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Generalized Form of Difference Sequence Space of Fuzzy Real Number

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Abstract: *The idea of generalized sequence space by using Orlicz function was firstly initiated by Lindenstrauss and Tzafriri. This study explores new sequence space of fuzzy real numbers by using an special type of Orlicz function called modulus functions. We have built a double sequence space and tried to establish some of the properties like linearity, completeness and some containment relations. The fuzzy number sequence space, the fuzzy number difference sequence space, and the fuzzy number multiplier of double sequence space are all the subject of extensive study.*

Keywords: Generalized difference sequence space, Fuzzy real number, Modulus functions, Complete metric space.

1. Introduction

A sequence space is a linear space in which real or complex sequences may be represented. In this paper, the set of real numbers ($\mathbb{R}$) is contrasted with the set of complex numbers($\mathbb{C}$), while ($\mathbb{N}$) denotes the set of nonnegative integers. Let ω stands for the set of all real or complex sequences; ℓ_∞ and c for the set of all bounded sequences and the space of convergent sequences, respectively.
Ganie A. H. [1] developed the notions of fuzzy sets and fuzzy set operations. Then, many writers have addressed different aspects of fuzzy set theory and applications. They studied about fuzzy topological spaces, similarity relations, fuzzy orderings, fuzzy event measurement, and fuzzy mathematical programming. In particular, quantum particle physics is profoundly affected by the concept of fuzzy topology. Most notably, the concept of an intuitionistic fuzzy normed space was presented with regard to string theory. All convergent fuzzy number series were shown to be bounded, and their properties were analyzed to establish this. Numerous authors, not only Altinok et al.[3], have discussed fuzzy number sequences in their own works.

Antesar [2] independently conceptualized the notion of convergence in statistics. Over the years, and under many labels, researchers in subjects as diverse as Fourier analysis, ergodic theory, and number theory have studied statistical convergence. He showed Furthermore, there is a strong relationship between the concepts of probability convergence and statistical convergence and studied the Stone-ech compactification of the natural numbers is related to subsets of statistics and its extensions. Connor, Fridy, Alát, Tripathy and others looked at its ties to summability theory from a sequence space perspective [2]. Recent statistical convergence generalizations have focused on both strong integral summability and the structure of ideals of bounded continuous functions on locally compact spaces.

The Orlicz sequence spaces were defined by Lindenstrauss and Tzafriri [7] in 1979. The study of Orlicz sequence spaces ℓ_M by experts in Banach space theory began for a very specific reason. Lindberg's interest in Orlicz spaces stemmed from his discovery of Banach space with symmetric Schauder bases and complementary subspaces isomorphic to them c_0 or $\ell_p(1 \le p < \infty)$. Further research into Orlicz sequence spaces led them to the conclusion that each and every one of these spaces ℓ_M contain a subspace isomorphic to $\ell_p(1 \le p < \infty)$. Difference sequence space was firstly initiated by Kizmaz [6] in 1981. He generalized the classical sequence spaces ℓ_∞, c_0 and c such that

$$Z(\Delta) = \{x = (x_k) \in \omega : (\Delta x_k) = (x_k - x_{k-1}) \in Z\}$$

where $Z = \ell_\infty, c_0$ and c. Many mathematicians like Et and Kolak, Pahari, Ghimire and Pahari generalized the difference sequence spaces and defined more generalized forms of difference sequence spaces and studied their different linear and topological properties. In 2024, Kaphle et. al. [5] defined the sequence space $\ell_M(X, \Delta, \alpha, P)$ defined by the Orlicz function M and studied the different containment relation between the spaces.

Hardy [11] developed the concept of regular convergence for double sequences in the sense that the double sequence has a limit in the Pringsheim sense and has one-sided limits. In 2005, Altay and Basar [12] defined the double sequence spaces $BS, BS(t), CS_p, CS_{bp}, CS_r$ and BV, studied some of their properties, and demonstrated that they are total paranormed or normed spaces under certain conditions. In order to study the double statistical convergence of a sequence of fuzzy numbers, Savas [14] introduced some new double sequence spaces of fuzzy numbers in 2010. Das, M. [13] introduced certain vector-valued difference double sequences defined by the Orlicz function in 2012 to investigate their various properties. Similarly, Savas and Patterson [15] developed some new double sequence spaces and investigated some of their features in 2007. Fruitful contributions towards the sequence spaces of fuzzy real numbers were given by Tripathy and Sharma [16] [17][18], Talo[19], Dabbas and Battor [20], Mansoor and Battor[21]. Paudel et. al. [8][9][10] studied about various types of difference sequence spaces, studied their linear, topological, completeness, solidness properties and studied about various applications of sequence of fuzzy real numbers in engineering and medical field.

