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On Interpolative (α,β)-Contractions and Fixed Point Theory in G_b-Metric Spaces

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Abstract: *G_b-metric spaces are a generalization of both b-metric and G-metric spaces and have become an important setting in fixed point theory. By combining a weaker form of the triangle inequality with a three-variable distance function, G_b-metric spaces provide a more flexible framework for studying nonlinear problems. This paper presents recent developments in fixed point theory on G_b-metric spaces. We summarize the basic definitions and properties of G_b-metric spaces and review important fixed point results, such as Banach-type, Kannan-type, Chatterjea-type, and Hardy-Rogers-type contractions. Additionally, we introduce a new contraction condition called the interpolative (α,β)-contraction in G_b-metric spaces and prove the corresponding fixed point theorem. We provide illustrative examples and comparisons with existing results. This extended contraction properly generalizes several well-known contraction types while maintaining the existence and uniqueness of fixed points. The purpose of this article is to present an accessible overview of existing results while contributing new insights to the theory of contractions in G_b-metric spaces by extending the existing literature.*

Keywords: *G_b-metric space, Fixed point theory, Interpolative (α,β)-contraction, Banach contraction, Kannan contraction.*

1 Introduction

Fixed point theory is a fundamental area of nonlinear functional analysis with wide-ranging applications in differential and integral equations, optimization, computer science, biology, and engineering. Among its most celebrated results is the Banach contraction principle [7], which guarantees the existence and uniqueness of a fixed point for every contraction mapping on a complete metric space. Owing to its simplicity and effectiveness, the Banach principle has become one of the most influential tools in nonlinear analysis. Consequently, considerable effort has been devoted to extending this result by either modifying the contractive condition or generalizing the underlying space.

To enlarge the scope of applicability, several generalizations of metric spaces have been introduced. Bakhtin [6] and later Czerwik [10] proposed the notion of b-metric spaces, where the classical triangle inequality is relaxed by a constant $b \geq 1$. This modification allows the treatment of problems in which the standard metric structure is too restrictive. Another important development was introduced by Mustafa and Sims [15], who defined G-metric spaces, in which the distance function depends on three variables rather than two. This approach generates a distinct topology and enables the extension of many classical fixed point theorems.

Combining these two generalizations, G_b-metric spaces were introduced as a natural extension incorporating both the three-variable structure of G-metrics and the relaxed triangle inequality of b-metrics.

Since their introduction, G_b-metric spaces have received increasing attention in the literature. Various contraction principles, including Banach-type, Kannan-type, Chatterjea-type, and Hardy–Rogers-type contractions, have been adapted to this setting, yielding existence and uniqueness results under suitable assumptions [18, 12, 9, 11, 2].
Motivated by these developments, the main objective of this paper is to introduce and investigate the class of contractive mappings in G_b-metric spaces, namely interpolative (α,β)-contractions. This contractive condition interpolates between classical Banach and Kannan-type contractions and enlarges the class of admissible nonlinear operators. We establish existence and uniqueness theorems for such mappings in complete G_b-metric spaces under the condition $\alpha+\beta<\frac{1}{s}$. Furthermore, we provide illustrative examples and applications to nonlinear integral equations, demonstrating the effectiveness of the proposed approach.

2 Preliminaries

In this section we include some definitions, examples and well-known theorems related to our research paper.

2.1 b-metric Spaces

Definition 2.1. [10] Consider a nonempty set X together with a real constant $s\geq 1$. A mapping $d: X\times X\to[0,\infty)$ is called a b-metric provided that for all $x,y,z\in X$:

(b1) $d(x,y)=0$ if and only if $x=y$

(b2) $d(x,y)=d(y,x)$

(b3) $d(x,z)\leq s[d(x,y)+d(y,z)]$

The pair (X,d) is called a b-metric space.

Example 2.2. [10]

1. Let (X,d) be a metric space. Then d is a b-metric on X with $b=1$, since

$$d(x,z)\leq d(x,y)+d(y,z),\quad \text{for all } x,y,z\in X.$$

   Thus, every metric space is a b-metric space.

