A PRIORI ESTIMATES IN TERMS OF THE MAXIMUM NORM FOR THE SOLUTION OF THE NAVIER-STOKES EQUATIONS WITH PERIODIC INITIAL DATA

SANTOSH PATHAK

Department of Mathematics and Statistics
University of New Mexico, Albuquerque, NM 87131, USA
spathak@unm.edu

Abstract: In this paper, we consider the Cauchy problem for the incompressible Navier-Stokes equations in $\mathbb{R}^n$ for $n \geq 3$ with smooth periodic initial data and derive a priori estimtes of the maximum norm of all derivatives of the solution in terms of the maximum norm of the initial data. This paper is a special case of a paper by H-O Kreiss and J. Lorenz which also generalizes the main result of their paper to higher dimension.

Key Words: Incompressible Navier-Stokes equation; Maximum norm estimates, Periodic initial data

AMS (MOS) Subject Classification. 35G25, 35Q30, 76D03, 76D05

1. INTRODUCTION

We consider the Cauchy problem of the Navier-Stokes equations in $\mathbb{R}^n, n \geq 3$:

$$(1.1) \qquad u_t + u \cdot \nabla u + \nabla p = \triangle u, \qquad \nabla \cdot u = 0,$$

with initial condition

$$(1.2) \qquad u(x,0) = f(x), \qquad x \in \mathbb{R}^n,$$

where $u = u(x,t) = (u_1(x,t), \cdots u_n(x,t))$ and $p = p(x,t)$ stand for the unknown velocity vector field of the fluid and its pressure, while $f = f(x) = (f_1(x), \cdots f_n(x))$ is the given initial velocity vector field. In what follows, we will use the same notations for the space of vector valued and scalar functions for convenience in writing.

There is a large literature on the existence and uniqueness of solution of the Navier-Stokes equations in $\mathbb{R}^n$. For given initial data, solutions of (1.1) and (1.2) have been constructed in various function spaces. For example, if $f \in L^r$ for some r with $3 \leq r < \infty$, then it is well known that there is a unique classical solution in some maximum interval of time $0 \leq t < T_f$ where $0 < T_f \leq \infty$. But for the uniqueness of the pressure one requires $|p(x,t)| \to 0$ as $|x| \to \infty$. (See [6] and [11] for $r = 3$ and [1] for $3 < r < \infty$.)

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If $f \in L^{\infty}(\mathbb{R}^n)$ then existence of a regular solution follows from [2]. The solution is only unique if one puts some growth restrictions on the pressure as $|x| \to \infty$. A simple example of non-uniqueness is demonstrated in [7] where the velocity u is bounded but $|p(x,t)| \leq C|x|$. In addition, an estimate $|p(x,t)| \leq C(1 + |x|^{\sigma})$ with $\sigma < 1$ (see [3]) implies uniqueness. Also the assumption $p \in L^1_{loc}(0,T;BMO)$ (see [5]) implies uniqueness.

In this paper we consider the initial function $f \in C^{\infty}_{per}(\mathbb{R}^n)$ which is the space of smooth 2π periodic functions. Since $C^{\infty}_{per}(\mathbb{R}^n)$ is a closed subspace of the Banach space $L^{\infty}(\mathbb{R}^n)$, the existence of a regular solution of the Navier-Stokes equations (1.1) and (1.2) can be guaranteed by [2]. In addition, in a paper by Giga and others [5] they consider $f \in BUC(\mathbb{R}^n)$ where $BUC(\mathbb{R}^n)$ is the space of bounded uniformly continuous functions. In the paper they construct a regular solution of the Navier-Stokes equations in some maximum interval of time $0 \leq t < T_f$ where $T_f \leq \infty$. Clearly, our case is also a special case of their paper where we put extra assumption of "smooth periodic" on their initial function $f \in BUC(\mathbb{R}^n)$. Moreover, for smooth periodic initial data, the existence of smooth periodic solution is proved by H-O Kreiss and J. Lorenz in their book [9] for $n = 3$ where they use successive iteration using the the vorticity formulation. On the other hand, Giga and others use iteration on the integral equation of the transformed abstract ordinary differential equations to construct a mild solution of the Navier-Stokes equations and later prove such mild solution is indeed a regular solution (local in time) of the Navier-Stokes equations (1.1) and (1.2) for initial function in the space of bounded uniformly continous functions (BUC). That is $f \in BUC(\mathbb{R}^n)$. Readers are referred to the paper by Giga and others [5] for details on existence of the smooth periodic solution of the Navier-Stokes equations of (1.1) and (1.2) for $f \in C^{\infty}_{per}(\mathbb{R}^n)$ with necessary alternations in their proofs of case $f \in BUC(\mathbb{R}^n)$.

