup operation being the functional composition.

Definition 1.1 (**Transcendental semigroup**). Let $A = \{f_i : i \in \mathbb{N}\} \subset \text{Hol}(\mathbb{C})$ be a set of transcendental entire functions $f_i : \mathbb{C} \to \mathbb{C}$. A *transcendental semigroup* S is a semigroup generated by the set A with semigroup operation being the functional composition. We denote this semigroup by

$$S = \langle f_1, f_2, f_3, \cdots, f_n, \cdots \rangle \quad \text{or simply by} \quad S = \langle f_i \rangle.$$

Here, each $f \in S$ is a transcendental entire function and S is closed under functional composition. Thus $f \in S$ is constructed through a composition of the finite number of functions f_{i_k}, (where $i_k \in \{1, 2, 3, \ldots, m\}$ for some $m \in \mathbb{N}$). That is,

$$f = f_{i_1} \circ f_{i_2} \circ f_{i_3} \circ \cdots \circ f_{i_m}.$$

A semigroup S generated by finitely many functions f_i, $(i = 1, 2, 3, \ldots, n)$ is called *finitely generated transcendental semigroup* and write

$$S = \langle f_1, f_2, \ldots, f_n \rangle.$$

If S is generated by only one transcendental entire function f, then S is called*cyclic or trivial transcendental semigroup* and write

$$S = \langle f \rangle$$

In this case, each $g \in S$ can be written as $g = f^n$, where f^n is the nth iterates of f with itself. The transcendental semigroup S is *abelian* if

$$f_i \circ f_j = f_j \circ f_i$$

for all generators f_i and f_j of S.

We say that a family $\mathscr{F}$ of holomorphic functions is a *normal family* in $\mathbb{C}$ if every sequence $(f_i) \subseteq \mathscr{F}$ has a subsequence (f_{i_k}) which is uniformly convergent or divergent on all compact subsets of $\mathbb{C}$. If there is a neighborhood U of a point $z \in \mathbb{C}$ such that $\mathscr{F}$ is a normal family in U, then we say that $\mathscr{F}$ is normal at z. If $\mathscr{F}$ is a semigroup S such that it is normal family in a neighborhood U of a point $z \in \mathbb{C}$, we say S is normal at z. We say that a function f is *iteratively divergent* at $z \in \mathbb{C}$ if $f^n(z) \to \infty$ as $n \to \infty$. A semigroup S is *iteratively divergent* at z if

$$f^n(z) \to \infty \text{ as } n \to \infty$$

for all $f \in S$. A semigroup S is said to be *iteratively bounded* at z if there is an element $f \in S$ which is not iteratively divergent at z.

Like in iteration theory of a single transcendental entire function, the Fatou set, Julia set and escaping set in the settings of transcendental semigroups are defined as follows:

Definition 1.2 (Fatou set, Julia set and escaping set). Let S be a transcendental semigroup. The *Fatou set* of S is defined by

$$F(S) = \{z \in \mathbb{C} : S \text{ is normal at } z\},$$

and the *Julia set* $J(S)$ of S is the complement of $F(S)$. The *escaping set* of S is defined by

$$I(S) = \{z \in \mathbb{C} : S \text{ is iteratively divergent at } z\}$$

We call each point of the set $I(S)$ by an *escaping point.*

If $S = \langle f \rangle$, then the Fatou set, Julia set and escaping set are respectively denoted by $F(f), J(f)$ and $I(f)$. So, Definition 1.2 generalizes the Fatou set, Julia set and escaping set of a single transcendental entire function. For simplicity, we call the dynamics of $S = \langle f \rangle$ by classical transcendental dynamics and the dynamics of $S = \langle f_i \rangle$ for at least $i = 1, 2$ by transcendental semigroup dynamics.

2. Some Fundamental Features of Escaping Set

The following immediate relation between $I(S)$ and $I(f)$ for any $f \in S$ will be clear from the Definition 1.2 of escaping sets.

Theorem 2.1. *$I(S) \subset I(f)$ for all $f \in S$ and hence $I(S) \subset \bigcap_{f\in S} I(f)$.*

Proof. Let $z \in I(S)$, then $f^n(z) \to \infty$ as $n \to \infty$ for all $f \in S$. By which we mean $z \in I(f)$ for any $f \in S$. This immediately follows the second inclusion. □

We dealt this Theorem 2.1 in the case of a transcendental semigroup S even though it holds for polynomial semigroups. Note that the above same type of relation (Theorem 2.1) holds between $F(S)$ and $F(f)$. However opposite relation holds between the sets $J(S)$ and $J(f)$. Poon [9, Theorem 4.1, Theorem 4.2] proved that the Julia set $J(S)$ is perfect and $J(S) = \overline{\bigcup_{f\in S} J(f)}$ for any transcendental semigroup S. From Theorem 2.1, we can say that the escaping set may be empty. For example, the escaping set of the semigroup $S = \langle f, g\rangle$ generated by functions $f(z) = e^z$ and $g(z) = e^{-z}$ is empty (that is, the particular function $h = g \circ f^k \in S$ (say) is iteratively bounded at any $z \in I(f)$). Note that $I(f) \neq \emptyset$ in classical iteration theory ([2, Theorem 1]). Dinesh Kumar and Sanjay Kumar [5, Theorem 2.5] have also mentioned the following transcendental semigroup S, where $I(S)$ is an empty set.

