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In-Depth of Euler's Work on the Zeta Function

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Abstract: *Euler developed many theorems and formulae in diverse fields of mathematics. One of the most significant contributions is the product formula. The study focuses on elaborating the fundamental concept of the zeta function. This expository paper interprets "Euler's original proofs" of the product formula and examines his efforts in solving the Basel problem using this product formula.*

Keywords: Bassel problem, Convergent, Prime numbers, Product formula

1. Introduction

Leonard Euler (1707-1783) was the first mathematician to rigorously define the Zeta function in the domain [1, ∞). Euler was the renowned mathematician who made significant contributions to various branches of mathematics. In 1737, he published a paper in which he established the first proof that the infinite sum of the reciprocals of the prime numbers diverges. This work is the prominent beginning of a new era of innovation and the founding document of analytic number theory [7]. *Introductio in analysin infinitorum* is an important book that describes Euler's work in analysis [2]. His relevant works on infinite series, both divergent and convergent series, made him known to public. In his work, we found his approach was heuristic giving priority in discovery and symbolic manipulation which needs further mathematical treatment. He showed $1-1+1-1+1-1+\ldots=\frac{1}{2}$ and $1+2+4+8+\ldots=-1$ which were controversial and abstract. He also established the formula for the Riemann zeta function given by

$$\zeta(s)=\sum_1^\infty \frac{1}{n^s} \qquad (1.1)$$

where s is a positive integer greater than 1. In 1730, Euler began the study on the solution of the Bassel problem proposed by Pietro Mengoli in 1644 and, he continued this work until he described ζ(s) [2]. In 1749, he published a paper on it, which was almost 110 years before Riemann [1]. In 1779, Roger Apery proved the irrationality of $\zeta(3)$ by using Euler's formula. The Euler product formula is one of the important contributions to the theory of the zeta function. It is given by

$$\zeta(s)=\sum_{n=1}^\infty \frac{1}{n^s}=\prod_p \left(1-\frac{1}{p^s}\right)^{-1} \qquad (1.2)$$

Euler not only evaluated this function for even positive integers, he also linked it with π. This insight marked a significant step in understanding the relationship between infinite series and number theory. He also discovered the methods to link the Euler product formula to prime numbers. Bernhard Riemann (1826–

1866) applied Euler's ideas to the complex domain and developed the new ideas of the Riemann zeta function. Riemann investigated the analytic continuation of ζ(s) to almost all complex values of s, except for s = 1. Riemann's research on the zeta function played a vital role in analysis and number theory forming the foundation of modern mathematics. Analysis as an autonomous mathematical discipline was only a part of Euler's conception [4,5]. Its special properties continue to inspire research into the profound mysteries of prime numbers. This paper is very useful for glorifying the Euler's work on infinite series, which makes the clear path for summing infinite series, especially, in summability theory and number theory.

2. Fundamental Concept of Zeta Function

In the course of studying infinite series, Euler also introduced a power series given by

$$x+\frac{x^2}{2^2}+\frac{x^3}{3^2}+\frac{x^4}{4^2}+\frac{x^5}{5^2}+\cdots=\sum_{n=1}^{\infty}\frac{x^n}{n^2} \qquad (2.1)$$

which reduces to $1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\cdots=\sum_{n=1}^{\infty}\frac{1}{n^2}$ when x =1.As he replaced exponent 2 by s and symbolized ζ(s) then we can have $\zeta(s)=\sum_{1}^{\infty}\frac{1}{n^s}$ where $s>1$ which is the zeta function developed by Euler. By extending its domain as complex number, Riemann deduced its extended form after approximately 110 years later. Now this expression is popularly known as Riemann zeta function. In 1735 Euler proved that $\zeta(2)=\frac{\pi^2}{6}$ in different ways [3,6,8,9,10]. One of the proofs, he acted in his early days can be treated as follows.