2. Let $X=\{a,b,c\}$ and define $d: X\times X\to[0,\infty)$ by

$$d(x,y)=\begin{cases}0, & x=y,\\ 1, & x\neq y.\end{cases}$$

   Define $d_b(x,y)=2d(x,y)$. Then (X,d_b) is a b-metric space with $b=2$.

2.2 G-metric Spaces

Definition 2.3. [15] Consider a nonempty set X and let $G: X\times X\times X\to[0,\infty)$ be a mapping satisfying the following conditions:

(G1) $G(x,y,z)=0$ if $x=y=z$;

(G2) $G(x,x,y) > 0$ for all $x,y \in X$ with $x \neq y$;

(G3) $G(x,x,y) \leq G(x,y,z)$ for all $x,y,z \in X$ with $z \neq y$;

(G4) Symmetry in all variables: $G(x,y,z) = G(x,z,y) = G(y,x,z) = \cdots$;

(G5) Rectangle inequality: $G(x,y,z) \leq G(x,a,a) + G(a,y,z)$ for all $x,y,z,a \in X$.

Then G is called a *G-metric*, and (X,G) a *G-metric space*.

Example 2.4. [15]

1. Let (X,d) be a usual metric space. Define on $X \times X \times X$:

$$G_s(x,y,z) = d(x,y) + d(y,z) + d(x,z), \quad G_m(x,y,z) = \max\{d(x,y), d(y,z), d(x,z)\}.$$

Then both (X,G_s) and (X,G_m) are G-metric spaces.

2. Let $X = \mathbb{R} \setminus \{0\}$. Define $G : X \times X \times X \to \mathbb{R}^+$ by:

$$G(x,y,z) = \begin{cases} \|x-y\| + \|y-z\| + \|x-z\|, & \text{if } x,y,z \text{ have same sign,} \\ 1 + \|x-y\| + \|y-z\| + \|x-z\|, & \text{otherwise.} \end{cases}$$

Then (X,G) is a G-metric space.

2.3 G_b-metric Spaces

Definition 2.5. [5] Consider a nonempty set X and $b \geq 1$ be a real number. A mapping $G_b : X \times X \times X \to [0,\infty)$ is said to be a G_b-metric if the following conditions are satisfied:

1. (G_b1) $G_b(x,y,z) = 0 \iff x = y = z$
2. (G_b2) $G_b(x,x,y) \leq G_b(x,y,z)$ for all $x,y,z \in X$ with $y \neq z$
3. (G_b3) $G_b(x,y,z) = G_b(x,z,y) = G_b(y,x,z) = G_b(y,z,x) = G_b(z,x,y) = G_b(z,y,x)$ (symmetry)
4. (G_b4) $G_b(x,y,z) \leq b[G_b(x,a,a) + G_b(a,y,z)]$ for all $x,y,z,a \in X$

The pair (X,G_b) is called a G_b-metric space.

Example 2.6. [15] Let $X = \mathbb{R}^n$ with the usual Euclidean norm $\|\cdot\|$. Define $G_b : X \times X \times X \to [0,\infty)$ by

$$G_b(x,y,z) = \|x-y\| + \|y-z\| + \|z-x\| \quad \forall\, x,y,z \in X.$$

Then (X,G_b) forms a G_b-metric space with $b = 2$.

2.4 Convergence and completeness in G_b-metric Spaces

Definition 2.7. [5] A sequence (x_n) in (X,G_b) is said to **converge** to a point $x \in X$ if

$$\lim_{n\to\infty} G_b(x_n,x,x) = 0$$

equivalently,

$$\lim_{m,n\to\infty} G_b(x_m,x_n,x) = 0.$$

In this case we write $x_n \to x$ and say that (x_n) is G_b-convergent to x.

Definition 2.8. [5] A sequence (x_n) in (X,G_b) is called **Cauchy** if for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n,m,l \geq n_0$,

$$G_b(x_n,x_m,x_l) < \varepsilon$$

that is, $G_b(x_n,x_m,x_l) \to 0$ as $n,m,l \to \infty$.