The work in this paper reproves Theorem 4.1 of the Kreiss and Lorenz paper [8] in periodic case assuming smooth periodic solution exists for some maximum interval of time $0 \leq t < T_f$. Since we are in a special case of their paper, result of Theorem 4.1 must be true for smooth periodic initial data as well, but what makes our work interesting and different is the approach taken to handle the pressure term of the Navier-Stokes equations while deriving the result of Theorem 4.1 of the Kreiss and Lorenz paper as I have adopted in my first paper [10] Notice, pressure term of the Navier-Stokes equations can be determined from the Poisson equation

$$(1.3) \qquad\qquad \triangle p = -\nabla \cdot (u \cdot \nabla)u$$

which is given by

$$(1.4) \qquad\qquad p = \sum_{i,j} R_i R_j(u_i u_j),$$

where $R_i = (-\triangle)^{-1/2} D_i$ is the i-th Riesz transform. Since the Riesz transforms are not bounded in $L^{\infty}(\mathbb{R}^n)$, the pressure term $p \in L^1_{loc}(0,T;BMO)$ where BMO is the space of functions of bounded mean oscillation. Because of the non-local nature of the pressure, the proof of Theorem 4.1 of the Kreiss and Lorenz paper is complicated, however.

The main objective of this paper is to derive a priori estimates of the maximum norm of the derivatives of u in terms of the maximum norm of the initial function, $u(x, 0) = f(x)$, *assuming* the solution to exist and to be $C_{per}^\infty(\mathbb{R}^n), n \geq 3$ for $0 \leq t < T_f$. Before we start formulating the problem, we introduce the following notations

$$|f|_\infty = \sup_x |f(x)| \quad \text{with} \quad |f(x)|^2 = \sum_i f_i^2(x),$$

and $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}, D_i = \partial/\partial x_i$ for a multiindex $\alpha = (\alpha_1, \cdots, \alpha_n)$. In what follows, for any $j = 0, 1, \cdots$, if $|\alpha| = j$ then we will denote D^α by D^j. We also set

$$|\mathcal{D}^j u(t)|_\infty := |\mathcal{D}^j u(\cdot, t)|_\infty = \max_{|\alpha|=j} |D^\alpha u(\cdot, t)|_\infty.$$

Clearly, $|\mathcal{D}^j u(t)|_\infty$ measures all space derivatives of order j in maximum norm.

Proving the following theorem is the main goal of this paper whereas Kreiss and Lorenz in their paper [8] prove the same theorem for $f \in L^\infty(\mathbb{R}^n)$ for $n = 3$ from rather difficult approach while dealing with the pressure term $p(x, t)$.

Theorem 1.1. *Consider the Cauchy problem for the Navier-Stokes equations (1.1), (1.2), where $f \in C_{per}^\infty(\mathbb{R}^n)$ for $n \geq 3$ with $\nabla \cdot f = 0$. There is a constant $c_0 > 0$ and for every $j = 0, 1, \cdots$ there is a constant K_j so that*

$$(1.5) \qquad t^{j/2}|\mathcal{D}^j u(t)|_\infty \leq K_j |f|_\infty \quad \text{for} \quad 0 < t \leq \frac{c_0}{|f|_\infty^2}.$$

The constants c_0 and K_j are independent of t and f.

For the purpose of proving Theorem 1.1, we start by transforming the momentum equation (1.1) of the Navier-Stokes equations into the abstract ordinary differential equation for u

$$(1.6) \qquad u_t = \triangle u - \mathbb{P}(u \cdot \nabla)u$$

by eliminating the pressure, where $\mathbb{P}$ is the Leray projector defined by

$$\mathbb{P} = (\mathbb{P}_{ij})_{1 \leq i,j \leq n}, \quad \mathbb{P}_{ij} = \delta_{ij} + R_i R_j;$$

where R_i is same as in (1.4) and δ_{ij} is the Kronecker delta function. Note that the equation (1.6) is obtained from (1.1) by applying the Leray projector with the properties $\mathbb{P}(\nabla p) = 0, \mathbb{P}(\triangle u) = \triangle u$, since $\nabla \cdot u = 0$.