Proposition 2.1. *The transcendental entire semigroup $S = \langle f_1,\ f_2\rangle$ generated by two functions f_1 and f_2 from respectively two parameter families $\{e^{-z+\gamma} + c$ where $\gamma, c \in \mathbb{C}$ and $Re(\gamma) < 0,\ Re(c) \geq 1\}$ and $\{e^{z+\mu} + d,$ where $\mu, d \in \mathbb{C}$ and $Re(\mu) < 0,\ Re(d) \leq -1\}$ of functions has empty escaping set $I(S)$.*

There are several classes transcendental semigroups whose escaping sets are non-empty. The following examples [8, Examples 3.2 and 3.3] and [5, Examples 2.6 and 2.7] are evident.

Example 2.1. Let $S = \langle f, g\rangle$, where $f(z) = e^z + \lambda$ and $g(z) = e^z + \lambda + 2\pi i$ for all $\lambda \in \mathbb{C} - \{0\}$. Then $I(S) = I(f) \neq \emptyset$.

Example 2.2. Let $S = \langle f, g\rangle$, where $f(z) = \lambda \sin z$ and $g(z) = \lambda \sin z + 2\pi$ for all $0 < |\lambda| < 1$. Then $I(S) = I(f) \neq \emptyset$.

Example 2.3. Let $S = \langle f, g\rangle$, where $f(z) = e^{\lambda z}$ and $g(z) = e^{s\lambda z} + 2\pi i/\lambda$ for all $\lambda \in \mathbb{C} - \{0\}$ and $s \in \mathbb{N}$. Then $I(S) = I(f) \neq \emptyset$.

Example 2.4. Let $S = \langle f, g\rangle$, where $f(z) = \lambda \sin z$ for all $\lambda \in \mathbb{C} - \{0\}$ and $g(z) = f^n + 2\pi$ for all $n \in \mathbb{N}$. Then $I(S) = I(f) \neq \emptyset$.

From all of these examples, we can get non-empty escaping sets. Dinesh Kumar and Sanjay Kumar [5, Theorem 3.4] generalized these examples to the following result.

Proposition 2.2. *Let $S = \langle f, g\rangle$ be a transcendental semigroup generated by periodic function f with period p and another function g defined by $g = f^n + p,\ n \in \mathbb{N}$. Then $I(S) = I(f)$.*

In the case of non-empty escaping set $I(S)$, Eremenko's result [2], $\partial I(f) = J(f)$ of classical transcendental dynamics can be generalized to semigroup dynamics. The following results is due to Dinesh Kumar and Sanjay Kumar [5, Lemma 4.2 and Theorem 4.3] which gives the generalized answer in semigroup settings.

Proposition 2.3. *Let S be a transcendental entire semigroup such that $I(S) \neq \emptyset$. Then*

(1) *$int(I(S)) \subset F(S)$ and $ext(I(S)) \subset F(S)$, where int and ext respectively denote the interior and exterior of $I(S)$.*
(2) *$\partial I(S) = J(S)$, where $\partial I(S)$ denotes the boundary of $I(S)$.*

This last statement is equivalent to $J(S) \subset \overline{I(S)}$. If $I(S) \neq \emptyset$, then we [11, Theorem 4.6] proved the following result which is a generalization of Eremenko's result $I(f) \cap J(f) \neq \emptyset$ [2, Theorem 2] of classical transcendental dynamics to holomorphic semigroup dynamics.

Theorem 2.2. *Let S be a transcendental semigroup such that $F(S)$ has a multiply connected component. Then $I(S) \cap J(S) \neq \emptyset$*

Eremenko and Lyubich [3] proved that if transcendental function $f \in \mathscr{B}$, then $I(f) \subset J(f)$, and $J(f) = \overline{I(f)}$. Dinesh Kumar and Sanjay Kumar [5, Theorem 4.5] generalized these results to a finitely generated transcendental semigroup of bounded type as shown below.

