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TRANSFORMING ANALYTICALLY INTRACTABLE DYNAMICAL SYSTEMS WITH A CONTROL PARAMETER INTO A TRACTABLE GINZBURG-LANDAU EQUATION: FEW ILLUSTRATIONS

C. KANCHANA [1] AND P. G. SIDDHESHWAR [2]

[1] *College of Science, Harbin Institute of Technology, Shenzhen Graduate School, Shenzhen, Nanshan-518055, China*

[2] *Department of Mathematics, Bangalore University, Jnanabharathi Campus, Bangalore 560056, India*

Abstract: In the paper a means of making a simplified study of dynamical systems with a control parameter is presented. The intractable, third-order classical Lorenz system, the Lorenz-like Chen system and two topologically dissimilar fifth-order Lorenz systems are considered for illustration. Using the multi-scale method, these systems are reduced to an analytically tractable first-order Ginzburg-Landau equation(GLE) in one of the amplitudes. The analytical solution of the GLE is used to find the remaining amplitudes.

Key Words: Dynamical systems, Lorenz, Chen, multiscale method, Ginzburg-Landau.
AMS (MOS) Subject Classification. 34Nxx.

1. Introduction

Chaos appears in the solution of nonlinear dynamical systems that are sensitive to initial conditions. Poincare's [1] fundamental work on chaos theory paved the way for Lorenz [2] to study the chaotic attractor in a system of three ordinary, non-linear, autonomous differential equations with a control paramter. The intractable system which Lorenz studied numerically is now well known as the Lorenz model [2]. Later Rössler [3] came up with another such third-order system (simpler than Lorenz model) while modeling the equilibrium in chemical reactions. More than two decades later Chen [4] proposed a Lorenz-like system which is now used for the problem of masking modulation of sinusoidal data and also for security improvement[5]. Siddheshwar et al. [6] derived a fifth-order Lorenz model in their study of Rayleigh-Bénard convection(RBC) in nanoliquids using the two-phase model. Siddheshwar and Titus [7] used additional modes in the study of RBC and arrived at a fifth-order Lorenz model which is a generalization of the classical Lorenz model but is topologically different from the one derived by Siddheshwar et al.[6].

A result on autonomous ODEs indicates that higher-order non-linear systems can be transformed to equations of lower order. As a consequence of this, successful attempts were made to reduce the classical Lorenz system of third-order into GLE by the following methods:

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i) Reduction of Lorenz equation in three amplitudes into a single equation in one amplitude by eliminating the other two amplitudes [7].
ii) Method of multiscales [8].
iii) Differential geometry method based on central manifold theorem [9] and
iv) Renormalization method [10].

This paper presents the multiscale method to reduce the following four analytically intractable dynamical systems with a control parameter

(1) Third-order Lorenz system,
(2) Chen system and
(3) Two topologically different penta-modal Lorenz systems.

to a GLE in one of the amplitudes.

2. Reduction of dynamical systems

2.1. **Third-order(classical) Lorenz system (ρ : control paramter).** Consider the Lorenz system [2] in the well-known form:

$$\left.\begin{aligned}\frac{dX}{d\tau} &= \sigma(Y - X)\\ \frac{dY}{d\tau} &= \rho X - Y - XZ\\ \frac{dZ}{d\tau} &= -\beta Z + XY\end{aligned}\right\}, \tag{2.1}$$

where X, Y and Z are amplitudes, τ is time, σ, β are real numbers and ρ is the control parameter.

To obtain an analytical solution we reduce the order of the Lorenz system by using the multiscale method and the same is discussed below.

We use the following regular perturbation expansion in the Lorenz system (2.1):

$$\left.\begin{aligned}X &= \epsilon X_1 + \epsilon^2 X_2 + \epsilon^3 X_3 + \cdots\\ Y &= \epsilon Y_1 + \epsilon^2 Y_2 + \epsilon^3 Y_3 + \cdots\\ Z &= \epsilon W_1 + \epsilon^2 W_2 + \epsilon^3 W_3 + \cdots\\ \rho &= \rho_0 + \epsilon \rho_1 + \epsilon^2 \rho_2 + \cdots\end{aligned}\right\}, \tag{2.2}$$

where ϵ is a small amplitude.