2. Preliminaries and Definitions

2.1 Orlicz Function

Every continuous, non-decreasing, convex function $M: \mathbb{R}^+ \to \mathbb{R}^+$which also satisfies;

1. $M(t) > 0$,
2. $M(t)$ approaches to ∞, as t approaches to ∞
   is said to be an Orlicz function.

If we replace the convexity property of M by the inequality

$$M(x+y) \leq M(x) + M(y),$$

then the function M is called a modulus function.

Let D be the set of all bounded intervals $A = [a,b]$ on the real line $\mathbb{R}$. For any $A, B \in D$ with $A = [a_1, b_1]$ and $B = [a_2, b_2]$, $A \subseteq B$ if $a_2 \leq a_1$ and $b_1 \leq b_2$. Define a relation d on D by

$$d(A,B) = \max\{|a_1 - a_2|, |b_1 - b_2|\}$$

Then clearly, d defines a metric on D and obviously (D, d) is a complete metric space.

2.2 Fuzzy Real Number

A fuzzy real number is a fuzzy set i.e a mapping $\chi: \mathbb{R} \to I = [0,1]$ associating each real number $t \in \mathbb{R}$ with its membership value $\chi(t)$ satisfying that X is

1. normal if there exists a real number t such that $\chi(t) = 1$;
2. convex if for $t, s \in \mathbb{R}$ and $0 \leq \lambda \leq 1, \chi(\lambda t + (1-\lambda)s) \geq \min\{\chi(t), \chi(s)\}$;
3. upper semi continuous if for each $\varepsilon > 0, \chi^{-1}([0, a+\varepsilon))$ is open for all $a \in I$ in the usual topology of $\mathbb{R}$.

2.3 Definitions

The α-level set on a fuzzy set χ is denoted by $\mathcal{X}_\alpha$ and defined by

$$\chi_\alpha = \{t \in \mathbb{R}: \chi(t) \geq \alpha\}$$

The support of a fuzzy number is the set of all those elements of the fuzzy number having membership value greater than zero.

Suppose $R(I)$ denotes the set of all real fuzzy numbers which are upper semi-continuous and have compact support. In other words, if $\chi \in R(I)$ then for any $\alpha \in [0,1]$,

$$\chi_\alpha = \begin{cases} \{t: \chi(t) \geq \alpha\} & \text{for } \alpha \in (0,1] \\ \{t: \chi(t) < \alpha\} & \text{for } \alpha = 0 \end{cases} \tag{2.1}$$

The addition and scalar multiplication on $R(I)$ are defined as

$$(\chi + \mathcal{Y})_\alpha = (\chi_\alpha) + (\mathcal{Y}_\alpha) \text{ and } (\alpha\chi)_\alpha = \alpha(\mathcal{X})_\alpha \text{ for all } \alpha \in [0,1].$$

Consider a mapping $\bar{d}: R(I) \times R(I) \to \mathbb{R}$ by the relation

$$\bar{d}(\chi, \mathcal{Y}) = \sup d(\chi_\alpha, \mathcal{Y}_\alpha) \text{ for } 0 \leq \alpha \leq 1$$

Then $\bar{d}$ defines a metric on $R(I)$ and $(R(I), \bar{d})$ forms a complete metric space. Also, for any $\chi, \mathcal{Y} \in R(I), \chi \leq \mathcal{Y}$ if and only if $\chi_\alpha \leq \mathcal{Y}_\alpha$ for $\alpha \in [0,1]$ and $\chi_\alpha = [x_1^\alpha, x_2^\alpha]$ and $\mathcal{Y}_\alpha = [y_1^\alpha, y_2^\alpha]$.

Let $\lambda: R(I) \times R(I) \to \mathbb{R}$ be defined by $\lambda(\chi, \mathcal{Y}) = \sup\limits_{0 \leq \alpha \leq 1} \lambda_\alpha(\chi_\alpha, \mathcal{Y}_\alpha)$ where $\lambda_\alpha: R(I) \times R(I) \to \mathbb{R}$ is defined by

$$\lambda_\alpha(\chi_\alpha, \mathcal{Y}_\alpha) = \min\{|\chi_1^\alpha - \mathcal{Y}_1^\alpha|, |\chi_2^\alpha - \mathcal{Y}_2^\alpha|\}.$$

Similarly, $\rho: R(I) \times R(I) \to \mathbb{R}$ be defined by $\rho(\chi, \mathcal{Y}) = \sup\limits_{0 \leq \alpha \leq 1} \rho_\alpha(\chi_\alpha, \mathcal{Y}_\alpha)$ where $\rho_\alpha: R(I) \times R(I) \to \mathbb{R}$ is defined by

$$\rho_\alpha(\chi_\alpha, \mathcal{Y}_\alpha) = \max\{|\chi_1^\alpha - \mathcal{Y}_1^\alpha|, |\chi_2^\alpha - \mathcal{Y}_2^\alpha|\}.$$

A sequence of fuzzy numbers $\mathcal{X} = (\mathcal{X}_k)$ is a function $\chi: \mathbb{N} \to R(I)$, where $\mathbb{N} = \{0,1,2, \dots\}$. The number $\mathcal{X}_k$ is the k^{th} value of the function at $k \in \mathbb{N}$ and is the k^{th} term of the sequence.