Since $\mathbb{P}(u \cdot \nabla u) = \sum_i D_i \mathbb{P}(u_i u)$, therefore it is very appropriate to consider an analogous system of (1.6) as below:

$$(1.7) \qquad u_t = \triangle u + D_i \mathbb{P}g(u) \quad x \in \mathbb{R}^n, \quad t > 0$$

with initial condition

$$(1.8) \qquad u(x, 0) = f(x) \quad \text{where} \quad f \in C_{per}^\infty(\mathbb{R}^n).$$

Here $g : \mathbb{R}^n \to \mathbb{R}^n$ is assumed to be quadratic in u. The maximal interval of existence is again $0 \le t < T_f$. We would like to prove the estimates of the maximum norm of the derivatives of the solution of (1.7) and (1.8) in terms of the maximum norm of the initial data.

Theorem 1.2. *Under the above assumptions on f and g the solution of (1.7) and (1.8) satisfies the following*
(a) There is a constant $c_0 > 0$ with

$$(1.9) \qquad\qquad T(f) > \frac{c_0}{|f|_\infty^2}$$

and

$$(1.10) \qquad\qquad |u(t)|_\infty \le 2|f|_\infty \quad for \quad 0 \le t \le \frac{c_0}{|f|_\infty^2}.$$

(b) For every $j = 1, 2, \cdots$, there is a constant $K_j > 0$ with

$$(1.11) \qquad\qquad t^{j/2}|\mathcal{D}^j u(.,t)|_\infty \le K_j|f|_\infty \quad for \quad 0 < t \le \frac{c_0}{|f|_\infty^2}.$$

The constant c_0 and K_j are independent of t and f.

In section 2, we will introduce some auxiliary results for the solution of the heat equation and few other important estimates which are used later in section 3 and 4. Proof of Theorem 1.2 will be provided in section 3. Then we prove Theorem 1.1 in section 4. Finally, we outline some remarks on the use of the result obtained in Theorem 1.1.

2. Some Auxiliary results

Let us consider $f \in C_{per}^\infty(\mathbb{R}^n)$. The solution of

$$u_t = \triangle u, \quad u = f \quad \text{at} \quad t = 0,$$

is denoted by

$$u(t) := u(\cdot, t) = e^{\triangle t}f = \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} \theta(x - y, t)f(y)dy$$

where

$$(2.1) \qquad\qquad \theta(x, t) = \sum_{k \in \mathbb{Z}^n} e^{-|k|^2 t}e^{ik \cdot x}, \quad t > 0$$

is the periodic heat kernel in $\mathbb{R}^n$. Using the Poisson summation formula, (2.1) can be written as

$$(2.2) \qquad\qquad \theta(x, t) = \sum_{k \in \mathbb{Z}^n} \left(\frac{\pi}{t}\right)^n \exp\left[\frac{-|x + 2\pi k|^2}{4t}\right], \quad t > 0.$$

With the use of (2.2), it is well known that

$$(2.3) \qquad\qquad |e^{t\triangle}f|_\infty \le |f|_\infty, \quad t \ge 0$$

and

(2.4) $$|\mathcal{D}^j e^{t\triangle} f|_\infty \le C_j t^{-j/2} |f|_\infty$$

for some $C_j > 0$ independent of t and f.

Lemma 2.1. *Let* $f \in C^\infty_{per}(\mathbb{R}^n)$ *then for any* $j \ge 1$

(2.5) $$|\mathcal{D}^j e^{\triangle t} \mathbb{P} f|_\infty \le C_j t^{-j/2} |f|_\infty \quad for \quad t > 0$$

for some constant $C_j > 0$ *independent of* t *and* f.

Proof. Let us first denote $e^{\triangle t} f = \theta * f$ where $\theta(x, t)$ is given by (2.2). Let us denote the Fourier coefficient of a function by $\mathcal{F}$. For $\xi \in \mathbb{Z}^n$, notice $\mathcal{F}(\theta(x, t))(\xi) = C e^{-t|\xi|^2}, t > 0$, where C depends on the normalizing constant in the definition of the Fourier coefficient. In the proof of this lemma, we will allow the constant C to change line to line as per the need. Now, for any $t > 0$, any choice of $k, l \in \{1, 2, \cdots, n\}$; and for any multiindex α such that $|\alpha| = j$, the operator $D^j e^{\triangle t} \mathbb{P}_{kl}$ on the Fourier side is given by

$$\mathcal{F}(D^j e^{\triangle t} \mathbb{P}_{kl} f_l)(\xi) = (-i\xi)^\alpha \mathcal{F}(e^{\triangle t} \mathbb{P}_{kl} f_l)(\xi)$$

$$= (-i\xi)^\alpha \mathcal{F}(\theta * \mathbb{P}_{kl} f_l)(\xi)$$

$$= C(-i\xi)^\alpha \mathcal{F}(\theta(x, t))(\xi) \mathcal{F}(\mathbb{P}_{kl} f_l)(\xi)$$

$$= C(-i\xi)^\alpha e^{-t|\xi|^2} \left(\delta_{kl} - \frac{\xi_k \xi_l}{|\xi|^2} \right) \mathcal{F}(f_l)(\xi)$$

$$= C(-i\xi)^\alpha e^{-t|\xi|^2} \delta_{kl} \mathcal{F}(f_l)(\xi)$$

$$- C(-i\xi)^\alpha \xi_k \xi_l \mathcal{F}(f_l)(\xi) \int_t^\infty e^{-\tau|\xi|^2} d\tau.$$