For our convenience we define the operators :

$$L = \begin{bmatrix} -\sigma & -\sigma & 0\\ \rho_0 & -1 & 0\\ 0 & 0 & -\beta \end{bmatrix} \text{ and } M_i = [X_i, Y_i, Z_i]^{Tr},\ i = 1, 2, 3, \tag{2.3}$$

where Tr denotes transpose of a matrix.

Substituting Eq. (2.2) in the Lorenz system (2.1) and using the time variation only at the slow time scale which is taken to be $\tau_1 = \epsilon^2 \tau$ and on comparing the like powers of ϵ on either side of the resulting equations, we get the following system of equations at various orders :

First-order system:

$$LM_1 = 0, \tag{2.4}$$

Second-order system:

$$LM_2 = [R_{21}, R_{22}, R_{23}]^{Tr}, \tag{2.5}$$

Third-order system:

$$LM_3 = [R_{31}, R_{32}, R_{33}]^{Tr}, \tag{2.6}$$

where

$$\left. R_{21} = 0, R_{22} = -\rho_1 X_1 + X_1 Z_1, R_{23} = -X_1 Y_1, \right\}, \tag{2.7}$$

$$\left. \begin{aligned} R_{31} = \frac{dX_1}{d\tau_1}, R_{32} = \frac{dY_1}{d\tau_1} - \rho_2 X_1 + X_1 Z_2 + X_2 Z_1, \\ R_{33} = \frac{dZ_1}{d\tau_1} - X_1 Y_2 - X_2 Y_1, \end{aligned} \right\}. \tag{2.8}$$

The solution of the first- and second-order systems is given by :

$$M_1 = [X_1, \ \ \rho_0 X_1, \ \ 0]^{Tr}, \tag{2.9}$$

$$M_2 = [0, \ \ 0, \ \ \frac{\rho_0}{\beta} X_1^2]^{Tr}. \tag{2.10}$$

At the third-order, we consider the Fredholm solvability condition to get the condition for existence of its solution as :

$$\sum_{j=1}^{3} R_{ij} \hat{M}_1 = 0, \qquad (i = 2, 3), \tag{2.11}$$

where $\hat{M}_1$ represents the solution of the adjoint system of Eq. (2.4).

To find the value of ρ_0 we take determinant(L)=0 and this yields $\rho_0 = 1$ and on substituting $i = 2$ in Eq. (2.11) and using Eqs. (2.7) and (2.9) in the resulting equation, we get $\rho_1 = 0$.

Substituting $i = 3$ in Eq. (2.11) and on using Eqs. (2.8) and (2.9) in the resulting equation, we get the Ginzburg-Landau equation in the form:

$$\frac{dX_1}{d\tau_1} = Q_1 X_1 - Q_2 X_1^3, \tag{2.12}$$

where

$$Q_1 = \frac{\rho_2 \sigma}{1 + \sigma}, \ \ Q_2 = \frac{\sigma}{\beta(1 + \sigma)}, \ \ \rho_2 = \frac{\rho - 1}{\epsilon^2}. \tag{2.13}$$

Solving Eq. (2.12) subject to initial condition $X_1(0) = 1$, we get

$$X_1(\tau_1) = \frac{1}{\sqrt{\frac{Q_2}{Q_1} + \left(1 - \frac{Q_2}{Q_1}\right) e^{-2Q_1 \tau_1}}}. \tag{2.14}$$

Using Eqs. (2.9), (2.10) and (2.14) in Eq. (2.2) we get the solution for X, Y and Z as

$$X = \epsilon X_1, \quad Y = \epsilon X_1, \quad Z = \epsilon^2 \frac{1}{\beta} X_1^2. \tag{2.15}$$

Having found the analytical solution of the classical Lorenz system, in the next section we show that the solution of a Lorenz-like system (Chen [4] system) can be obtained from the Lorenz system by redefining parameters.

2.2. **Chen system (c : control parameter).** Consider the Chen system [4] in the standard form:

$$\left.\begin{aligned} \frac{dx}{dt} &= a(y - x) \\ \frac{dy}{dt} &= (c - a)x - xz + cy \\ \frac{dz}{dt} &= xy - bz \end{aligned}\right\}, \tag{2.16}$$

where x, y and z are amplitudes, t is time, a, b are real numbers and c is the control parameter.