In this article we let ω as the set of all real or complex-valued double sequences which is a vector space with coordinate wise addition and scalar multiplication. Then any vector subspace of ω is called sequence space.

3. Double Sequence Space Research for Fuzzy Numbers

A double sequence of fuzzy numbers $\mathcal{X} = (\mathcal{X}_{nk})$ is a function $\mathcal{X}: \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{R}(I)$, the set of fuzzy real numbers. The fuzzy number $\mathcal{X}_{nk}$ is the value of the function at the point $(n,k) \in \mathbb{N} \times \mathbb{N}$ and is called $(n,k)^{th}$-term of the double sequence.

Example 3.1. The function $\mathcal{X}: \mathbb{N} \times \mathbb{N} \longrightarrow$ defined by $\mathcal{X}(n,k) = \frac{n}{n+k}$ is a double sequence. A double sequence $\mathcal{X} = (\mathcal{X}_{nk})$ of fuzzy numbers is said to be bounded if there exist fuzzy numbers M and m such that $m \leq \mathcal{X}_{nk} \leq M$ for all $n,k \in \mathbb{N}$.
A double sequence $\mathcal{X} = (\mathcal{X}_{nk})$ of fuzzy numbers is said to be Cauchy double sequence if $\forall \epsilon > 0\ \exists n_0 \in \mathbb{N}: d\left(\mathcal{X}_{nk}^{i}, \mathcal{X}_{nk}^{j}\right) < \epsilon$ for $\min(i,j) \geq n_0$.
We also say that the double sequence $\mathcal{X} = (\mathcal{X}_{nk})$ of fuzzy numbers converges to a fuzzy number $\mathcal{X}_o$ if $\mathcal{X}_{nk}$ tends to $\mathcal{X}_o$ as both n and k independently tend to ∞.

A double sequence space $\omega^{\mathbb{F}}$ of fuzzy numbers is said to be solid if $(Y_{nk}) \in \omega^{\mathbb{F}}$, whenever $|\mathcal{Y}_{nk}| \leq |\mathcal{X}_{nk}|$ for all $n,k \in \mathbb{N}$ for some $(\mathcal{X}_{nk}) \in \omega^{\mathbb{F}}$.

Tripathy and Sharma [10] in 2011 defined classes of double sequence spaces of fuzzy numbers defined by Orlicz function as follows:

$$\left(2^{\ell}\infty\right)_{\mathbb{F}}(M) = \left\{\mathcal{X} = (\mathcal{X}_{nk}) \in \mathbb{R}(I): \sup_{n,k} \mathcal{M}\left(\frac{d(\mathcal{X}_{nk}, 0)}{\rho}\right) < \infty, \text{ for some } \rho > 0\right\}$$

$$2^{C_{\mathbb{F}}(M)} = \left\{\mathcal{X} = (\mathcal{X}_{nk}) \in \mathbb{R}(I): \lim_{n,k} \mathcal{M}\left(\frac{d(\mathcal{X}_{nk}, \mathcal{X})}{\rho}\right) = 0, \text{ for some } \rho > 0\right\}$$

$$(2^{C_0})_{\mathbb{F}}(M) = \left\{\mathcal{X} = (\mathcal{X}_{nk}) \in \mathbb{R}(I): \lim_{n,k} \mathcal{M}\left(\frac{d(\mathcal{X}_{nk}, 0)}{\rho}\right) = 0, \text{ for some } \rho > 0\right\}$$

