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The Multi-Symplectic Integrator: A Tool for Certain Classes of Nonlinear Evolution of Wave Equations

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Abstract: *The non-linear evolution of wave equation by the Multi-Symplectic Integrator (M-S-I) which is alike finite difference structure (further six-point structure) is derived for certain classes as Telegraph - Klein Gordon (T-K-G) system, which is combination of two wave equation. The numerical simulation & the conservation stuff to be deliberated.*

Keywords: Multi-symplectic integrator (M-S-I), Telegraph - Klein Gordon (T-K-G) equations, New six-point scheme.

1. Introduction

In engineering and technical field, the non-linear wave spectacles appear in nearly form or alternatively almost wholly. The nonlinear evolution equations (NLEEs) elucidate these spectacles [1, 2,5,18]. We go thru the NLEEs grouping of two wave equation (T-K-G system)

$$\Upsilon_{tt} - \Upsilon_{xx} + \Upsilon + \ell|\Upsilon|^2\,\Upsilon + \vartheta\Upsilon_t = 0 \tag{1.1}$$

where, $\Upsilon(x,t)$ is complex function with t & x are space & time capricious. If $\ell = 0, \vartheta = 1$ it describes telegraph Equation [7, 12,13,14, 15, 17] and $\ell = -1, \vartheta = 0$ its nonlinear K-G Type Equation [12, 13,14,15, 17]. In solid state physics, plasma physics, nonlinear optics and quantum field theory the K-G equations play a substantial character. Examine the explanations of non-linear partial differential structure or NLEEs many powerful methods [13,14,17] were cast-off. The idea of a multi-symplectic integrator, in the form of a finite difference thingy, and its weird speculative outcomes directed by Bridge & Reich. They said it's a super local idea and, like, written out as a finite difference scheme [3,4, 6, 8,9, 10, 11,16]. We kind of break down the structure with finite difference schemes to, like, show how the multi-Symplectic thing works and prove the system's lead thru numerical mock-up.

2. Multi-Symplectic Integrator of Nonlinear Evolution Wave Equations

We consider the following Nonlinear Evolution Wave Equations eq. (1.1)

$$\frac{d^2\Upsilon}{dt^2} - \frac{d^2\Upsilon}{dx^2} + \Upsilon + \ell|\Upsilon|^2\,\Upsilon + \vartheta\frac{d\Upsilon}{dt} = 0$$

If $\Upsilon = r + ih$ where Υ, r, h are function of $x \& t$, and comparing real and imaginary parts equal to 0

$$r_{tt} - r_{xx} + r(1 + \ell(r^2 + h^2)) + \vartheta r_t = 0$$

$$h_{tt} - h_{xx} + h(1 + \ell(r^2 + h^2)) + \vartheta h_t = 0$$

Introducing the canonical momenta

$$r_x = a, \quad r_t = b, \quad h_x = c, \quad h_t = d$$

$$\Rightarrow \qquad \mathrm{K}\omega_t + L\omega_x = \nabla_\omega S(\omega) \tag{2.1}$$

State variable $\omega \in R^{\hbar}$, $\hbar \geq 2$. Hither $\mathrm{K}, L \in R^{q\times q}$ are skew-symmetric matrices and $S : R^q \to R$ is a function of class C^∞. ∇_ω is the typical gradient in R^q. For $S(\omega)$ & $\nabla_\omega S(\omega)$, the scheme is multi-symplectic in the intellect that *K & L* are skew-symmetric matrix archetypal of the *t & x* directions. *S* denotes a Hamiltonian utility [3,10,19].