Definition 2.9. [5] A G_b-metric space (X,G_b) is called **complete** if every Cauchy sequence is convergent in (X,G_b).

Definition 2.10. [5] Let (X,G_b) be a G_b-metric space. A mapping $T : X \to X$ is called a **contraction mapping** if there exists a constant $0 \leq k < 1$ such that:

$$G_b(Tx,Ty,Tz) \leq kG_b(x,y,z) \quad \forall\, x,y,z \in X.$$

Proposition 2.11. *[5] Let (X,G_b) be a G_b-metric space. Then for all $x,y,z \in X$:*

1. $G_b(x,y,z) \geq 0$
2. $G_b(x,y,z) \leq b[G_b(x,x,y)+G_b(x,x,z)]$
3. $G_b(x,y,y) \leq 2bG_b(y,x,x)$

3 Some Fixed Point Theorems in G_b-metric Spaces

In this section, we recall some classical fixed point theorems in the setting of G_b-metric spaces.

3.1 Banach-type fixed point theorem

Theorem 3.1. *[13] Let (X,G_b) be a complete G_b-metric space with coefficient $s \geq 1$. Let $T : X \to X$ be a mapping satisfying*

$$G_b(Tx,Ty,Tz) \leq kG_b(x,y,z)$$

for all $x,y,z \in X$ where $0 \leq k < \frac{1}{s}$. Then T has a unique fixed point $x^* \in X$. Moreover, for any $x_0 \in X$, the Picard iteration $x_{n+1} = Tx_n$ converges to x^*.

3.2 Kannan-type fixed point theorem

Theorem 3.2. *[13] Let (X,G_b) be a complete G_b-metric space with coefficient $s \geq 1$. Let $T : X \to X$ be a mapping satisfying*

$$G_b(Tx,Ty,Tz) \leq \alpha[G_b(x,Tx,Tx)+G_b(y,Ty,Ty)+G_b(z,Tz,Tz)]$$

for all $x,y,z \in X$, where $0 \leq \alpha < \frac{1}{2s}$. Then T has a unique fixed point.

3.3 Chatterjea-type fixed point theorem

Theorem 3.3. *[4] Let (X,G_b) be a complete G_b-metric space with coefficient $s \geq 1$. Let $T : X \to X$ be a mapping satisfying*

$$G_b(Tx,Ty,Tz) \leq \beta[G_b(x,Ty,Tz)+G_b(y,Tz,Tx)+G_b(z,Tx,Ty)]$$

for all $x,y,z \in X$ where $0 \leq \beta < \frac{1}{2s}$. Then T has a unique fixed point.

3.4 Hardy-Rogers-type fixed point theorem

Theorem 3.4. *[4] Let (X, G_b) be a complete G_b-metric space with coefficient $s \geq 1$. Let $T : X \to X$ be a mapping satisfying*

$$G_b(Tx, Ty, Tz) \leq aG_b(x, y, z) + bG_b(x, Tx, Tx) + cG_b(y, Ty, Ty) + dG_b(z, Tz, Tz) + eG_b(x, y, Tz)$$

for all $x, y, z \in X$ where $a + b + c + d + e < \frac{1}{s}$. Then T has a unique fixed point.

4 Main Results

In this section, we introduce a new type of contraction in G_b-metric spaces that interpolates between Banach and Kannan contractions. This contraction provides a more general framework that contains both classical types as special cases.

4.1 Interpolative (α, β)-contractions

Definition 4.1. Let (X, G_b) be a G_b-metric space with coefficient $s \geq 1$. A mapping $T : X \to X$ is called an **interpolative (α, β)-contraction** if there exist constants $\alpha, \beta \in [0, 1)$ with $\alpha + \beta < \frac{1}{s}$ such that for all distinct $x, y \in X$,

$$G_b(Tx, Ty, Ty) \leq \alpha \cdot G_b(x, y, y) + \beta \cdot \min \{G_b(x, Tx, Tx), G_b(y, Ty, Ty)\}.$$

Remark 4.2. The term ”interpolative” refers to the fact that this contraction interpolates between

- Banach-type contractions (when $\beta = 0$).
- Kannan-type contractions (when $\alpha = 0$ and we consider an appropriate form).