Using Fourier expansion we can write

$$D^j (e^{\triangle t} \mathbb{P}_{kl} f_l)(x) = C \sum_{\xi \in \mathbb{Z}^n} (-i\xi)^\alpha \delta_{kl} e^{-t|\xi|^2} \mathcal{F}(f_l)(\xi) e^{i\xi \cdot x}$$

$$+ C \sum_{\xi \in \mathbb{Z}^n} (-i\xi)^\alpha (i\xi_k)(i\xi_l) \mathcal{F}(f_l)(\xi) e^{i\xi \cdot x} \int_t^\infty e^{-\tau|\xi|^2} d\tau$$

$$= (-1)^j C \delta_{kl} D^\alpha \sum_{\xi \in \mathbb{Z}^n} e^{-t|\xi|^2} \mathcal{F}(f_l)(\xi) e^{i\xi \cdot x}$$

$$+ C(-1)^j \int_t^\infty \sum_{\xi \in \mathbb{Z}^n} e^{-\tau|\xi|^2} (i\xi)^\alpha (i\xi_k)(i\xi_l) \mathcal{F}(f_l)(\xi) e^{i\xi \cdot x} d\tau$$

$$= (-1)^j C \delta_{kl} D^\alpha e^{\triangle t} f_l + (-1)^j C \int_t^\infty D^\alpha D_k D_l e^{\triangle \tau} f_l d\tau$$

$$= I_1 + I_2.$$

From (2.4) we have $|I_1|_\infty \le C_j t^{-j/2} |f_l|_\infty$. By the use of (2.4) one more time we obtain

$$|I_2|_\infty \le C_j |f_l|_\infty \int_t^\infty \tau^{-(j+2)/2} d\tau$$

$$\le C_j t^{-j/2} |f_l|_\infty.$$

Therefore

$$|D^j e^{\triangle t}\mathbb{P}_{kl}f_l|_\infty \leq |I_1|_\infty + |I_2|_\infty$$
$$\leq C_j t^{-j/2}|f_l|_\infty.$$

Hence Lemma 2.1 is proved. $\qquad\square$

Corollary 2.2. *Let $g \in C_{per}^\infty(\mathbb{R}^n \times [0,T])$ for some $T > 0$, then the solution of*

$$(2.6) \qquad\qquad u_t = \triangle u + D_i\mathbb{P}g, \qquad u = 0 \quad at \quad t = 0$$

satisfies

$$(2.7) \qquad\qquad |u(t)|_\infty \leq Ct^{1/2}\max_{0\leq s\leq t}|g(s)|_\infty.$$

Proof. The solution of (2.6) is given by

$$u(t) = \int_0^t e^{\triangle(t-s)}D_i\mathbb{P}g(u))(s)ds$$

and

$$|u(t)|_\infty \leq \int_0^t |e^{\triangle(t-s)}D_i\mathbb{P}g(u)(s)|_\infty ds.$$

After commuting D_i with the heat semi-group, we can use Lemma 2.1 to obtain

$$|u(t)|_\infty \leq \max_{0\leq s\leq t}|g(s)|_\infty \int_0^t (t-s)^{-1/2}ds.$$

Hence we obtain

$$|u(t)|_\infty \leq Ct^{1/2}\max_{0\leq s\leq t}|g(s)|_\infty.$$

$\qquad\square$

3. Estimates for $u_t = \triangle u + D_i\mathbb{P}g(u)$: proof of Theorem 1.2

In this section we consider the system $u_t = \triangle u + D_i\mathbb{P}g(u)$ with the initial condition $u = f$ at $t = 0$ where $f \in C_{per}^\infty(\mathbb{R}^n)$. It is well-known that the solution is smooth 2π periodic in a maximal interval $0 \leq t < T_f$ where $0 < T_f \leq \infty$.