In 2020, Dabbas and Battor[20] defined the classes of sequences of fuzzy real numbers using double Orlicz functions as follows:

$$\ell_{\infty}^{\mathbb{F}}(\mathcal{X}, \mathcal{P}) = \left\{(\mathcal{X}_{nk}, \mathcal{Y}_{nk}) \in \omega_F: \sup_{n,k} \left\{\left(\lambda\left(\frac{d(\mathcal{X}_{nk}, 0)}{r}\right)\right\}^{P_{nk}} \vee \left(\rho\left(\frac{d(\mathcal{Y}_{nk}, 0)}{r}\right)\right)^{P_{nk}} < \infty\right\} \text{ for some } r > 0\right\}$$

$$C^{\mathbb{F}}(\mathcal{X}, \mathcal{P}) = \left\{(\mathcal{X}_{nk}, \mathcal{Y}_{nk}) \in \omega_F: \lim_{n,k} \left\{\left(\lambda\left(\frac{d(\mathcal{X}_{nk}, X_1)}{r}\right)\right)^{P_{nk}} \vee \left(\rho\left(\frac{d(\mathcal{Y}_{nk}, X_2)}{r}\right)\right)^{P_{nk}} = 0\right\} \text{ for some } r > 0\right\}$$

$$C_o^{\mathbb{F}}(\mathcal{X}, \mathcal{P}) = \left\{(\mathcal{X}_{nk}, \mathcal{Y}_{nk}) \in \omega_F: \lim_{n,k} \left\{\left(\lambda\left(\frac{d(\mathcal{X}_{nk}, 0)}{r}\right)\right\}^{P_{nk}} \vee \left(\rho\left(\frac{d(\mathcal{Y}_{nk}, 0)}{r}\right)\right)^{P_{nk}} = 0\right\} \text{ for some } r > 0\right\}$$

and studies different properties of the classes. Paudel et.al. [8] defined the sequence spaces of fuzzy real numbers by using the concept of Orlicz function as follows:

$$\ell_\infty^{\mathbb{F}}(\mathcal{M},\lambda,\rho)=\left\{(\mathcal{X}_{nk})\in\omega^F:\sup_{n,k}\mathcal{M}\left(\frac{\lambda(\mathcal{X}_{nk},0)}{r}\right)<\infty;\sup_{n,k}\mathcal{M}\left(\frac{\rho(\mathcal{X}_{nk},0)}{r}\right)<\infty \text{ for some } r>0\right\}$$

$$C_r^{\mathbb{F}}(\mathcal{M},\lambda,\rho)=\left\{(\mathcal{X}_{nk})\in\omega^F:\lim_{n,k}\mathcal{M}\left(\frac{\lambda(\mathcal{X}_{nk},\mathcal{L})}{r}\right)=0;\lim_{n,k}\mathcal{M}\left(\frac{\rho(\mathcal{X}_{nk},\mathcal{L})}{r}\right)=0 \text{ for some } r>0\right\}$$

$$C_o^{\mathbb{F}}(\mathcal{M},\lambda,\rho)=\left\{(\mathcal{X}_{nk})\in\omega^F:\lim_{n,k}\mathcal{M}\left(\frac{\lambda(\mathcal{X}_{nk},0)}{r}\right)=0;\lim_{n,k}\mathcal{M}\left(\frac{\rho(\mathcal{X}_{nk},0)}{r}\right)=0 \text{ for some } r>0\right\}$$

They studied different properties like linearity, completeness and solidity of these spaces.

In this article we define the difference sequence spaces of fuzzy real numbers $W^{\mathbb{F}}(\Delta_v^m,F,u,p)$. A series of modulus functions and a series of multipliers are used to broaden the scope of this investigation into fuzzy numbers. Here, we define following class as follows:

$$W^{\mathbb{F}}(\Delta_v^m,F,u,p)=\left\{\mathcal{X}=(\mathcal{X}_k)\in W(E):\frac{1}{n}\sum_{s=1}^{n}\left[f_k\left(\sup_k\bar{d}_k\left(u_k\Delta_v^m\mathcal{X}_{k,s},L_k\right)\right)\right]^{p_k}\to 0 \text{ as } n\to\infty\right\}$$

where,

$\Delta_v^m X_{k,s}=\sum_{i=0}^{m}(-1)^i\binom{m}{i}\mathcal{X}_{k+vi,s}$ and $F=(f_k)$ be a sequence of modulus functions, $p=(p_k)$ consists of a finite series of positive integers, and $u=(u_k)$ consists only of positive integers.

Let $p=(p_{k,l})$ constitute a pairwise sequence of positive real numbers with

$$0\le p_{k,l}\le\sup_{k,l}p_{k,l}=H_1, \text{ and let } K=\max\{1,2^{H_1-1}\}.$$

Then we use the following inequality in this article

$$\left|a_{k,l}+b_{k,l}\right|^{p_{k,l}}\le K\left(\left|a_{k,l}\right|^{p_{k,l}}+\left|b_{k,l}\right|^{p_{k,l}}\right) \tag{3.1}$$

for the Complex-plane factorable sequences $a_{k,l}$ and $b_{k,l}$. For isolated sequence spaces, we will employ the next inequality.