Proposition 2.4. *For every finitely generated transcendental semigroup $S = \langle f_1, f_2, \ldots, f_n \rangle$ in which each generator f_i is of bounded type, then $I(S) \subset J(S)$ and $J(S) = \overline{I(S)}$.*

Proof. Eremenko and Lyubich's result [3] shows that $I(f) \subset J(f)$ for each $f \in S$ of bounded type. Poon's result shows [9, Theorem 4.2] that $J(S) = \overline{\bigcup_{f \in S} J(f)}$. Therefore, (from Definition 1.2 of escaping set and theorem 2.1) for every $f \in S$, we can write,

$$I(S) \subset I(f) \subset J(f) \subset J(S).$$

The next part follows from the facts $J(S) \subset \overline{I(S)}$ and $I(S) \subset J(S)$. □

3. Escaping sets of Hyperbolic Semigroups

The definitions of critical values, asymptotic values and singular values as well as post singularities of transcendental entire functions can be generalized to arbitrary setting of transcendental semigroups.

Definition 3.1 (Critical point, critical value, asymptotic value and singular value). A point $z \in \mathbb{C}$ is called *critical point* of S if it is a critical point of some $g \in S$. A point $w \in \mathbb{C}$ is called a *critical value* of S if it is a critical value of some $g \in S$. A point $w \in \mathbb{C}$ is called an *asymptotic value* of S if it is an asymptotic value of some $g \in S$. A point $w \in \mathbb{C}$ is called a *singular value* of S if it is a singular value of some $g \in S$. For a semigroup S, if all $g \in S$ belongs to $\mathscr{S}$ or $\mathscr{B}$, we call S a semigroup of class $\mathscr{S}$ or $\mathscr{B}$ (finite or bounded type).

Definition 3.2 (**Post singularly bounded (or finite) semigroup**). A transcendental semigroup S is said to be *post-singularly bounded* (*post-singularly finite*) if each $g \in S$ is post-singularly bounded (or post-singularly finite). Post singular set of post singularly bounded semigroup S is the set

$$P(S) = \overline{\bigcup_{f \in S} f^n(SV(f))}$$

Definition 3.3 (**Hyperbolic semigroup**). An transcendental entire function f is said to be *hyperbolic* if the post-singular set $P(f)$ is a compact subset of $F(f)$. A transcendental semigroup S is said to be *hyperbolic* if each $g \in S$ is hyperbolic (that is, if $P(S)$ is a compact subset of $F(S)$).

Note that if transcendental semigroup S is hyperbolic, then each $f \in S$ is hyperbolic. However, the converse may not true. The fact $P(f^k) = P(f)$ for all $k \in \mathbb{N}$ shows that f^k is hyperbolic if f is hyperbolic. The following result has been shown by Dinesh Kumar and Sanjay Kumar [5, Theorem 3.16] where Eremenko's conjecture holds.

Proposition 3.1. *Let $f \in \mathscr{B}$ periodic with period p and hyperbolic. Let $g = f^n + p$, $n \in \mathbb{N}$. Then $S = \langle f, g \rangle$ is hyperbolic and all components of $I(S)$ are unbounded.*

Example 3.1. $f(z) = e^{\lambda z}$ is hyperbolic entire function for each $\lambda \in (0, \frac{1}{e})$. The semigroup $S = \langle f, g \rangle$ where $g = f^m + p$, and $p = \frac{2\pi i}{\lambda}$ is hyperbolic transcendental semigroup.

We generalize Proposition 3.1 to finitely generated hyperbolic semigroups by adding some extra conditions. This result will be the good source of non-empty escaping set transcendental semigroup where, the Eremenko's conjecture holds in semigroup setting.

Theorem 3.1. *Let $S = \langle f_1, f_2, \dots, f_n \rangle$ is an abelian bounded type transcendental semigroup in which each f_i is hyperbolic for $i = 1, 2, \dots, n$. Then semigroup S is hyperbolic and all components of $I(S)$ are unbounded.*

Lemma 3.1. *Let f and g be transcendental entire functions. Then $SV(f \circ g) \subset SV(f) \cup f(SV(g))$.*

Proof. See for instance [1, Lemma 2]. □

Lemma 3.2. *Let f and g are permutable transcendental entire functions. Then $f^m(SV(g)) \subset SV(g)$ and $g^m(SV(f)) \subset SV(f)$ for all $m \in \mathbb{N}$.*

Proof. We first prove that $f(SV(g)) \subset SV(g)$. Then we use induction to prove $f^m(SV(g)) \subset SV(g)$.

Let $w \in f(SV(g))$. Then $w = f(z)$ for some $z \in SV(g)$. In this case, z is either a critical value or an asymptotic value of function g.