Lemma 1: $\int_0^1\frac{\ln(1-x)}{x}dx=-\zeta(2)$

Proof: Let, $I=\int_0^1\frac{\ln(1-x)}{x} = \int_0^1\frac{1}{x}\left(-x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}-\frac{x^5}{5}-\cdots\right)dx$

$$=\int_0^1\left(-1-\frac{x}{2}-\frac{x^2}{3}-\frac{x^3}{4}-\frac{x^4}{5}-\cdots\right)dx$$

$$=-\left[x+\frac{x^2}{2^2}+\frac{x^3}{3^2}+\frac{x^4}{4^2}+\frac{x^5}{5^2}+\cdots\right]_0^1$$

$$=-\left[1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\cdots\right]$$

$$=-\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\therefore\int_0^1\frac{\ln(1-x)}{x} = -\zeta(2) \qquad (2.2)$$

Lemma 2: $\zeta(2)=(ln2)^2+2\sum_{n=1}^{\infty}\frac{1}{2^n\,n^2}$

Proof: From (2.2) we have

$$-\zeta(2)=\int_0^1\frac{\ln(1-x)}{x}$$

Put , $1-x=v\Rightarrow x=1-v\Rightarrow dx=-dv$

When x = 0, v =1 and when x = 1, v = 0

On substitution, we get

$$-\zeta(2)=\int_1^0 \frac{\ln v}{1-v}(-dv)$$
$$=\int_0^x \frac{\ln v}{1-v}dv + \int_x^1 \frac{\ln v}{1-v}dv \ ; x\epsilon(0,1)$$
$$= I_1+I_2 \text{(say) where } \quad I_1=\int_0^x \frac{\ln v}{1-v}dv \quad \text{and} \quad I_2=\int_x^1 \frac{\ln v}{1-v}dv$$

Now,

$$I_1=\int_0^x \frac{\ln v}{1-v}\ dv \ =\int_0^x lnv\,(1-v)^{-1}\ dv\ =\int_0^x lnv\,(1+v+v^2+v^3+\cdots)dv$$

On integration by parts, we get

$$I_1= -lnx \ln(1-x) - \sum_{n=1}^{\infty}\frac{x^n}{n^2}$$

Again, $I_2=\int_x^1 \frac{\ln v}{1-v}dv$

Put , $1-v=u \Rightarrow v=1-v \Rightarrow dv=-du$

When $v = x$, u =1-x and when u= 1, v = 0

On substitution, we get

$$I_2 = \int_{1-x}^0 \frac{\ln(1-u)}{u}(-du) \quad = -\int_{1-x}^0 \frac{\ln(1-u)}{u}(-du)$$
$$= \int_{1-x}^0 \frac{u+\frac{u^2}{2}+\frac{u^3}{3}+\cdots}{u}du \qquad = -\int_0^{1-x}(1+\frac{u}{2}+\frac{u^2}{2}+\cdots)du$$
$$= -\{\frac{(1-x)}{1^2}+\frac{(1-x)^2}{2^2}+\frac{(1-x)^3}{3^2}+\frac{(1-x)^4}{4^2}+\cdots\}$$
$$\therefore \ I_2 = -\sum_{n=1}^{\infty}\frac{(1-x)^n}{n^2}$$

So, $-\zeta(2)= I_1 + \ I_2 =-lnx\ln(1-x) - \sum_{n=1}^{\infty}\frac{x^n}{n^2} - \sum_{n=1}^{\infty}\frac{(1-x)^n}{n^2}$

$$\Rightarrow \zeta(2)= lnx\ln(1-x) + \sum_{n=1}^{\infty}\frac{x^n}{n^2} + \sum_{n=1}^{\infty}\frac{(1-x)^n}{n^2}$$

As x lies between 0 and 1, Euler chose $x=\frac{1}{2}$ then on substitution, he obtained

$$\zeta(2) = (ln2)^2 + \sum_{n=1}^{\infty}\frac{1}{2^n n^2} + \sum_{n=1}^{\infty}\frac{1}{2^n n^2}$$
$$\therefore \ \zeta(2) = (ln2)^2 + 2\sum_{n=1}^{\infty}\frac{1}{2^n\, n^2} \tag{2.3}$$

which is a remarkable formula derived by Euler [11].