The equation (2.1) is multi-symplectic in fauna with $\omega = (\kappa, a, b, \nu, c, d)^T \in R^6$

Then we get

$$b_t - a_x + \vartheta b = -\left\{1 + \ell(r^2 + h^2)\right\} r = -pr$$
$$d_t - c_x + \vartheta d = -\left\{1 + \ell(r^2 + h^2)\right\} h = -ph$$
$$r_x = a, \quad r_t = b,$$
$$h_x = c, \quad h_t = d$$

So that $\nabla_\omega S(\omega) = (pr,\ ph,\ a,\ b,\ c,\ d)^T$, Where

$$K = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad L = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Exploiting midpoint difference scheme

$$\frac{b_{q+1/2}^{p+1} - b_{q+1/2}^{p}}{\Delta t} - \frac{a_{q+1}^{p+1/2} - a_{q}^{p+1/2}}{\Delta x} + \vartheta\hat{b} = -\left\{1 + \ell(\hat{r}^2 + \hat{h}^2)\right\}\hat{r}$$

$$\frac{d_{q+1/2}^{p+1} - d_{q+1/2}^{p}}{\Delta t} + \frac{c_{q+1}^{p+1/2} - c_{q}^{p+1/2}}{\Delta x} + \vartheta\hat{d} = -\left\{1 + \ell(\hat{r}^2 + \hat{h}^2)\right\}\hat{h}$$

$$\frac{r_{q+1}^{p+1/2} - r_{q}^{p+1/2}}{\Delta x} = a_{q+1/2}^{p+1/2}$$

$$\frac{r_{q+1/2}^{p+1} - r_{q+1/2}^{p}}{\Delta t} = b_{q+1/2}^{p+1/2}$$

$$\frac{h_{q+1}^{p+1/2} - h_{q}^{p+1/2}}{\Delta x} = c_{q+1/2}^{p+1/2}$$

$$\frac{h_{q+1/2}^{p+1} - h_{q+1/2}^{p}}{\Delta t} = d_{q+1/2}^{p+1/2}$$

Where $\hat{r} = r_{q+1/2}^{p+1/2}, \hat{h} = h_{q+1/2}^{p+1/2}, \hat{b} = b_{q+1/2}^{p+1/2}, \hat{d} = d_{q+1/2}^{p+1/2}$

Eliminating a, *b, c & d*, we have

$$2\frac{r_{q+1/2}^{p+2} - 2r_{q+1/2}^{p+1} - r_{q+1/2}^{p}}{(\Delta t)^2} - \frac{2\left(r_{q+2}^{p+1/2} - 2r_{q+1}^{p+1/2} + r_{q}^{p+1/2}\right)}{(\Delta x)^2}$$
$$+\vartheta\frac{r_{q+1/2}^{p+1} - r_{q+1/2}^{p} + r_{q+3/2}^{p+1} - r_{q+3/2}^{p}}{\Delta t} + \left\{1 + \ell\left(\left(r_{q+1/2}^{p+1/2}\right)^2 + \left(h_{q+1/2}^{p+1/2}\right)^2\right)\right\} r_{q+1/2}^{p+1/2} \tag{2.2}$$
$$+\left\{1 + \ell\left(\left(r_{q+3/2}^{p+1/2}\right)^2 + \left(h_{q+3/2}^{p+1/2}\right)^2\right)\right\} r_{q+3/2}^{p+1/2} = 0$$

and

$$2\frac{h_{q+1/2}^{p+2} - 2h_{q+1/2}^{p+1} - h_{q+1/2}^{p}}{(\Delta t)^2} - \frac{2\left(h_{q+2}^{p+1/2} - 2h_{q+1}^{p+1/2} + h_{q}^{p+1/2}\right)}{(\Delta x)^2}$$
$$+\vartheta\frac{h_{q+1/2}^{p+1} - h_{q+1/2}^{p} + h_{q+3/2}^{p+1} - h_{q+3/2}^{p}}{\Delta t} + \left\{1 + \ell\left(\left(r_{q+1/2}^{p+1/2}\right)^2 + \left(h_{q+1/2}^{p+1/2}\right)^2\right)\right\} h_{q+1/2}^{p+1/2} \tag{2.3}$$
$$+\left\{1 + \ell\left(\left(r_{q+3/2}^{p+1/2}\right)^2 + \left(h_{q+3/2}^{p+1/2}\right)^2\right)\right\} h_{q+3/2}^{p+1/2} = 0$$