Let us consider u is the solution of the inhomogeneous equation $u_t = \triangle u + D_i\mathbb{P}(g(u(x,t)))$ and recall $g(u)$ is quadratic in u. Thus, there is a constant C_g such that we have the following:

$$(3.1) \qquad\qquad |g(u)| \leq C_g|u|^2, \qquad |g_u(u)| \leq C_g|u|, \qquad \text{for all} \quad u \in \mathbb{R}^n$$

We first estimate the maximum norm of u.

Lemma 3.1. *Let C_g denote the constant in (3.1) and let C denote the constant in (2.7); set $c_0 = \frac{1}{16C^2C_g^2}$. Then we have $T_f > c_0/|f|_\infty^2$ and*

$$(3.2) \qquad\qquad |u(t)|_\infty < 2|f|_\infty \qquad for \qquad 0 \leq t < \frac{c_0}{|f|_\infty^2}.$$

Proof. Suppose (3.2) does not hold, then we can find the smallest time t_0 such that $|u(t_0)|_\infty = 2|f|_\infty$. Since t_0 is the smallest time so we have $t_0 < c_0/|f|_\infty^2$. Now by (2.3) and (2.7) we have

$$2|f|_\infty = |u(t_0)|_\infty$$
$$\leq |f|_\infty + Ct_0^{1/2} \max_{0 \leq s \leq t_0} |g(s)|_\infty$$
$$\leq |f|_\infty + CC_g t_0^{1/2} \max_{0 \leq s \leq t_0} |u(s)|_\infty^2$$
$$\leq |f|_\infty + CC_g t_0^{1/2} 4|f|_\infty^2.$$

This gives

$$1 \leq 4CC_g t_0^{1/2} |f|_\infty,$$

therefore $t_0 \geq 1/(16C^2 C_g^2 |f|_\infty^2) = c_0/|f|_\infty^2$ which is a contradiction. There (3.2) must hold. The estimate $T_f > c_0/|f|_\infty^2$ is valid since $\limsup_{t \to T_f} |u(t)|_\infty = \infty$ if T_f is finite. $\qquad \square$

Now we prove estimate (1.11) of Theorem 1.2 by induction on j. Let $j \geq 1$, and assume

$$(3.3) \qquad t^{k/2} |\mathcal{D}^k u(t)|_\infty \leq K_k |f|_\infty, \quad \text{for} \quad 0 \leq t \leq \frac{c_0}{|f|_\infty^2} \quad \text{and} \quad 0 \leq k \leq j - 1.$$

Let us apply D^j to the equation $u_t = \triangle u + D_i \mathbb{P} g(u)$ to obtain

$$v_t = \triangle v + D^{j+1} \mathbb{P} g(u), \quad v := D^j u,$$

$$v(t) = D^j e^{\triangle t} f + \int_0^t e^{\triangle(t-s)} D^{j+1} (\mathbb{P} g(u))(s) ds.$$

Using (2.4) we get

$$(3.4) \qquad t^{j/2} |v(t)|_\infty \leq C|f|_\infty + t^{j/2} \left| \int_0^t e^{\triangle(t-s)} D^{j+1} (\mathbb{P} g(u))(s) ds \right|_\infty.$$

We split the integral into

$$\int_0^{t/2} + \int_{t/2}^t =: I_1 + I_2$$

and obtain

$$|I_1(t)| = \left| \int_0^{t/2} D^{j+1} e^{\triangle(t-s)} (\mathbb{P} g(u))(s) ds \right|_\infty$$
$$\leq \int_0^{t/2} |D^{j+1} e^{\triangle(t-s)} (\mathbb{P} g(u))(s) ds|_\infty ds.$$

Using the inequality (2.5) in Lemma 2.1, we get

$$|I_1(t)|_\infty \leq C \int_0^{t/2} (t-s)^{-(j+1)/2} |g(u(s))|_\infty ds$$
$$\leq C|f|_\infty^2 t^{(1-j)/2}.$$

The integrand in I_2 has singularity at $s = t$. Therefore, we can move only one derivative from $D^{j+1}\mathbb{P}g(u)$ to the heat semigroup.(If we move two or more derivatives then the singularity becomes non-integrable.) Thus, we have

$$|I_2(t)|_\infty = \left| -\int_{t/2}^t De^{\triangle(t-s)}(D^j\mathbb{P}g(u))(s)ds \right|_\infty.$$

Since the Leray projector commutes with any order derivatives, therefore

$$|I_2(t)|_\infty = \left| -\int_{t/2}^t De^{\triangle(t-s)}(\mathbb{P}D^jg(u))(s)ds \right|_\infty.$$