Let $p=(p_k)$ be a sequence of positive real numbers with

$$0\le p_k\le\sup_k p_k=H, D=\max\{1,2^{H-1}\}$$

then

$$|a_k+b_k|^{p_k}\le D(|a_k|^{p_k}+|b_k|^{p_k}) \tag{3.2}$$

For all k and $(a_k),(b_k)\in\mathbb{C}$. Also, for all $a\in\mathbb{C}$.

$$|a|^{p_k}\le\max\{1,|a|^H\} \tag{3.3}$$

4. Main Results

Lemma 4.1. *Let* $a_k\ge 0, b_k\ge 0$ *for all* k *consist of an unending succession of complex numbers, and* $1\le p_k\le supp_k<\infty$, *then*

$$\left(\sum_k|a_k+b_k|^{p_k}\right)^{1/M}\le\left(\sum_k|a_k|^{p_k}\right)^{\frac{1}{M}}+\left(\sum_k|b_k|^{p_k}\right)^{\frac{1}{M}}, where$$

$M=max(1,H), H=supp_k$.

Theorem 4.2. *Let $p = (p_k)$ exist as a finite series of positive real numbers and $u = (u_k)$ consist solely of positive integers. Then $W^{\mathbb{F}}(\Delta_v^m, F, u, p)$ forms a linear space over the real number.*

Proof:

Let $\mathcal{X} = (\mathcal{X}_k)$ and $\mathcal{Y} = (\mathcal{Y}_k) \in W^{\mathbb{F}}$ and $\alpha, \beta \in \mathbb{R}$. Then

$$\frac{1}{n}\sum_{s=1}^{n}\left[f_k\left(\sup_k \bar{d}_k\left(u_k\Delta_v^m\mathcal{X}_{k,s}, L_k\right)\right)\right]^{p_k} \to 0 \text{ as } n \to \infty \quad \text{and}$$

$$\frac{1}{n}\sum_{s=1}^{n}\left[f_k\left(\sup_k \bar{d}_k\left(u_k\Delta_v^m\mathcal{Y}_{k,s}, L_k\right)\right)\right]^{p_k} \to 0 \text{ as } n \to \infty$$

$$\text{So, } \frac{1}{n}\sum_{s=1}^{n}\left[f_k\left(\sup_k \bar{d}_k\left(u_k\Delta_v^m(\alpha\mathcal{X}_{k,s} + \beta\mathcal{Y}_{k,s}), L_k\right)\right)\right]^{p_k}$$

$$= \alpha\frac{1}{n}\sum_{s=1}^{n}\left[f_k\left(\sup_k \bar{d}_k\left(u_k\Delta_v^m\mathcal{X}_{k,s}, L_k\right)\right)\right]^{p_k} + \beta\frac{1}{n}\sum_{s=1}^{n}\left[f_k\left(\sup_k \bar{d}_k\left(u_k\Delta_v^m\mathcal{Y}_{k,s}, L_k\right)\right)\right]^{p_k}$$

$$\to 0 \text{ as } n \to \infty$$

So as a consequence of modulus functions' subadditivity property $f(\lambda x) \leq (1 + [|\lambda|])f(x)$.

$$\alpha\mathcal{X} + \beta\mathcal{Y} \in W^{\mathbb{F}}(\Delta_v^m, F, u, p).$$

Hence $W^{\mathbb{F}}(\Delta_v^m, F, u, p)$ is a linear space.

Theorem 4.3. *Let (E_k, d_k) follow a regularity in the set of all closed metric spaces and let us assume that there is a finite sequence (p_k) of positive real integers such that $\inf p_k > 0$. Then $W^{\mathbb{F}}(\Delta_v^m, F, u, p)$ is a complete metric space, defined from the point of view of the metric*

$$g(\mathcal{X}, \mathcal{Y}) = \sum_{i=1}^{m} f_k\left(\sup_k \bar{d}_k(\mathcal{X}_{k,i}, \mathcal{Y}_{k,i})\right) + \sup_n\left[\frac{1}{n}\sum_{s=1}^{n}\left(f_k\left(\sup_k \bar{d}_k\left(u_k\Delta_v^m\mathcal{X}_{k,s}, u_k\Delta_v^m\mathcal{Y}_{k,s}\right)\right)\right)^{p_k}\right]^{1/M}$$

Proof:

Let $(\mathcal{X}^{(q)})$ be a Cauchy sequence in $W^{\mathbb{F}}(\Delta_v^m, F, u, p)$ where

$$\mathcal{X}^{(q)} = \left(\left(\mathcal{X}_{k,s}^{(q)}\right)_{s=1}^{\infty}\right)_{k=1}^{\infty} \in W^{\mathbb{F}}(\Delta_v^m, F, u, p) \text{ for each } q \in \mathbb{N}.$$