Combining (2.2) & (2.3) we come to be

$$2\frac{\Upsilon_{q+1/2}^{p+2} - 2\Upsilon_{q+1/2}^{p+1} + \Upsilon_{q+1/2}^{p}}{(\Delta t)^2} - \frac{2\left(\Upsilon_{q+2}^{p+1/2} - 2\Upsilon_{q+1}^{p+1/2} + \Upsilon_{q}^{p+1/2}\right)}{(\Delta x)^2}$$
$$+\vartheta\frac{\Upsilon_{q+1/2}^{p+1} - \Upsilon_{q+1/2}^{p} + \Upsilon_{q+3/2}^{p+1} - \Upsilon_{q+3/2}^{p}}{\Delta t} + \left\{1 + \ell\left(\left|\Upsilon_{q+1/2}^{p+1/2}\right|^2\right)\right\} \Upsilon_{q+1/2}^{p+1/2} +$$
$$\left\{1 + \ell\left(\left|\Upsilon_{q+3/2}^{p+1/2}\right|^2\right)\right\} \Upsilon_{q+3/2}^{p+1/2} = 0 \tag{2.4}$$

Ensuing the discretization form

$$z_{n+1/2}^{m+1} = \frac{z_{n+1}^{m+1} + z_{n}^{m+1}}{2}, \quad z_{n+1}^{m+1/2} = \frac{z_{n+1}^{m+1} + z_{n+1}^{m}}{2}, \quad z_{n+1/2}^{m+1/2} = \frac{z_{n+1}^{m} + z_{n}^{m} + z_{n}^{m+1} + z_{n+1}^{m+1}}{4}$$

We will come to the ensuing procedure

$$\frac{\Upsilon_{q+1}^{p+2} + \Upsilon_{q}^{p+2} - 2\left(\Upsilon_{q+1}^{p+1} + \Upsilon_{q}^{p+1}\right) + \Upsilon_{q+1}^{p} + \Upsilon_{q}^{p}}{(\Delta t)^2} - \frac{2\left(\Upsilon_{q+1}^{p+1} + \Upsilon_{q+1}^{p} - 2\left(\Upsilon_{q}^{p+1} + \Upsilon_{q}^{p}\right) + \Upsilon_{q-1}^{p+1} + \Upsilon_{q-1}^{p}\right)}{(\Delta x)^2}$$
$$+\vartheta\frac{\Upsilon_{q-1}^{p+1} + 2\Upsilon_{q}^{p+1} + \Upsilon_{q+1}^{p+1} - \left(\Upsilon_{q-1}^{p} + 2\Upsilon_{q}^{p} + \Upsilon_{q+1}^{p}\right)}{\Delta t} + \left\{1 + \ell\left(\left|\Upsilon_{q+1/2}^{p+1/2}\right|^2\right)\right\} \Upsilon_{q+1/2}^{p+1/2} \tag{2.5}$$
$$+\left\{1 + \ell\left(\left|\Upsilon_{q-1/2}^{p+1/2}\right|^2\right)\right\} \Upsilon_{q-1/2}^{p+1/2} = 0$$

3. Conservation Stuff for Multi-Symplectic Integrators

The M-S-I of T-K-G system eq. (2.4) can be rewritten as

$$2\frac{\Upsilon_{q+1/2}^{p+2}-2\Upsilon_{q+1/2}^{p+1}+\Upsilon_{q+1/2}^{p}}{(\Delta t)^2}-\frac{2\left(\Upsilon_{q+2}^{p+1/2}-2\Upsilon_{q+1}^{p+1/2}+\Upsilon_{q}^{p+1/2}\right)}{(\Delta x)^2}$$
$$+\vartheta\frac{\Upsilon_{q+1/2}^{p+1}-\Upsilon_{q+1/2}^{p}+\Upsilon_{q+3/2}^{p+1}-\Upsilon_{q+3/2}^{p}}{\Delta t}+\left\{1+\ell\left(\left|\Upsilon_{q+1/2}^{p+1/2}\right|^2\right)\right\}\Upsilon_{q+1/2}^{p+1/2}+\left\{1+\ell\left(\left|\Upsilon_{q+3/2}^{p+1/2}\right|^2\right)\right\}\Upsilon_{q+3/2}^{p+1/2}=0 \quad (3.1)$$