From this result (2.3), Euler became more enthusiastic that he found the series $\sum_{n=1}^{\infty}\frac{1}{2^n\, n^2}$ is fast converging series and hence he estimated its value by adding some terms and obtained its approximate value equal to 0.582241 and the value of $(ln2)^2$ is approximately 0.480453.Thus he declared by using (2.3) that the value of $\zeta(2)$ is approximately 1.644934 [1].In a brief study of the works done by Leonard Euler, we observed that there was an ocean of Mathematics in his mind. Even in Zeta function, he has done many revolutionary works out of which it is a small piece of work that we are studying. Such problems concerning to Zeta function have already been issued by Pietro Mengoli (1625−1686) but he couldn't solve them in rigorous form. Similarly, John Wallis (1616-1703) computed the value of $\zeta(2)$ to 3 decimal places as 1.645 while Euler found its value is $\frac{\pi^2}{6}$ [1].

3. Euler Product Formula and Zeta Function

Theorem: If S is a set of all complex numbers s and Re(s) >1, then every zeta function can be expressed as the infinite products given by $\prod_{p,prime}(1-\frac{1}{p^s})^{-1}$

Proof: We have zeta function,

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s} \quad , n \in N$$

$$\Rightarrow \zeta(s) = \frac{1}{1^s}+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{4^s}+\frac{1}{5^s}+\frac{1}{6^s}+\frac{1}{7^s}+\cdots \qquad (3.1)$$

Multiplying both sides of (3.1) by $\frac{1}{2^s}$,we get

$$\frac{1}{2^s}\zeta(s) = \frac{1}{2^s}+\frac{1}{4^s}+\frac{1}{6^s}+\frac{1}{8^s}+\frac{1}{10^s}+\frac{1}{12^s}+\frac{1}{14^s}+\cdots \qquad (3.2)$$

Subtracting (3.2) from (3.1) we get

$$(1-\frac{1}{2^s})\zeta(s) = 1+\frac{1}{3^s}+\frac{1}{5^s}+\frac{1}{7^s}+\frac{1}{9^s}+\frac{1}{11^s}+\frac{1}{13^s}\ldots \qquad (3.3)$$

Multiplying both sides of (3.3) by $\frac{1}{3^s}$ and subtracting resulting one from (3.3) itself, we get

$$(1-\frac{1}{2^s})(1-\frac{1}{3^s})\ \zeta(s) = 1+\frac{1}{5^s}+\frac{1}{7^s}+\frac{1}{11^s}+\frac{1}{13^s}+\frac{1}{17^s}+\cdots \qquad (3.4)$$

Multiplying both sides of (3.4) by $\frac{1}{5^s}$ and subtracting resulting one from (3.4) itself, we get

$$(1-\frac{1}{2^s})(1-\frac{1}{3^s})\ (1-\frac{1}{5^s})\zeta(s) = 1+\frac{1}{7^s}+\frac{1}{11^s}+\frac{1}{13^s}+\frac{1}{17^s}+\cdots \qquad (3.5)$$

If we continue this process infinitely many times, we remain 1 only on the right side while the left side becomes $\{(1-\frac{1}{2^s})\left(1-\frac{1}{3^s}\right)\left(1-\frac{1}{5^s}\right)\ldots\}\ \zeta(s)$.

$$\text{i.e., } \{(1-\frac{1}{2^s})\left(1-\frac{1}{3^s}\right)\left(1-\frac{1}{5^s}\right)\ldots\}\ \zeta(s)=1$$

Let p represents all the prime numbers 2,3,5,7,11, ... then we get

$$\prod_{p,prime}\left(1-\frac{1}{p^s}\right)\ \zeta(s)=1$$

$$\therefore\ \zeta(s)=\prod_p(1-\frac{1}{p^s})^{-1} \qquad (3.6)$$

This completes the proof. It is noted that the identity (3.6) is known as Euler product formula.