Then $g(\mathcal{X}^{(q)}, \mathcal{X}^{(r)}) \to 0$ as $r \to \infty$

This means

$$\sum_{i=1}^{m} f_k\left(\sup_k \bar{d}_k\left(\mathcal{X}_{k,i}^{(q)}, \mathcal{X}_{k,i}^{(r)}\right)\right) + \sup_n\left[\frac{1}{n}\sum_{s=1}^{n}\left(f_k\left(\sup_k \bar{d}_k\left(u_k\Delta_v^m\mathcal{X}_{k,s}^{(q)}, u_k\Delta_v^m\mathcal{X}_{k,s}^{(r)}\right)\right)\right)^{p_k}\right]^{\frac{1}{M}} \to 0 \text{ as } r \to \infty$$

Therefore, it follows that

$$\sum_{i=1}^{m} f_k\left(\sup_k \bar{d}_k\left(\mathcal{X}_{k,i}^{(q)}, \mathcal{X}_{k,i}^{(r)}\right)\right) \to 0 \text{ as } q, r \to \infty. \tag{4.1}$$

and

$$\sup_n \left[\frac{1}{n}\sum_{s=1}^{n}\left(f_k\left(\sup_k \bar{d}_k\left(u_k\Delta_v^m \mathcal{X}_{k,s}^{(q)}, u_k\Delta_v^m \mathcal{X}_{k,s}^{(r)}\right)\right)\right)^{p_k}\right]^{1/M} \to 0 \text{ as } q, r \to \infty \qquad (4.2)$$

The resulting equation gives us

$$f_k\left(\sup_k \bar{d}_k\left(\mathcal{X}_{k,i}^{(q)}, \mathcal{X}_{k,i}^{(r)}\right)\right) \to 0 \qquad (4.3)$$

$$\text{as } q, r \to \infty \text{ for each } i = 1,2,\dots,m$$

In contrast, (f_k) is a sequence of modulus functions, so we must have

$$\sup_k \bar{d}_k\left(\mathcal{X}_{k,i}^{(r)}\right) \to 0 \text{ as } q, r \to \infty \text{ for each } i = 1,2,\dots,m$$

Therefore, $\left\{\mathcal{X}_{k,i}^{(q)}\right\}$ is a Cauchy sequence in E_k such that for any $k, i = 1,2,\dots,m$. Because (f_k) is a sequence of modulus functions, we again get the following from equation:

$$\sup_k \bar{d}_k\left(u_k\Delta_v^m \mathcal{X}_{k,s}^{(q)}, u_k\Delta_v^m \mathcal{X}_{k,s}^{(r)}\right) \to 0 \text{ as } q, r \to \infty \text{ for each } s = 1,2,\dots,n$$

Thus, $\left(u_k\Delta_v^m \mathcal{X}_{k,s}^{(q)}\right)$ is a Cauchy sequence in the field E_k for all $s = 1,2,\dots,n$ and all $k \in \mathbb{N}$. However, considering that every E_k exists in its whole. So let $\mathcal{X}_{k,i}^{(q)} \to \mathcal{X}_{k,i}$ as $q \to \infty$ for each $i = 1,2,\dots,m$ and for all k and $u_k\Delta_v^m \mathcal{X}_{k,s}^{(q)} \to u_k\Delta_v^m \mathcal{X}_{k,s}$ as $q \to \infty$ for each $s = 1,2,\dots,n$ and for all k. Thus, utilizing equations (4.3), we obtain

$$\sum_{i=1}^{m} f_k\left(\sup_k \bar{d}_k\left(\mathcal{X}_{k,i}^{(q)}, \mathcal{X}_{k,i}\right)\right) \to 0, \text{as } q \to \infty \qquad (4.4)$$

and

$$\sup_n \left[\frac{1}{n}\sum_{s=1}^{n}\left(f_k\left(\sup_k \bar{d}_k\left(u_k\Delta_v^m \mathcal{X}_{k,s}^{(q)}, u_k\Delta_v^m \mathcal{X}_{k,s}^{(r)}\right)\right)\right)^{p_k}\right]^{1/M} \to 0 \text{ as } q \to \infty \qquad (4.5)$$

i.e.

$$g\left(\mathcal{X}^{(q)}, \mathcal{X}\right) \to 0 \text{ as } r \to \infty$$

Now,

Let $\mathcal{X} \in W^{\mathbb{F}}(\Delta_v^m, F, u, p)$.Then,

$$\frac{1}{n}\sum_{s=1}^{n}\left(f_k\left(\sup_k \bar{d}_k\left(u_k\Delta_v^m \mathcal{X}_{k,s}^{(q)}, u_k\Delta_v^m \mathcal{X}_{k,s}^{(r)}\right)\right)\right)^{p_k} \to 0 \text{ as } q \to \infty \text{ for all } n \in \mathbb{N}$$

i.e. for given $\varepsilon > 0$, there exists $q_0 \in \mathbb{N}$ such that

$$\frac{1}{n}\sum_{s=1}^{n}\left(f_k\left(\sup_k \bar{d}_k\left(u_k\Delta_v^m \mathcal{X}_{k,s}^{(q)}, u_k\Delta_v^m \mathcal{X}_{k,s}\right)\right)\right)^{p_k} < \frac{\varepsilon}{3} \qquad (4.6)$$

for all $q > q_0$ and for all $n \in \mathbb{N}$.