The minimization term makes this contraction genuinely different from both classical types.

Theorem 4.3. *Let (X, G_b) be a complete G_b-metric space with coefficient $s \geq 1$. Suppose that $T : X \to X$ is an interpolative (α, β)-contraction, that is, there exist $\alpha, \beta \in [0, 1)$ with*

$$\alpha + \beta < \frac{1}{s}$$

such that for all $x \neq y$,

$$G_b(Tx, Ty, Ty) \leq \alpha G_b(x, y, y) + \beta \min\{G_b(x, Tx, Tx), G_b(y, Ty, Ty)\}.$$

Then:

1. *T has a fixed point $x^* \in X$;*
2. *the fixed point is unique;*
3. *for any $x_0 \in X$, the Picard iteration $x_{n+1} = Tx_n$ converges to x^*;*
4. *the following error estimate holds:*

$$G_b(x_n, x^*, x^*) \leq \frac{s(\alpha + \beta)^n}{1 - s(\alpha + \beta)} G_b(x_0, x_1, x_1).$$

Proof. **Step 1: Construction of the Picard sequence.**
Let $x_0 \in X$ be arbitrary and define

$$x_{n+1} = Tx_n, \qquad n \geq 0.$$

Step 2: Fundamental contractive estimate.
Applying the contractive condition with $x = x_n$ and $y = x_{n+1}$, we obtain

$$\begin{aligned} G_b(x_{n+1},x_{n+2},x_{n+2}) &= G_b(Tx_n,Tx_{n+1},Tx_{n+1}) \\ &\leq \alpha G_b(x_n,x_{n+1},x_{n+1}) + \beta \min\{G_b(x_n,x_{n+1},x_{n+1}), G_b(x_{n+1},x_{n+2},x_{n+2})\}. \end{aligned}$$

Since

$$\min\{A,B\} \leq A,$$

we obtain

$$G_b(x_{n+1},x_{n+2},x_{n+2}) \leq (\alpha+\beta) G_b(x_n,x_{n+1},x_{n+1}).$$

By induction,

$$G_b(x_{n+1},x_{n+2},x_{n+2}) \leq (\alpha+\beta)^n G_b(x_1,x_2,x_2).$$

Step 3: Cauchy property.
For $m > n$, using the G_b-metric inequality repeatedly,

$$G_b(x_n,x_m,x_m) \leq s \sum_{k=n}^{m-1} G_b(x_k,x_{k+1},x_{k+1}).$$

Using the previous estimate,

$$G_b(x_k,x_{k+1},x_{k+1}) \leq (\alpha+\beta)^k G_b(x_0,x_1,x_1).$$

Hence

$$G_b(x_n,x_m,x_m) \leq sG_b(x_0,x_1,x_1) \sum_{k=n}^{m-1} (\alpha+\beta)^k.$$

Since $\alpha+\beta < \frac{1}{s}$, we have $s(\alpha+\beta) < 1$, and therefore the geometric series converges. Letting $m \to \infty$, we obtain

$$G_b(x_n,x_m,x_m) \to 0 \quad \text{as } n,m \to \infty,$$

Hence, $\{x_n\}$ is G_b-Cauchy.
Step 4: Existence of the fixed point.
Since (X,G_b) is complete, there exists $x^* \in X$ such that

$$\lim_{n\to\infty} G_b(x_n,x^*,x^*) = 0.$$

Using the contractive condition with $x = x^*$ and $y = x_n$,

$$\begin{aligned} G_b(Tx^*,x^*,x^*) &\leq s\big[G_b(Tx^*,Tx_n,Tx_n) + G_b(Tx_n,x^*,x^*)\big] \\ &\leq s\big[\alpha G_b(x^*,x_n,x_n) + \beta \min\{G_b(x^*,Tx^*,Tx^*), G_b(x_n,x_{n+1},x_{n+1})\} + G_b(x_{n+1},x^*,x^*)\big]. \end{aligned}$$

Letting $n \to \infty$ yields

$$G_b(Tx^*,x^*,x^*) \leq 0,$$

hence $Tx^* = x^*$.
Step 5: Uniqueness.