Taking conjugate

$$2\frac{\overline{\Upsilon_{q+1/2}^{p+2}-2\Upsilon_{q+1/2}^{p+1}+\Upsilon_{q+1/2}^{p}}}{(\Delta t)^2}-\frac{2\overline{\left(\Upsilon_{q+2}^{p+1/2}-2\Upsilon_{q+1}^{p+1/2}+\Upsilon_{q}^{p+1/2}\right)}}{(\Delta x)^2}$$
$$+\vartheta\frac{\overline{\Upsilon_{q+1/2}^{p+1}-\Upsilon_{q+1/2}^{p}+\Upsilon_{q+3/2}^{p+1}-\Upsilon_{q+3/2}^{p}}}{\Delta t}+\overline{\left\{1+\ell\left(\left|\Upsilon_{q+1/2}^{p+1/2}\right|^2\right)\right\}\Upsilon_{q+1/2}^{p+1/2}}+\overline{\left\{1+\ell\left(\left|\Upsilon_{q+3/2}^{p+1/2}\right|^2\right)\right\}\Upsilon_{q+3/2}^{p+1/2}}=0 \quad (3.2)$$

For equation (3.1) take summation over l from 1 to N and multiple of $\overline{\Upsilon_q^{p+1/2}}$

$$\frac{2}{(\Delta t)^2}\left(\sum_{q=1}^{N}\left\{\Upsilon_{q+1/2}^{p+2}-2\Upsilon_{q+1/2}^{p+1}+\Upsilon_{q+1/2}^{p}\right\}\overline{\Upsilon_q^{p+1/2}}\right)-\frac{2}{(\Delta x)^2}\left(\sum_{q=1}^{N}\left\{\Upsilon_{q+2}^{p+1/2}-2\Upsilon_{q+1}^{p+1/2}+\Upsilon_{q}^{p+1/2}\right\}\overline{\Upsilon_q^{p+1/2}}\right)$$
$$+\frac{\vartheta}{\Delta t}\left(\sum_{q=1}^{N}\left\{\Upsilon_{q+1/2}^{p+1}-\Upsilon_{q+1/2}^{p}+\Upsilon_{q+3/2}^{p+1}-\Upsilon_{q+3/2}^{p}\right\}\overline{\Upsilon_q^{p+1/2}}\right)+\left(\sum_{q=1}^{N}\left\{\left\{1+\ell\left(\left|\Upsilon_{q+1/2}^{p+1/2}\right|^2\right)\right\}\Upsilon_{q+1/2}^{p+1/2}\right\}\overline{\Upsilon_q^{p+1/2}}\right) \quad (3.3)$$
$$+\left(\sum_{q=1}^{N}\left\{\left\{1+\ell\left(\left|\Upsilon_{q+3/2}^{p+1/2}\right|^2\right)\right\}\Upsilon_{q+3/2}^{p+1/2}\right\}\overline{\Upsilon_q^{p+1/2}}\right)=0$$