Since $\mathcal{X}^{(q)} \in W^{\mathbb{F}}(\Delta_v^m, F, u, p)$, we can find $L^{(q)}$ such that

$$\frac{1}{n}\sum_{s=1}^{n}\left(f_k\left(\sup_k \bar{d}_k\left(u_k\Delta_v^m \mathcal{X}_{k,s}^{(q)}, L_k^{(q)}\right)\right)\right)^{p_k} < \frac{\varepsilon}{3} \qquad (4.7)$$

for all $n > n_0$ where $L_k^{(q)} \in E_k$.

Similarly, for $\mathcal{X}^{(r)} \in W^{\mathbb{F}}(\Delta_v^m, F, u, p)$ we can find $L^{(r)}$ such that

$$\frac{1}{n}\sum_{s=1}^{n}\left(f_k\left(\sup_k \bar{d}_k\left(u_k\Delta_v^m\mathcal{X}_{k,s}^{(r)}, L_k^{(r)}\right)\right)\right)^{p_k} < \frac{\varepsilon}{3} \tag{4.8}$$

for all $n > n_1$ where $L_k^{(r)} \in E_k$.

Consider $n_2 = \max(q_0, n_0, n_1)$. Then

$$\begin{aligned} f_k\left(\sup_k \bar{d}_k\left(L_k^{(q)}, L_k^{(r)}\right)\right) &= \frac{1}{n}\sum_{s=1}^{n}\left(f_k\left(\sup_k \bar{d}_k\left(L_k^{(q)}, L_k^{(r)}\right)\right)\right)^{p_k} \\ &\leq \frac{1}{n}\sum_{s=1}^{n}\left(f_k\left(\sup_k \bar{d}_k\left(u_k\Delta_v^m\mathcal{X}_{k,s}^{(q)}, L_k^{(q)}\right)\right)\right)^{p_k} \\ &+ \frac{1}{n}\sum_{s=1}^{n}\left(f_k\left(\sup_k \bar{d}_k\left(u_k\Delta_v^m\mathcal{X}_{k,s}^{(q)}, u_k\Delta_v^m\mathcal{X}_{k,s}^{(r)}\right)\right)\right)^{p_k} \\ &+ \tfrac{1}{n}\textstyle\sum_{s=1}^{n}\left(f_k\left(\sup_k \bar{d}_k\left(u_k\Delta_v^m\mathcal{X}_{k,s}^{(r)}, L_k^{(r)}\right)\right)\right)^{p_k} < \varepsilon, \end{aligned} \tag{4.9}$$

for all $q, r \geq n_2$

Choose $\varepsilon = f(\varepsilon_1), \varepsilon_1 > 0$ and Then $\bar{d}_k\left(L_k^{(q)}, L_k^{(r)}\right) < \varepsilon_1$ for all $q, r \geq n_2$

i.e. $L_k^{(q)}$ constitutes a Cauchy sequence in E_k. So $L_k^{(q)} \to L_k$ as $q \to \infty$. From equation (4.9), we get

$$\frac{1}{n}\sum_{s=1}^{n}\left(f_k\left(\sup_k \bar{d}_k\left(L_k^{(q)}, L_k\right)\right)\right)^{p_k} < \varepsilon \ \forall q \geq n_2$$

Hence we have

$$\begin{aligned} &\frac{1}{n}\sum_{s=1}^{n}\left(f_k\left(\sup_k \bar{d}_k\left(u_k\Delta_v^m\mathcal{X}_{k,s}, u_k\Delta_v^m\mathcal{Y}_{k,s}\right)\right)\right)^{p_k} \\ &\leq D\frac{1}{n}\sum_{s=1}^{n}\left(f_k\left(\sup_k \bar{d}_k\left(u_k\Delta_v^m\mathcal{X}_{k,s}^{(q)}, u_k\Delta_v^m\mathcal{X}_{k,s}\right)\right)\right)^{p_k} \\ &+ D\frac{1}{n}\sum_{s=1}^{n}\left(f_k\left(\sup_k \bar{d}_k\left(u_k\Delta_v^m\mathcal{X}_{k,s}^{(q)}, L_k^{(q)}\right)\right)\right)^{p_k} \\ &+ D\frac{1}{n}\sum_{s=1}^{n}\left(f_k\left(\sup_k \bar{d}_k\left(L_k^{(q)}, L_k^{(r)}\right)\right)\right)^{p_k} \\ &\leq \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3} + \varepsilon = \tfrac{5\varepsilon}{3} \text{ for all } n \geq n_2. \end{aligned} \tag{4.10}$$