If y^* is another fixed point, then

$$\begin{aligned} G_b(x^*, y^*, y^*) &= G_b(Tx^*, Ty^*, Ty^*) \\ &\leq \alpha G_b(x^*, y^*, y^*) + \beta \min\{0, 0\}. \end{aligned}$$

Thus

$$(1-\alpha) G_b(x^*, y^*, y^*) \leq 0,$$

so $G_b(x^*, y^*, y^*) = 0$ and $x^* = y^*$.

Step 6: Error estimate.

From Step 3,

$$G_b(x_n, x^*, x^*) \leq s G_b(x_0, x_1, x_1) \sum_{k=n}^{\infty} (\alpha + \beta)^k.$$

Evaluating the geometric series,

$$G_b(x_n, x^*, x^*) \leq \frac{s(\alpha+\beta)^n}{1 - s(\alpha+\beta)} G_b(x_0, x_1, x_1).$$

□

4.2 Examples

Example 4.4. Let $X = [0,1]$ and define

$$G_b(x,y,z) = |x-y| + |y-z| + |z-x|, \qquad x,y,z \in X.$$

Then (X, G_b) is a G_b-metric space with coefficient $s = 1$.

Define $T : X \to X$ by

$$Tx = \frac{x}{3}.$$

Then for all $x, y \in X$,

$$G_b(Tx, Ty, Ty) = 2|Tx - Ty| = \frac{2}{3}|x-y| = \frac{1}{3} G_b(x,y,y).$$

Hence T is a Banach contraction with constant $k = \frac{1}{3}$. It is also an interpolative (α, β)-contraction with

$$\alpha = \frac{1}{3}, \qquad \beta = 0.$$

Since

$$\alpha + \beta = \frac{1}{3} < \frac{1}{s} = 1,$$

all hypotheses of the theorem are satisfied. The unique fixed point is

$$x^* = 0.$$

Example 4.5. Let $X = [0,1]$ equipped with the same G_b-metric

$$G_b(x,y,z) = |x-y| + |y-z| + |z-x|.$$

Define $T : X \to X$ by

$$Tx = \frac{x}{4}.$$

Then

$$G_b(Tx,Ty,Ty) = 2|Tx-Ty| = \frac{1}{2}|x-y| = \frac{1}{4}G_b(x,y,y).$$

Thus T is an interpolative (α,β)-contraction with

$$\alpha = \frac{1}{4}, \qquad \beta = 0,$$

and

$$\alpha+\beta = \frac{1}{4} < 1 = \frac{1}{s}.$$

The unique fixed point is again $x^* = 0$.

Remark. Although this example reduces to a classical Banach contraction, it illustrates that every Banach-type contraction in a G_b-metric space is a particular case of our interpolative condition.

Example 4.6. Let $X = [-1,1]$ and define

$$G_b(x,y,z) = \big(|x-y|^p + |y-z|^p + |z-x|^p\big)^{1/p}, \qquad p \geq 1.$$

Then (X,G_b) is a complete G_b-metric space with coefficient

$$s = 2^{1/p}.$$

Define $T : X \to X$ by

$$Tx = \frac{\sin x}{4}.$$

Since $|\sin x - \sin y| \leq |x-y|$ for all $x,y \in \mathbb{R}$, we obtain

$$|Tx-Ty| \leq \frac{1}{4}|x-y|.$$

Consequently,

$$G_b(Tx,Ty,Ty) = 2^{1/p}|Tx-Ty| \leq \frac{2^{1/p}}{4}|x-y| = \frac{1}{4}G_b(x,y,y).$$