Applying the following scaling to the Chen system (2.16) :

$$x = -cX, \quad y = -cY, \quad z = -cZ, \quad \tau = -ct, \tag{2.17}$$

we get :

$$\left.\begin{aligned} \frac{dX}{d\tau} &= \frac{-a}{c}(Y - X) \\ \frac{dY}{d\tau} &= \frac{-(c-a)}{c}X - Y - XZ \\ \frac{dZ}{d\tau} &= \frac{b}{c}Z + XY \end{aligned}\right\}. \tag{2.18}$$

On comparing the system (2.18) with the Lorenz system (2.1), we get :

$$\sigma = \frac{-a}{c}, \quad \beta = \frac{-b}{c}, \quad \rho = -\left(1 - \frac{a}{c}\right). \tag{2.19}$$

Thus, the solution of the Chen system (2.16) can be obtained from those of the Lorenz system (2.1) by using the relation (2.19). The GLE corresponding to the Chen [4] system is :

$$\frac{dx_1}{dt} = Q_1' x_1 - Q_2' x_1^3, \tag{2.20}$$

where

$$Q_1' = \frac{c_2 a}{c_0 - a}, \quad Q_2' = \frac{a}{b(a - c_0)}. \tag{2.21}$$

Using multiscale expansion (2.2), we get solution the for x, y and z as

$$x = \epsilon x_1, \quad y = \epsilon x_1, \quad z = \epsilon^2 \frac{1}{b} x_1^2. \tag{2.22}$$

where x_1 is given by

$$x_1(\tau) = \frac{1}{\sqrt{\dfrac{Q_2'}{Q_1'} + \left(1 - \dfrac{Q_2'}{Q_1'}\right) e^{-2Q_1'\tau}}}. \tag{2.23}$$

We next consider a fifth-order Lorenz model for illustration.

2.3. **Penta-modal Lorenz systems (ρ : control parameter). Type I :** Consider five-mode Lorenz model derived by Siddheshwar et al. [6] :

$$\frac{dX}{d\tau} = \sigma(Y - X - rS), \tag{2.24}$$

$$\frac{dY}{d\tau} = \rho X - Y - XZ, \tag{2.25}$$

$$\frac{dZ}{d\tau} = -\beta Z + XY, \tag{2.26}$$

$$\frac{dS}{d\tau} = X - \frac{\epsilon^2}{\beta}\frac{N_A}{Le}Y - \frac{1}{Le}S + XP, \tag{2.27}$$

$$\frac{dP}{d\tau} = \frac{\epsilon^2}{\beta}\frac{N_A}{\rho Le}Z - \frac{\beta}{Le}P - XS, \tag{2.28}$$

where N_A, Le and r are real numbers, ρ is the control parameter.

The individual equations in the system (2.24)-(2.28) are topologically dissimilar to each other and a procedure similar to what was adopted in third-order systems earlier may be used. We apply the multiscale method in the Lorenz system (2.24)-(2.28) as follows:

$$\begin{bmatrix} X \\ Y \\ Z \\ S \\ P \\ \rho \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \rho_0 \end{bmatrix} + \epsilon \begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \\ S_1 \\ P_1 \\ \rho_1 \end{bmatrix} + \epsilon^2 \begin{bmatrix} X_2 \\ Y_2 \\ Z_2 \\ S_2 \\ P_2 \\ \rho_2 \end{bmatrix} + \epsilon^3 \begin{bmatrix} X_3 \\ Y_3 \\ Z_3 \\ S_3 \\ P_3 \\ 0 \end{bmatrix} + \cdots \tag{2.29}$$

Let us define operators

$$L = \begin{bmatrix} -1 & 1 & 0 & -r & 0 \\ \rho_0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -\beta 1 & 0 & 0 \\ \rho_0 & 0 & 0 & -\dfrac{\rho_0}{Le} & 0 \\ 0 & 0 & 0 & 0 & -\dfrac{\beta\rho_0}{Le} \end{bmatrix} \text{ and } M_i = \begin{bmatrix} X_i \\ Y_i \\ Z_i \\ S_i \\ P_i \end{bmatrix}, i = 1(1)5. \tag{2.30}$$

Substituting Eq. (2.29) in the fifth-order Lorenz system (2.24)-(2.28), using the time variation only at the slow time scale which is taken to be $\tau_1 = \epsilon^2\tau$ and on comparing the like powers of ϵ on either side of the resulting equations, we get the following equations at various orders :