For equation (3.2) take summation over l from 1 to N and multiple of $\Upsilon_q^{p+1/2}$

$$\frac{2}{(\Delta t)^2}\left(\sum_{q=1}^{N}\left\{\overline{\Upsilon_{q+1/2}^{p+2}-2\Upsilon_{q+1/2}^{p+1}+\Upsilon_{q+1/2}^{p}}\right\}\Upsilon_q^{p+1/2}\right)-\frac{2}{(\Delta x)^2}\left(\sum_{q=1}^{N}\left\{\overline{\Upsilon_{q+2}^{p+1/2}-2\Upsilon_{q+1}^{p+1/2}+\Upsilon_{q}^{p+1/2}}\right\}\Upsilon_q^{p+1/2}\right)$$
$$+\frac{\vartheta}{\Delta t}\left(\sum_{q=1}^{N}\left\{\overline{\Upsilon_{q+1/2}^{p+1}-\Upsilon_{q+1/2}^{p}+\Upsilon_{q+3/2}^{p+1}-\Upsilon_{q+3/2}^{p}}\right\}\Upsilon_q^{p+1/2}\right)+\left(\sum_{q=1}^{N}\left\{\overline{\left\{1+\ell\left(\left|\Upsilon_{q+1/2}^{p+1/2}\right|^2\right)\right\}\Upsilon_{q+1/2}^{p+1/2}}\right\}\Upsilon_q^{p+1/2}\right) \quad (3.4)$$
$$+\left(\sum_{q=1}^{N}\left\{\overline{\left\{1+\ell\left(\left|\Upsilon_{q+3/2}^{p+1/2}\right|^2\right)\right\}\Upsilon_{q+3/2}^{p+1/2}}\right\}\Upsilon_q^{p+1/2}\right)=0$$

By equation 3.3, we have

$$\sum_{l=1}^{N}\left(\overline{\Upsilon_{q+1/2}^{p+2}-2\Upsilon_{q+1/2}^{p+1}+\Upsilon_{q+1/2}^{p}}\right)\Upsilon_q^{p+1/2}=\sum_{l=1}^{N}\left(\overline{\Upsilon_{q+1/2}^{p+2}}\Upsilon_q^{p+1/2}+\overline{\Upsilon_{q+1/2}^{p}}\Upsilon_q^{p+1/2}\right)-\sum_{l=1}^{N}\left|\Upsilon_q^{p+1}+\Upsilon_q^{p}\right|+\overline{\Upsilon_0^{p+1/2}}\Upsilon_1^{p+1/2}-\overline{\Upsilon_N^{p+1/2}}\Upsilon_{N+1}^{p+1/2} \quad (3.5)$$

$$\sum_{q=1}^{N}\left(\overline{\Upsilon_{q+2}^{p+1/2}-2\Upsilon_{q+1}^{p+1/2}+\Upsilon_{q}^{p+1/2}}\right)\Upsilon_q^{p+1/2}=\sum_{q=1}^{N}\left(\overline{\Upsilon_{q+2}^{p+1/2}}\Upsilon_q^{p+1/2}+\overline{\Upsilon_{q}^{p+1/2}}\Upsilon_q^{p+1/2}\right)-\sum_{q=1}^{N}\left|\Upsilon_{q+1}^{p}+\Upsilon_q^{p}\right|+\overline{\Upsilon_0^{p+1/2}}\Upsilon_1^{p+1/2}-\overline{\Upsilon_N^{p+1/2}}\Upsilon_{N+1}^{p+1/2} \quad (3.6)$$