If we use Lemma 2.1 for $j = 1$, we obtain

$$(3.5) \qquad |I_2(t)|_\infty \le C\int_{t/2}^t (t-s)^{-1/2}|D^jg(u)(s)|_\infty ds.$$

Since $g(u)$ is quadratic in u, therefore

$$|D^jg(u)|_\infty \le C|u|_\infty|\mathcal{D}^ju|_\infty + \sum_{k=1}^{j-1}|\mathcal{D}^ku|_\infty|\mathcal{D}^{j-k}u|_\infty.$$

By induction hypothesis (3.3) we obtain

$$(3.6) \qquad \sum_{k=1}^{j-1}|\mathcal{D}^ku(s)|_\infty|\mathcal{D}^{j-k}u(s)|_\infty \le Cs^{-j/2}|f|_\infty^2.$$

Expression in (3.5) can be estimated as below:

$$|I_2(t)|_\infty \le C\int_{t/2}^t (t-s)^{-1/2}\left(C|u(s)|_\infty|\mathcal{D}^ju(s)|_\infty + \sum_{k=1}^{j-1}|\mathcal{D}^ku(s)|_\infty|\mathcal{D}^{j-k}u(s)|_\infty \right)ds$$

$$= J_1 + J_2.$$

Using (3.6), and since $\int_{t/2}^t(t-s)^{-1/2}s^{-j/2}ds = Ct^{(1-j)/2}$, where C is independent of t, we obtain $|J_2(t)|_\infty \le C|f|_\infty^2 t^{(1-j)/2}$.

For J_1, we have

$$|J_1(t)|_\infty = C\int_{t/2}^t (t-s)^{-1/2}|u(s)|_\infty|\mathcal{D}^ju(s)|_\infty ds$$

$$\le C|f|_\infty\int_{t/2}^t (t-s)^{-1/2}s^{-j/2}s^{j/2}|\mathcal{D}^ju(s)|_\infty ds$$

$$\le C|f|_\infty t^{(1-j)/2}\max_{0\le s\le t}\{s^{j/2}\mathcal{D}^ju(s)|_\infty\}.$$

We use these bounds to bound the integral in (3.4). We have $v = D^ju$. Then maximizing the resulting estimate for $t^{j/2}|D^ju(t)|_\infty$ over all derivatives D^j of order j and setting

$$\phi(t) := t^{j/2}|\mathcal{D}^ju(t)|_\infty$$

and from (3.4), we obtain the following estimate

$$\phi(t) \le C|f|_\infty + Ct^{1/2}|f|_\infty^2 + C|f|_\infty t^{1/2}\max_{0\le s\le t}\phi(s) \quad \text{for} \quad 0 \le t \le \frac{c_0}{|f|_\infty^2}.$$

Since $t^{1/2}|f|_\infty \le \sqrt{c_0}$ then $Ct^{1/2}|f|_\infty^2 \le C\sqrt{c_0}|f|_\infty$. Therefore

(3.7) $$\phi(t) \le C_j |f|_\infty + C_j |f|_\infty t^{1/2} \max_{0 \le s \le t} \phi(s) \quad \text{for} \quad 0 \le t \le c_0/|f|_\infty^2.$$

Let us fix C_j so that the above estimate holds, and set

$$c_j = \min \left\{ c_0, \frac{1}{4C_j^2} \right\}.$$

First, let us prove the following

$$\phi(t) < 2C_j |f|_\infty \quad \text{for} \quad 0 \le t < \frac{c_j}{|f|_\infty^2}.$$

Suppose there is a smallest time t_0 such that $0 < t_0 < c_j/|f|_\infty^2$ with $\phi(t_0) = 2C_j|f|_\infty$. Then using (3.7) we obtain

$$2C_j|f|_\infty = \phi(t_0) \le C_j|f|_\infty + 2C_j^2|f|_\infty^2 t_0^{1/2},$$

thus

$$1 \le 2C_j |f|_\infty t_0^{1/2} \quad \text{gives} \quad t_0 \ge c_j/|f|_\infty^2$$

which contradicts the assertion. Therefore, we proved the estimate

(3.8) $$t^{j/2} |\mathcal{D}^j u(t)|_\infty \le 2C_j |f|_\infty \quad \text{for} \quad 0 \le t \le c_j/|f|_\infty^2.$$

If

(3.9) $$T_j := \frac{c_j}{|f|_\infty^2} < t \le \frac{c_0}{|f|_\infty^2} =: T_0$$

then we start the corresponding estimate at $t - T_j$. Using Lemma 3.1, we have $|u(t-T_j)|_\infty \le 2|f|_\infty$ and obtain

(3.10) $$T_j^{j/2} |\mathcal{D}^j u(t)|_\infty \le 4C_j |f|_\infty.$$

Finally, for any t satisfying (3.9)

$$t^{j/2} \le T_0^{j/2} = \left(\frac{c_0}{c_j} \right)^{j/2} T_j^{j/2}$$

and (3.10) yield

$$t^{j/2} |\mathcal{D}^j u(t)|_\infty \le 4C_j \left(\frac{c_0}{c_j} \right)^{j/2} |f|_\infty.$$

This completes the proof of Theorem 1.2.