This implies that $\mathcal{X} \in W^{\mathbb{F}}(\Delta_v^m, F, u, p)$ and hence $W^{\mathbb{F}}(\Delta_v^m, F, u, p)$ contains all possible metrics and is therefore a complete metric space.

Theorem 4.4. *Let $p = (p_k)$ and $t = (t_k)$ exist, as positive number sequences, such that $0 < p_k \leq t_k$ for all $k \in \mathbb{N}$ and the sequence (t_k/p_k) be bounded. Then $W^{\mathbb{F}}(\Delta_v^m, F, u, t) \subset W^{\mathbb{F}}(\Delta_v^m, F, u, p)$.*

Proof:

Let $\mathcal{X} \in W^{\mathbb{F}}(\Delta_v^m, F, u, t)$ which implies

$$\frac{1}{n}\sum_{s=1}^{n}\left(f_k\left(\sup_k \bar{d}_k\left(u_k\Delta_v^m\mathcal{X}_{k,s}^{(q)}, L_k\right)\right)\right)^{t_k} \to 0 \text{ as } n \to \infty. \qquad (4.11)$$

Consider $\mu_k = \left(f_k\left(\sup_k \bar{d}_k\left(u_k\Delta_v^m\mathcal{X}_{k,s}, L_k\right)\right)\right)^{t_k}$ and $\lambda_k = (p_k/t_k)$ be such that $0 < \lambda \leq \lambda_k \leq 1$.

Define

$$c_k = \begin{cases} \mu_k, & \text{if } \mu_k \geq 1 \\ 0, & \text{if } \mu_k < 1 \end{cases} \text{ and } d_k = \begin{cases} 0, & \text{if } \mu_k \geq 1 \\ \mu_k, & \text{if } \mu_k < 1 \end{cases}$$

Then we have $\mu_k = c_k + d_k$ and $\mu_k^{\lambda_k} = c_k^{\lambda_k} + d_k^{\lambda_k}$.

Thus, it follows that $c_k^{\lambda_k} \leq c_k \leq \mu_k$ and $d_k^{\lambda_k} \leq d_k^{\lambda}$.

Therefore,

$$\frac{1}{n}\sum_{s=1}^{n}\left(f_k\left(\sup_k \bar{d}_k\left(u_k\Delta_v^m\mathcal{X}_{k,s}^{(q)}, L_k\right)\right)\right)^{p_k} \leq \frac{1}{n}\sum_{s=1}^{n}\left(f_k\left(\sup_k \bar{d}_k\left(u_k\Delta_v^m\mathcal{X}_{k,s}^{(q)}, L_k\right)\right)\right)^{t_k} \to 0 \text{ as } n \to \infty$$

Which implies, $\mathcal{X} \in W^{\mathbb{F}}(\Delta_v^m, F, u, p)$.

Theorem 5. *Let $F = (f_k)$ and $G = (g_k)$ exist as a pair of modulus sequences. Then we have*

$$W^{\mathbb{F}}(\Delta_v^m, F, u, p) \cap W^{\mathbb{F}}(\Delta_v^m, G, u, p) \subset W^{\mathbb{F}}(\Delta_v^m, F + G, u, p)$$

$$W^{\mathbb{F}}(\Delta_v^m, F, u, p) = W^{\mathbb{F}}(\Delta_v^m, G, u, p) \;\; if\, 0 < inf\frac{F(x)}{G(x)} \leq sup\frac{F(x)}{G(x)} < \infty$$

Proof:

The proof is easy so we omit it.

5. Conclusion

In the study, we have used the modulus function to introduce a generalized difference sequence space $W^{\mathbb{F}}(\Delta_v^m, F, u, p)$of fuzzy real numbers. Also, we have shown linearity property of the space. Moreover, by defining a suitable metric, the completeness property is proved, and the inclusion relation $W^{\mathbb{F}}(\Delta_v^m, F, u, t) \subseteq W^{\mathbb{F}}(\Delta_v^m, F, u, p)$ is shown. The finding of the paper spread out the existing theory related to sequence space of fuzzy real numbers and gives a strong mathematical framework for the further investigation in the field of functional analysis and fuzzy application.

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