First-order system:

$$LM_1 = 0, \tag{2.31}$$

Second-order system:

$$LM_2 = [R_{21}, R_{22}, R_{23}, R_{24}, R_{25}]^{Tr}, \tag{2.32}$$

Third-order system:

$$LM_3 = [R_{31}, R_{32}, R_{33}, R_{34}, R_{35}]^{Tr}, \tag{2.33}$$

where

$$R_{21} = 0, R_{22} = X_1 Z_1, R_{23} = -X_1 Y_1, R_{24} = -X_1 P_1, R_{25} = -X_1 S_1, \tag{2.34}$$

$$\left.\begin{aligned} R_{31} &= \frac{1}{\sigma}\frac{dX_1}{d\tau_1}, \\ R_{32} &= \frac{dY_1}{d\tau_1} - \rho_2 X_1 + X_1 Z_2, \\ R_{33} &= -(X_1 Y_2 + X_2 Y_1) + \frac{dZ_1}{d\tau_1}, \\ R_{34} &= -(X_1 P_2 + X_2 P_1) + \frac{dS_1}{d\tau - 1} - \rho_2 X_1 + \frac{N_A}{Le} Y_1 + \frac{1}{Le} S_1, \\ R_{35} &= (X_1 S_2 + X_2 S_1) + \frac{dP_1}{d\tau_1} + \frac{\beta\rho_2}{Le} P_1. \end{aligned}\right\} . \tag{2.35}$$

The solution of the first- and second-order systems is given by

$$M_1 = [X_1,\ X_1,\ 0,\ LeX_1,\ 0]^{Tr}, \tag{2.36}$$

$$M_2 = [0,\ 0,\ \frac{1}{\beta}X_1,\ 0,\ \frac{-Le^2}{\beta}{X_1}^2]^{Tr}. \tag{2.37}$$

Consider the Fredholm solvability condition

$$\sum_{j=1}^{5} R_{ij}\hat{M}_1 = 0, \qquad (i = 2, 3), \tag{2.38}$$

where $\hat{M}_1$ represents the solution of the adjoint system of Eq. (2.36).

To find the value of ρ_0 we take determinant(L)=0 and this yields $\rho_0 = 1$ and substituting $i = 2$ in Eq. (2.38) and using Eqs. (2.34) and (2.36) in the resulting equation, we get $\rho_1 = 0$. Substituting $i = 3$ in Eq. (2.38) and using Eqs. (2.35) and (2.36) in the resulting equation, we get the Ginzburg-Landau equation in the form:

$$\frac{dX_1}{d\tau_1} = Q_1 X_1 - Q_2 X_1^3, \tag{2.39}$$

where

$$\left. Q_1 = \frac{\sigma(\rho_2 + N_A r)}{1 + \sigma(1 - Le^2 r)}, Q_2 = \frac{\sigma(1 - Le^3 r)}{\beta\left[1 + \sigma(1 - Le^2 r)\right]}, \right\} . \tag{2.40}$$

The solution of Eq. (2.39) subject to $X_1(0) = 1$ is given by Eq. (2.14).

We next consider a system which cannot be reduced to a single, real GLE. It gets reduced to a coupled system of GLEs.

Type II: Consider the five-mode Lorenz model derived by Siddheshwar and Titus [7] :

$$\frac{dX}{d\tau} = \sigma(Y - X), \tag{2.41}$$

$$\frac{dX'}{d\tau} = \sigma(Y' - X'), \tag{2.42}$$

$$\frac{dY}{d\tau} = \rho X - Y - XZ, \tag{2.43}$$

$$\frac{dY'}{d\tau} = \rho X' - Y' - X'Z, \tag{2.44}$$

$$\frac{dZ}{d\tau} = -\beta Z + XY + X'Y', \tag{2.45}$$

where ρ is the control parameter.

On observing the Eqs. (2.41)-(2.45), we note that Eqs. (2.41) and (2.42) are topologically similar and so are Eqs. (2.43) and (2.44). Equation (2.45) is a stand-alone type. In view of the above observation, we consider bunching of the equations into two sets :

i) Equations (2.41), (2.43) and (2.45), and
ii) Equations (2.42), (2.44) and (2.45).

We apply the multiscale method on the two bunches separately. The failure of the multiscale procedure when the entire lot of equations (2.41)-(2.45) was considered together lead us to this arrangement of bunching of equations.