4. Estimates For the Navier-Stokes Equations

Recall the transformed abstract ordinary differential equation (1.6)

(4.1) $$u_t = \triangle u - \mathbb{P}(u \cdot \nabla u), \quad \nabla \cdot u = 0$$

with

(4.2) $$u(x,0) = f(x).$$

Solution of (4.1) and (4.2) is given by

(4.3) $$u(t) = e^{\triangle t} f - \int_0^t e^{\triangle(t-s)} \mathbb{P}(u \cdot \nabla u)(s) ds.$$

Using (4.3) with previous estimates (2.3), (2.4) and (2.5), we prove the following lemma.

Lemma 4.1. *Set*

(4.4) $$V(t) = |u(t)|_\infty + t^{1/2} |\mathcal{D}u(t)|_\infty, \quad 0 < t < T(f).$$

There is a constant $C > 0$, independent of t and f, so that

(4.5) $$V(t) \le C|f|_\infty + Ct^{1/2} \max_{0 \le s \le t} V^2(s), \quad 0 < t < T(f).$$

Proof. Using estimate (2.3) of the heat equation in (4.3), we obtain

$$|u(t)|_\infty \le |f|_\infty + \left| \int_0^t e^{\triangle(t-s)} \mathbb{P}(u \cdot \nabla u)(s) ds \right|_\infty.$$

Apply identity $\mathbb{P}(u \cdot \nabla u) = \sum_i D_i \mathbb{P}(u_i u)$ with the fact, heat semi-group commutes with D_i, then use of inequality (2.5) in Lemma 2.1 for $j = 1$ to proceed

$$|u(t)|_\infty \le |f|_\infty + C \int_0^t (t-s)^{-1/2} |u(s)|_\infty^2 ds$$

$$= |f|_\infty + C \int_0^t (t-s)^{-1/2} s^{-1/2} s^{1/2} |u(s)|_\infty^2 ds$$

$$\le |f|_\infty + C \max_{0 \le s \le t} \{ s^{1/2} |u(s)|_\infty^2 \} \int_0^t (t-s)^{-1/2} s^{-1/2} ds.$$

Since $\int_0^t (t-s)^{-1/2} s^{-1/2} ds = C > 0$, which is independent of t, we have the following estimate

$$|u(t)|_\infty \le |f|_\infty + C \max_{0 \le s \le t} \{ s^{1/2} |u(s)|_\infty^2 \}$$

(4.6) $$|u(t)|_\infty \le |f|_\infty + Ct^{1/2} \max_{0 \le s \le t} V^2(s).$$

Apply D_i to (4.1), and the *Duhamel's principle* to obtain

(4.7) $$v(t) = D_i e^{\triangle t} f - \int_0^t e^{\triangle(t-s)} D_i \mathbb{P}(u \cdot \nabla) u(s) ds.$$

We can estimate the integral in (4.7) using Lemma 2.1 for $j = 1$ in the following way:

$$\left| \int_0^t D_i e^{\triangle(t-s)} \mathbb{P}(u \cdot \nabla u)(s) ds \right| \le \int_0^t |D_i e^{\triangle(t-s)} \mathbb{P}(u \cdot \nabla u)(s)| ds$$

$$\le C \int_0^t (t-s)^{-1/2} |u(s)|_\infty |\mathcal{D}u(s)|_\infty ds$$

$$= C \int_0^t (t-s)^{-1/2} s^{-1/2} s^{1/2} |u(s)|_\infty |\mathcal{D}u(s)|_\infty ds$$

$$\le C \max_{0 \le s \le t} \{ s^{1/2} |u(s)|_\infty |\mathcal{D}u(s)|_\infty \} \int_0^t (t-s)^{-1/2} s^{-1/2} ds$$

$$\le C \max_{0 \le s \le t} \{ |u(s)|_\infty^2 + s |\mathcal{D}u(s)|_\infty^2 \}.$$