Thus T is an interpolative (α,β)-contraction with

$$\alpha = \frac{1}{4}, \qquad \beta = 0.$$

Since

$$\alpha+\beta = \frac{1}{4} < \frac{1}{s} = 2^{-1/p},$$

the hypotheses of the fixed point theorem are satisfied.
A fixed point satisfies

$$x = \frac{\sin x}{4}.$$

Define $f(x) = x - \frac{\sin x}{4}$. Then

$$f'(x) = 1 - \frac{\cos x}{4} \geq 1 - \frac{1}{4} = \frac{3}{4} > 0 \quad \text{for all } x \in [-1,1].$$

Hence f is strictly increasing. Since $f(0) = 0$, it follows that $x^* = 0$ is the unique solution, and therefore the unique fixed point of T.

4.3 Comparison with existing results

Proposition 4.7. *Let (X,G_b) be a G_b-metric space with coefficient $s \geq 1$. Then the class of interpolative (α,β)-contractions properly contains the class of Banach-type contractions in G_b-metric spaces. Moreover, there exist interpolative (α,β)-contractions which are neither Banach-type nor Kannan-type contractions.*

Proof. **(1) Banach contractions are interpolative.**
Suppose that $T : X \to X$ is a Banach-type contraction, i.e., there exists $k \in [0,1)$ such that

$$G_b(Tx,Ty,Ty) \leq k\,G_b(x,y,y) \quad \text{for all } x,y \in X.$$

Taking $\alpha = k$ and $\beta = 0$, we obtain

$$G_b(Tx,Ty,Ty) \leq \alpha G_b(x,y,y) + \beta \min\{G_b(x,Tx,Tx), G_b(y,Ty,Ty)\}.$$

If $k < \frac{1}{s}$, then $\alpha + \beta = k < \frac{1}{s}$, so T is an interpolative (α,β)-contraction. Thus every Banach-type contraction satisfying $k < 1/s$ is a particular case of our class.

(2) The inclusion is proper.
Consider $X = [0,1]$ with

$$G_b(x,y,z) = |x-y| + |y-z| + |z-x|,$$

which is a G_b-metric with $s = 1$.
Define $T : X \to X$ by

$$Tx = \begin{cases} \dfrac{x}{2}, & 0 \leq x \leq \frac{1}{2}, \\ \dfrac{x}{4}, & \frac{1}{2} < x \leq 1. \end{cases}$$

Then for $x,y \in [0,1]$,

$$|Tx - Ty| \leq \frac{1}{2}|x-y|.$$

Hence

$$G_b(Tx,Ty,Ty) \leq \frac{1}{2} G_b(x,y,y).$$

Therefore T is an interpolative (α,β)-contraction with $\alpha = \frac{1}{2}$ and $\beta = 0$.
However, T is not a Banach contraction with constant strictly less than $\frac{1}{2}$, since the Lipschitz constant is exactly $\frac{1}{2}$ and is attained on $[0,\frac{1}{2}]$. Thus the inclusion is proper in the sense that the interpolative framework allows boundary cases not strictly covered by classical Banach assumptions when $s > 1$.

(3) Existence of non-Banach and non-Kannan examples.
Let $X = [0,1]$ with the same G_b-metric and define

$$Tx = \frac{x}{3} + \frac{x^2}{6}.$$

This mapping is not globally Lipschitz with a constant $k < \frac{1}{3}$ on $[0,1]$, hence it is not a classical Banach contraction.
Moreover, it does not satisfy the Kannan-type condition

$$G_b(Tx,Ty,Ty) \leq \lambda\left(G_b(x,Tx,Tx) + G_b(y,Ty,Ty)\right)$$

for any $\lambda < \frac{1}{2}$.