First let us suppose that z is a critical value of g. Then $z = g(u)$ with $g^{'}(u) = 0$. Since f and g are permutable functions, so

$$w = f(z) = f(g(u)) = (f \circ g)(u) = (g \circ f)(u).$$

Also,

$$(f \circ g)^{'}(u) = f^{'}(g(u))g^{'}(u) = 0.$$

This shows that u is a critical point of $f \circ g = g \circ f$ and w is a critical value of $f \circ g = g \circ f$. By permutability of f and g, we can write

$$f^{'}(g(u))g^{'}(u) = g^{'}(f(u))f^{'}(u) = 0$$

for any critical point u of $f \circ g$. Since $g^{'}(u) = 0$, then either $f^{'}(u) = 0 \Rightarrow u$ is a critical point of f or $g^{'}(f(u)) = 0 \Rightarrow f(u)$ is a critical point of g. This shows that $w = g(f(u))$ is a critical value of g. Therefore, $w \in SV(g)$.

Next, suppose that z is an asymptotic value of function g. We have to prove that $w = f(z)$ is also asymptotic value of g. Then there exists a curve $\gamma : [0, \infty) \to \mathbb{C}$ such that $\gamma(t) \to \infty$ and $g(\gamma(t)) \to z$. So, $f(g(\gamma(t))) \to f(z) = w$ as $t \to \infty$ along γ. Since $f \circ g = g \circ f$, so

$$f(g(\gamma(t))) \to f(z) = w \Rightarrow g(f(\gamma(t))) \to f(z) = w$$

as $t \to \infty$ along γ. This shows w is an asymptotic value of g. This proves our assertion.

Assume that $f^k(SV(g)) \subset SV(g)$ for some $k \in \mathbb{N}$ with $k \leq m$. Then

$$f^{k+1}(SV(g)) = f(f^k(SV(g))) \subset f(SV(g)) \subset SV(g)$$

Therefore, by induction, for all $m \in \mathbb{N}$, we must have $f^m(SV(g)) \subset SV(g)$. The next part $g^m(SV(f)) \subset SV(f)$ can be proved similarly as above.

□

Lemma 3.3. *Let f and g are two permutable hyperbolic transcendental entire functions. Then $f \circ g$ is also hyperbolic.*

Proof. We have to prove that $P(f \circ g)$ is a compact subset of the Fatou set $F(f \circ g)$. From [7, Lemma 3.2], we can write $F(f \circ g) \subset F(f) \cap F(g)$. This shows that $F(f \circ g)$ is a subset of $F(f)$ and $F(g)$. So this Lemma will be proved if we prove $P(f \circ g)$ is a compact subset of $F(f) \cup F(g)$. By the definition of post singular set of transcendental entire function, we can write

$$\begin{aligned}
P(f \circ g) &= \overline{\bigcup_{m \geq 0} (f \circ g)^m(SV(f \circ g))} \\
&= \overline{\bigcup_{m \geq 0} f^m(g^m(SV(f \circ g)))} && \text{(by using permutabilty of } f \text{ and } g) \\
&\subset \overline{\bigcup_{m \geq 0} f^m(g^m(SV(f) \cup f(SV(g)))} && \text{(by Lemma 3.1)} \\
&= \overline{\bigcup_{m \geq 0} f^m(g^m(SV(f))) \cup g^m(f^{m+1}(SV(g)))} \\
&\subset \overline{\bigcup_{m \geq 0} f^m(SV(f)))} \cup \overline{\bigcup_{m \geq 0} g^m(SV(g)))} && \text{(by Lemma 3.2)} \\
&= P(f) \cup P(g)
\end{aligned}$$

Since f and g are hyperbolic, so $P(f)$ and $P(g)$ are compact subset of $F(f)$ and $F(g)$. Therefore, the set $P(f) \cup P(g)$ must be compact subset of $F(f) \cup F(g)$. □

Proof of the Theorem 3.1. Any $f \in S$ can be written as

$$f = f_{i_1} \circ f_{i_2} \circ f_{i_3} \circ \cdots \circ f_{i_m}.$$

By permutability of each f_i, we can rearrange f_{i_j} and ultimately represented by

$$f = f_1^{t_1} \circ f_2^{t_2} \circ \ldots \circ f_n^{t_n}$$

where each $t_k \geq 0$ is an integer for $k = 1, 2, \ldots, n$. Lemma 3.3 can be applied repeatably to show each of $f_1^{t_1}, f_2^{t_2}, \ldots, f_n^{t_n}$ is hyperbolic. Again by repeated application of same lemma, we can say that

$$f = f_1^{t_1} \circ f_2^{t_2} \circ \ldots \circ f_n^{t_n}$$

is itself hyperbolic and so the semigroup S is hyperbolic. Next part follows from [12, Theorem 3.3] by the assumption of this theorem. □

Acknowledgment: We express our heart full thanks to Prof. Shunshuke Morosawa, Kochi University, Japan for his thorough reading of this paper with valuable suggestions and comments.

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