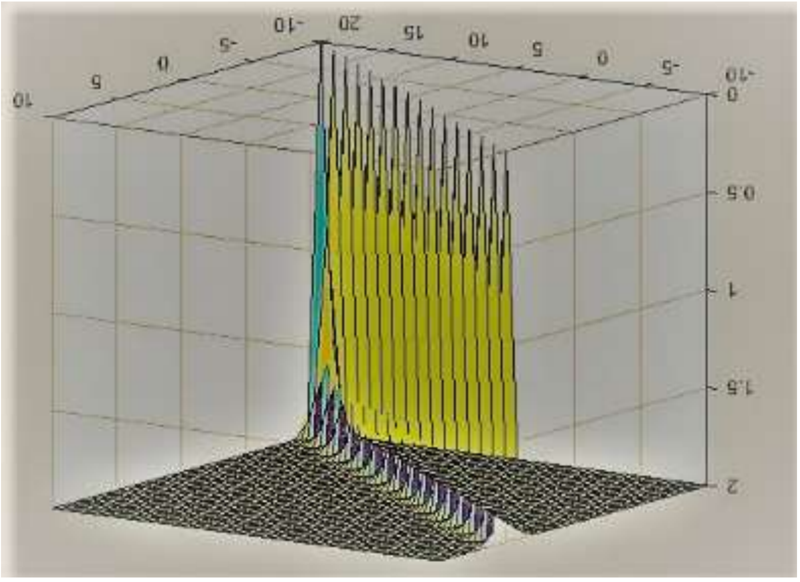
$$\sum_{q=1}^{N}\left(\Upsilon_{q+1/2}^{p+1}-\Upsilon_{q+1/2}^{p}\right)\overline{\Upsilon_q^{p+1}}=\sum_{q=1}^{N}\left(\Upsilon_{q+3/2}^{p+1}-\Upsilon_{q+3/2}^{p}\right)\overline{\Upsilon_q^{p+1/2}}+\left(\Upsilon_{1/2}^{p+1}-\Upsilon_{1/2}^{p}\right)\overline{\Upsilon_q^{p+1/2}}+\left(\Upsilon_{N+1/2}^{p+1}-\Upsilon_{N+1/2}^{p}\right)\overline{\Upsilon_q^{p+1/2}} \quad (3.7)$$

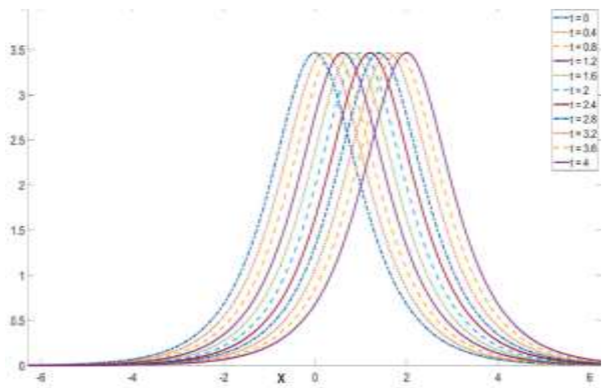
Putting the values from (3.5), (3.6), (3.7) in (3.3) we have

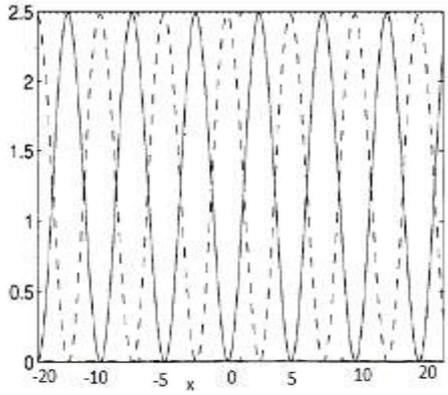
$$\sum_{q=1}^{N}\left|\Upsilon_q^{p+1}+\Upsilon_q^{p}\right|^2=\sum_{q=1}^{N}\left|\Upsilon_{q+1}^{p}+\Upsilon_q^{p}\right|^2.$$

4. Numerical Mock-up

We extant the numerical upshot of the prototypical structure consuming the M-S-I thru initial condition engaged as $\Upsilon(x,0)=\left(\frac{\ell+1}{w}\right)^{1/2}\tanh\left(\frac{(\ell+2w)}{2}x\right)e^{irx}$ with lattice size $\Delta x\otimes\Delta t=0.5\otimes 0.5$ and $-10\le x\le 10, -10\le t\le 10$, $\ell=1, r=-0.5, w=0.5$.







5. Conclusion

In this work, we demonstrate the M-S-I method for the Telegraph–Klein–Gordon system. We also present numerical experiments. The results show that the M-S-I method performs well in simulating soliton evolution and preserves energy effectively. It appears to be suitable for long-time simulations due to its strong energy conservation properties.

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