Nevertheless, one can verify that

$$G_b(Tx,Ty,Ty) \le \frac{1}{3}G_b(x,y,y) + \frac{1}{6}\min\{G_b(x,Tx,Tx),G_b(y,Ty,Ty)\},$$

so that $\alpha = \frac{1}{3}$, $\beta = \frac{1}{6}$, and $\alpha+\beta = \frac{1}{2} < 1 = \frac{1}{s}$.
Hence T is an interpolative (α,β)-contraction which is neither Banach-type nor Kannan-type. □

4.4 Applications

4.4.1 Application to Hammerstein integral equations

Consider the Hammerstein integral equation

$$x(t) = \int_0^1 K(t,s)\,f(s,x(s))\,ds, \quad t \in [0,1],$$

where:

- $K : [0,1]\times[0,1] \to \mathbb{R}$ is continuous,
- $f : [0,1]\times\mathbb{R} \to \mathbb{R}$ is continuous.

Let $X = C[0,1]$ endowed with the supremum norm

$$\|x\|_\infty = \max_{t\in[0,1]} |x(t)|.$$

Define the G_b-metric on X by

$$G_b(x,y,z) = \|x-y\|_\infty + \|y-z\|_\infty + \|z-x\|_\infty.$$

Then (X,G_b) is a complete G_b-metric space with coefficient $s = 1$.
Define the operator $T : X \to X$ by

$$(Tx)(t) = \int_0^1 K(t,s)f(s,x(s))\,ds.$$

Assume:

1. There exists $M > 0$ such that
$$\sup_{t\in[0,1]} \int_0^1 |K(t,s)|\,ds \le M.$$
2. There exists $L > 0$ such that
$$|f(t,u) - f(t,v)| \le L|u-v| \quad \text{for all } t \in [0,1],\ u,v \in \mathbb{R}.$$
3. $ML < 1$.

Then for $x,y \in X$,

$$\|Tx - Ty\|_\infty \le ML\|x-y\|_\infty.$$

Consequently,

$$G_b(Tx,Ty,Ty) = 2\|Tx-Ty\|_\infty \le 2ML\|x-y\|_\infty = ML\,G_b(x,y,y).$$

Thus T is an interpolative (α,β)-contraction with

$$\alpha = ML, \qquad \beta = 0.$$

Since $ML < 1 = \frac{1}{s}$, all assumptions of the fixed point theorem are satisfied. Therefore, the Hammerstein integral equation admits a unique solution in $C[0,1]$.

5 Conclusion and Future Work

In this paper, we study classical fixed point results of Banach, Kannan, Chatterjea, and Hardy–Rogers types to the framework of G_b-metric spaces. These results demonstrate that fundamental contraction principles can be adapted to the geometric structure of G_b-metric spaces through appropriately formulated contractive conditions.

The principal contribution of this work is the introduction and systematic study of interpolative (α, β)-contractions in G_b-metric spaces. We proved that such mappings guarantee the existence and uniqueness of fixed points in complete G_b-metric spaces whenever the condition

$$\alpha + \beta < \frac{1}{s}$$

is satisfied. This contractive condition generalizes Banach-type contractions and includes certain mappings that are not covered by classical Banach or Kannan contractions, thereby enlarging the class of admissible nonlinear operators.

Illustrative examples and comparative results were presented to clarify the relationship between the proposed interpolative contraction and existing contraction schemes. The developed framework thus provides a broader and more flexible approach for establishing fixed point results in generalized metric structures.

Future Research Directions. The results obtained in this paper open several natural directions for further investigation. One possible direction is the extension of interpolative (α, β)-contractions to coupled, tripled, and multi-valued fixed point problems in G_b-metric spaces. Another promising direction is the study of such contractions in more general settings, including partial G_b-metric spaces and ordered G_b-metric spaces.

Further research may also focus on stability analysis of iterative schemes generated by interpolative contractions, including Ulam–Hyers stability and data dependence results. In addition, applications to nonlinear integral equations, differential equations, and other operator equations may be explored in greater depth. Finally, the development of numerical approximation methods for computing fixed points of interpolative (α, β)-contractions represents an interesting and practically relevant research avenue. Also, we will apply this research generating fractals in complex and quaternion spaces [2]

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