Here, one may confuse about the value of p and hence we can also rewrite (3.6) as,

$$\zeta(s)=\frac{1}{1-\frac{1}{p_1^s}}\times\frac{1}{1-\frac{1}{p_2^s}}\times\frac{1}{1-\frac{1}{p_3^s}}\times\frac{1}{1-\frac{1}{p_4^s}}\times\frac{1}{1-\frac{1}{p_5^s}}\times\frac{1}{1-\frac{1}{p_6^s}}\times\frac{1}{1-\frac{1}{p_7^s}}\times\frac{1}{1-\frac{1}{p_8^s}}\times\ldots \qquad (3.7)$$

Every term of this product (3.7) must be a sum of infinite geometric series like

$$\frac{1}{1-\frac{1}{p_1^s}}=1+\frac{1}{p_1^s}+\frac{1}{p_1^{2s}}+\frac{1}{p_1^{3s}}+\frac{1}{p_1^{4s}}+\cdots$$

$$\frac{1}{1-\frac{1}{p_2^s}}=1+\frac{1}{p_2^s}+\frac{1}{p_2^{2s}}+\frac{1}{p_2^{3s}}+\frac{1}{p_2^{4s}}+\cdots$$

and so on. So, we can write right side of (3.7) as

$$\prod_{i=1}^{\infty} \frac{1}{1-\frac{1}{p_i^s}} = \prod_{i=1}^{\infty} \sum_{k=0}^{\infty} \frac{1}{p_i^{ks}}$$

$$= \left(1 + \frac{1}{p_1^s} + \frac{1}{p_1^{2s}} + \frac{1}{p_1^{3s}} + \cdots\right)\left(1 + \frac{1}{p_2^s} + \frac{1}{p_2^{2s}} + \frac{1}{p_2^{3s}} + \cdots\right)\left(1 + \frac{1}{p_2^s} + \frac{1}{p_2^{2s}} + \frac{1}{p_2^{3s}} + \cdots\right)$$

$$= 1 + \sum_{1\le i}^{\infty} \frac{1}{p_i^s} + \sum_{1\le i\le j}^{\infty} \frac{1}{p_i^s p_j^s} + \sum_{1\le i\le j\le k}^{\infty} \frac{1}{p_i^s p_j^s p_k^s} + \sum_{1\le i\le j\le k\le l}^{\infty} \frac{1}{p_i^s p_j^s p_k^s p_l^s} + \cdots \quad (3.8)$$

By rearranging (3.8), we get

$$\prod_{i=1}^{\infty} \frac{1}{1-\frac{1}{p_i^s}} = 1 + \frac{1}{p_1^s} + \frac{1}{p_2^s} + \frac{1}{p_1^{2s}} + \frac{1}{p_3^s} + \frac{1}{p_1^s p_2^s} + \frac{1}{p_4^s} + \frac{1}{p_1^{3s}} + \cdots$$

Substituting the prime numbers $p_1 = 2, p_2 = 3, p_3 = 5, p_4 = 7, p_5 = 11, p_6 = 13, p_7 = 17, \ldots$ we get

$$\prod_{i=1}^{\infty} \frac{1}{1-\frac{1}{p_i^s}} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \frac{1}{6^s} + \frac{1}{7^s} + \cdots = \zeta(s)$$

Thus, it verifies (3.6) and shows that Euler product is equal to Riemann Zeta function.

Corollary:1 The Riemann zeta function defined by (3.1) converges for $Re(s) > 1$.