Now considering Eqs. (2.41), (2.43) and (2.45) and applying multiscale method (2.2) and using the slow time scale, $\tau_1 = \epsilon^2\tau$, we get a system of equations at various orders. On comparing the like powers of ϵ on either side of the resulting equations, we get the following equations at various orders :

First-order system:

$$LM_1 = 0, \tag{2.46}$$

Second-order system:

$$LM_2 = [R_{21}, R_{22}, R_{23}]^{Tr}, \tag{2.47}$$

Third-order system:

$$LM_3 = [R_{31}, R_{32}, R_{33}]^{Tr}, \tag{2.48}$$

where

$$\left. R_{21} = 0, R_{22} = -\rho_1 X_1 + X_1 Y_1, R_{23} = -X_1 Y_1 - X_1' Y_1', \right\}, \tag{2.49}$$

$$\left. \begin{aligned} R_{31} = \frac{dX_1}{d\tau}, R_{32} = \frac{dY_1}{d\tau} - \rho_2 X_1 + X_1 Z_2 + X_2 Z_1, \\ R_{33} = \frac{dZ_1}{d\tau} - X_1 Y_2 - X_2 Y_1 - X_1' Y_2' - X_2' Y_1', \end{aligned} \right\}. \tag{2.50}$$

The solution of the first- and second-order systems subject to appropriate initial condition is given by :

$$M_1 = [X_1,\ \ \rho_0 X_1,\ \ 0]^{Tr}, \tag{2.51}$$

$$M_2 = \left[0,\ \ 0,\ \ \frac{\rho_0}{\beta} X_1^2 + \frac{1}{\beta} X_1' Y_1'\right]^{Tr}, \tag{2.52}$$

At the third-order, using the Fredholm solvability condition defined by Eq. (2.11), we get the Ginzburg-Landau equation in the form:

$$\frac{dX_1}{d\tau_1} = Q_1 X_1 - Q_2 ({X_1}^3 + X_1 {X_1'}^2). \tag{2.53}$$

where Q_1 and Q_2 are given in Eq. (2.13).

Next considering Eqs. (2.42), (2.44) and (2.45) and following a similar procedure to that considered earlier we get yet another Ginzburg-Landau equation in the form :

$$\frac{dX_1'}{d\tau_1} = Q_1 X_1' - Q_2 ({X_1'}^3 + X_1' {X_1}^2), \tag{2.54}$$

Equations (2.53) and (2.54) are a coupled system of real, Ginzburg-Landau equations which can be combined into the following:

$$\frac{d\widetilde{X_1}}{d\tau_1} = Q_1 \widetilde{X_1} - Q_2 \widetilde{X_1} \left|\widetilde{X_1}\right|^2, \tag{2.55}$$

by defining $\widetilde{X_1} = X_1 + iX_1'$, where $i = \sqrt{-1}$.

The phase-amplitude form of $\widetilde{X_1}$ is

$$\widetilde{X_1} = \left|\widetilde{X_1}\right| e^{i\Phi}. \tag{2.56}$$

Substituting Eq. (2.56) in Eq. (2.55), we get the following GLE in $|\widetilde{X_1}|$:

$$\frac{d\left|\widetilde{X_1}\right|}{d\tau_1} = Q_1 \left|\widetilde{X_1}\right| - Q_2 \left|\widetilde{X_1}\right|^3. \tag{2.57}$$

Equation (2.57) is essentially the GLE (2.12) for $\left|\widetilde{X_1}\right|$.

In the succeeding section we discuss the results obtained in the paper.

3. Results and Discussion

The four dynamical systems considered are nonlinear and analytically intractable. They possess a control parameter which regulates the dynamics in the system. Instability appears in such dynamical systems when the control parameter exceeds a critical value. Phase-space plots and phase-plane projections of them are normally obtained by using a numerical solution of the dynamical system based on the Runge-Kutta family of methods or predictor-corrector combinations. In the paper, we exploit the presence of a control parameter in the dynamical system and use it to define a slow time-scale and this in turn helps in obtaining the analytical solution for the dynamical system. The solution is valid for states of the system corresponding to values of the control parameter in the neighbourhood of its critical value. The multiscale method is used in the paper to reduce a higher-order dynamical system to a lower-order one and the latter is invariably a GLE for all the five illustrative dynamical systems considered. The procedure using the multiscale method in transforming an intractable dynamical system to a tractable one can be adopted with ease by similar means to many other dynamical systems [11],[12]. [13],[14], [15], [16],[17], [18], [19]). The most important aspect to be noted in the study is the variation in the method for topologically similar and dissimilar equations(see the illustrative penta-modal Lorenz models).

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