Therefore, using (2.4) $j = 1$ in expression (4.7), we arrive at

$$|v(t)|_\infty \leq Ct^{-1/2}|f|_\infty + C \max_{0 \leq s \leq t} \{|u(s)|_\infty^2 + s|\mathcal{D}u(s)|_\infty^2\}$$

(4.8) $$t^{1/2}|\mathcal{D}u(t)|_\infty \leq C|f|_\infty + Ct^{1/2} \max_{0 \leq s \leq t} V^2(t).$$

Using (4.6) and (4.8), we have proved Lemma 4.1. $\square$

Lemma 4.2. *Let $C > 0$ denote the constant in estimate (4.5) and set*

$$c_0 = \frac{1}{16C^4}.$$

Then $T_f > c_0/|f|_\infty^2$ and

(4.9) $$|u(t)|_\infty + t^{1/2}|\mathcal{D}u(t)|_\infty < 2C|f|_\infty \quad for \quad 0 \leq t < \frac{c_0}{|f|_\infty^2}.$$

Proof. We prove this lemma by contradiction after recalling the definition of $V(t)$ in (4.4). Suppose that (4.9) does not hold, then denote by t_0 the smallest time with $V(t_0) = 2C|f|_\infty$. Use (4.5) to obtain

$$2C|f|_\infty = V(t_0)$$
$$\leq C|f|_\infty + Ct_0^{1/2}4C^2|f|_\infty^2,$$

thus

$$1 \leq 4C^2 t_0^{1/2}|f|_\infty^2,$$

therefore $t_0 \geq c_0/|f|_\infty^2$. This contradiction proves (4.9) and $T_f > c_0/|f|_\infty^2$. $\square$

Lemma 4.2 proves Theorem 1.1 for $j = 0$ and $j = 1$. By an induction argument as in the proof of Theorem 1.2 one proves Theorem 1.1 for any $j = 0, 1, \cdots$

Remarks. We can apply estimate (1.5) of Theorem 1.1 for

(4.10) $$\frac{c_0}{2|f|_\infty^2} \leq t \leq \frac{c_0}{|f|_\infty^2}$$

and obtain

(4.11) $$|\mathcal{D}^j u(t)|_\infty \leq C_j|f|_\infty^{j+1}$$

in interval (4.10). Starting the estimate at $t_0 \in [0, T_f)$ we have

(4.12) $$|\mathcal{D}^j u(t_0 + t)|_\infty \leq C_j|u(t_0)|_\infty^{j+1}$$

for

(4.13) $$\frac{c_0}{2|u(t_0)|_\infty^2} \leq t \leq \frac{c_0}{|u(t_0)|_\infty^2}.$$

Then, if t_1 is fixed with

(4.14) $$\frac{c_0}{2|f|_\infty^2} \leq t_1 < T_f,$$

we can maximize both sides of (4.12) over $0 \le t_0 \le t_1$ and obtain

$$(4.15) \qquad \max\left\{|\mathcal{D}^j u(t)|_\infty : \frac{c_0}{2|f|_\infty^2} \le t \le t_1 + \tau\right\} \le C_j \max\{|u(t)|_\infty^{j+1} : 0 \le t \le t_1\}$$

with

$$\tau = \frac{c_0}{|u(t_1)|_\infty^2}$$

Estimate (4.15) says, essentially, that the maximum of the j-th derivatives of u measured by $|\mathcal{D}^j u|_\infty$, can be bounded in terms of $|u|_\infty^{j+1}$. The positive value of τ on the left-hand side of (4.15) shows that $|u|_\infty^{j+1}$ controls $|\mathcal{D}^j u|_\infty$ for some time into the future.

As is well known, if (u, p) solves the Navier-Stokes equations and $\lambda > 0$ is any scaling parameter, then the functions u_λ, p_λ defined by

$$u_\lambda(x, t) = \lambda u(\lambda x, \lambda^2 t), \qquad p_\lambda(x, t) = \lambda^2 p(\lambda x, \lambda^2 t)$$

also solve the Navier-Stokes equations. Clearly,

$$|u_\lambda(t)|_\infty = \lambda|u(\lambda^2 t)|_\infty, \qquad |\mathcal{D}^j u_\lambda(t)|_\infty = \lambda^{j+1}|\mathcal{D}^j u(\lambda^2 t)|_\infty.$$

Therefore, $|\mathcal{D}^j u|_\infty$ and $|u|_\infty^{j+1}$ both scale like λ^{j+1}, which is, of course, consistent with the estimate (4.15). We do not know under what assumptions $|u|_\infty^{j+1}$ can conversely be estimated in terms of $|\mathcal{D}^j u|_\infty$.

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