Here, we have

$$1 + \frac{1}{p_1^s} + \frac{1}{p_2^s} + \frac{1}{p_1^{2s}} + \frac{1}{p_3^s} + \frac{1}{p_1^s p_2^s} + \frac{1}{p_4^s} + \frac{1}{p_1^{3s}} + \cdots = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \frac{1}{6^s} + \cdots$$

And it is also true from (3.8) that

$$\left(1 + \frac{1}{p_1^s} + \frac{1}{p_2^s} + \frac{1}{p_3^s} + \frac{1}{p_4^s} + \frac{1}{p_5^s} + \cdots \right) < \left(1 + \frac{1}{p_1^s} + \frac{1}{p_2^s} + \frac{1}{p_1^{2s}} + \frac{1}{p_3^s} + \frac{1}{p_1^s p_2^s} + \frac{1}{p_4^s} + \frac{1}{p_1^{3s}} + \cdots \right)$$

$\Rightarrow \sum_{p,prime} p^{-Re(s)} < \sum_{n,natural} n^{-Re(s)}$

$\Rightarrow \sum_{p,prime} p^{-Re(s)}$ is convergent for Re(s)> 1

So, we found that the zeta function converges for Re(s)> 1

Corollary 2: Prove that $\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2}$ is $\frac{\pi^2}{8}$

Proof: Euler found, by research, that $\zeta(2)$ is $\frac{\pi^2}{6}$ **[7].** Now, put s =2 in (3.3) and we get,

$$\left(1 - \frac{1}{4}\right)\zeta(2) = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \frac{1}{11^2} + \frac{1}{13^2} + \cdots$$

$$\Rightarrow \frac{3}{4} \times \frac{\pi^2}{6} = \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2}$$

i.e. $\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{\pi^2}{8}$ [1].

Corollary 3: Is $\zeta(s)$ a rational number?

From (3.7), we can have

$$\zeta(s) = \frac{p_1^s}{p_1^s-1} \times \frac{p_2^s}{p_2^s-1} \times \frac{p_3^s}{p_3^s-1} \times \frac{p_4^s}{p_4^s-1} \times \frac{p_5^s}{p_5^s-1} \times \frac{p_6^s}{p_6^s-1} \times \frac{p_7^s}{p_7^s-1} \times \ldots$$

$$= \frac{2^s}{2^s-1} \times \frac{3^s}{3^s-1} \times \frac{5^s}{5^s-1} \times \frac{7^s}{7^s-1} \times \frac{11^s}{11^s-1} \times \frac{13^s}{13^s-1} \times \frac{17^s}{17^s-1} \ldots$$

$= \frac{2^s.3^s.5^s.7^s.11^s.13^s.17^s \ldots}{(2^s-1)(3^s-1)(5^s-1)(7^s-1)\ldots}$ which is not definite.

Moreover, the value of Riemann zeta function defined for Re$(s) > 1$ is associated with π for any even value like $\zeta(2) = \frac{\pi^2}{6}$ and $(4) = \frac{\pi^2}{90}$. Roger Apery had already proved the irrationality of ζ (3) in 1779. So, these values are not rational. In other values of s, it may be rational like $\zeta(0) = -\frac{1}{2}$.Thus, $\zeta(s)$ may or may not be a rational number.

4. Conclusion

Euler investigated the function $f(x) = x + \frac{x^2}{2^2} + \frac{x^3}{3^2} + \frac{x^4}{4^2} + \frac{x^5}{5^2} + \cdots$ as the generating function of zeta function at *s* =2.As a part of this generalization, Euler established the product formula which is equivalent to the Zeta function $\zeta(s)$ for Re$(s)>$ 1. Later, Bernhard Riemann extended Euler's ideas, analytically continuing ζ(*s*) beyond Re(s) >1 to the entire complex plane, $s \neq 1$. This extended function is now known as the Riemann zeta function, central to analytic number theory and the distribution of primes. The Euler product formula remains a cornerstone in the arrangement of prime numbers. This product formula is also a basic element in mathematics and it has numerous applications in physics and engineering, where zeta-function methods appear in quantum field theory, statistical mechanics, signal processing, and regularization methods.

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