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On Sequential Henstock Stieltjes delta Integral of Real Valued Functions on Time Scales

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Abstract: *The purpose of this paper is to introduce the concept of Sequential Henstock Stieltjes delta integral of real valued functions on time scales; which is a generalizations of some of the existing Henstock-delta type integrals in the literature and investigate a number of properties such as linearity, uniqueness and Cauchy criterion of this integral.*

Keywords: Sequential Henstock integral, Delta integral, Sequential Henstock integral, Time scales.

1. Introduction and Preliminaries

The Stieltjes-type Riemann integral has been deemed not sufficient enough for advanced mathematical analysis; since functions that are nowhere-continuous, extremely oscilliating and Dirichlet are not Riemann Stieltjes integrable. To remedy these shortcomings, a more potent and easier to handle integral than the Lebesgue integral was introduced independently by R. Henstock and J. Kurzweil in 1955 and 1957 respectively. This integral, well known as the Henstock integral generalizes the Riemann Integral and is equivalent to the Denjoy and Perron integrals. The Henstock integral is also easier and more reliable than the Wiener, Feynmann and Lebesgue integrals (see [1-25]). Interestingly, Guseinov [4-6] introduced the Riemann delta integral; then Park et al. [17] gave some conditions for the Riemann delta integrability of functions on compact interval [*a*, *b*]. Mozyrska et al. [16] studied the Riemann-Stieltjes delta integral.

The calculus of time scales was introduced by Stefan [23] . Thompson [24] applied time scales theory to the concept of Henstock-Kurzweil integral. Some of the basic properties of Henstock delta integral on time scales were introduced by Peterson and Thompson [20].

Recently, the authors [8-12] established the concept of the Sequential characterization of the Henstock integrals; developed by Paxton [19] for certain class of functions and obtained their properties. Then, Yang and Yabin [25] studied the sequential Henstock-Kurzweil delta integral on time scales.

However, most of the results involving Henstock-delta integral via sequence approach have been proved for real valued functions defined on the classical interval ([a,b]). Hence, there is need of extending to the Stieltjes setting of the Sequential Henstock delta integral, that involve integration of function *f* with respect to non decreasing function *g*; thereby providing better results and wider applications. In this paper, we introduce the concept and basic properties of the Sequential Henstock–Stieltjes delta integral for real-valued functions on time scales.

Let R denote the set of real numbers. A partition P of $[a,b]$ is a finite collection of interval point pairs $\theta=\{[w_{i-1},w_i],t_i\}$ where $t_i \in [a,b]$ for all i = 1,2,...,*n*.

Given a sequence of δ_n-fine partitions $\theta_n = \{[w_{(i-1)_n}, w_{i_n}], t_{i_n}\}_{i=1}^{m_n}$, we write

$S(h,\theta_n) = \sum_{i=1}^{n} h(t_{i_n})(w_{i_n} - w_{(i-1)_n})$ for Sequential Henstock sums over θ_n whenever $h:[a,b] \to R.$

A sequence of gauge on $[a, b]$ is a sequence of positive real-valued functions $\delta:[a,b] \to R^+$. Let $\delta_n > 0$, then a sequence of partitions θ_n is Sequential Henstock δ_n-fine if every *subinterval* $[w_{(i-1)_n}, w_{i_n}]$ satisfies $[w_{(i-1)_n}, w_{i_n}] \subset [t_{i_n} - \delta(t_{i_n}), t_{i_n} - \delta(t_{i_n})]$.

We give some notions and other preliminaries in this section.
A time scale T is a nonempty closed subset of R equipped with the topology inherited from the standard topology on *R*. We use $[a,b]_T$ to denote a time scale interval where

$[a,b]_T = T \cap [a,b]$ and $a, b \in T$.

Suppose $t \in T$ such that $a \le t \le b.$, then A time scale *T* is said to be isolated if *t* is right-scattered and left-scattered for all $t \in T$ where $t \ne \sup(t)$ and $t \ne \inf(t)$.

A function $h:[a,b] \to R$ is called *rd*–continuous provided it is continuous at all right-dense $t \in T$ and its' left sided limit exists (finite) at left dense points in *T*.

Definition 1.1 ([9]) (Henstock Stieltjes Integral) Let $\rho:[a,b] \to R$ be an increasing function. A function $h:[a,b] \to R$ is Henstock Stieltjes integrable with respect to ρ on [*a*, *b*] to a number $\alpha \in R$ if for any $\varepsilon > 0$ there is a positive gauge functions $\delta(x) > 0$ such that

$$|\sum_{i=1}^{n} h(t_i)(\rho(w_i) - \rho(w_{(i-1)})) - \alpha| < \varepsilon$$

whenever $\theta = \{[w_{i-1}, w_i], t_i\}$ is a δ-fine partition on [*a*,*b*]. We say that α is Henstock Stieltjes integral of *h* on [a,b]. i.e $\alpha = (HS)\int_a^b h d\rho$.

We use *HS* [*a*,*b*] to denote the set of all Henstock Stieltjes integrable functions on [*a*, *b*].

Remark 1.2 If ρ is an identity function in Definition 1.1, we obtain a definition for Henstock integral (see Iluebe and Mogbademu [10]).

Definition 1.3 ([12]). *Let* $\rho:[a,b] \to R$ be an increasing function. A function $h:[a,b] \to R$ is Sequential Henstock Stieltjes integrable with respect to ρ to a number $\alpha \in R$ if for any $\varepsilon > 0$ there is a sequence of positive gauge functions $\{\delta_n(x)\}_{n=1}^{\infty}$ such that

$$|\sum_{i=1}^{m_n} h(t_{i_n})(\rho(w_{i_n}) - \rho(w_{(i-1)_n})) - \alpha| < \varepsilon,$$

for each δ_n-fine tagged partition $\theta_n = \{[w_{(i-1)_n}, w_{i_n}], t_i\}_{i=1}^{m_n}$ on $[a,b]$. We say that α is a Sequential Henstock Stieltjes integral of *h* on $[a,b]$. i.e $\alpha = (HS)\int_a^b h d\rho$ and use $SH_h[a,b]$ to denote the set of all Sequential Henstock integrable functions defined on $[a,b]$.

Remark 1.4

i. If ρ is an identity function in Definition 1.3, we obtain a definition for Sequential Henstock integrals (see Iluebe and Mogbademu [8]).

ii. If $\delta_n = \delta$ for all *n* in Definition 1.3, we obtain a definition for Henstock Stieltjes integral (see Iluebe and Mogbademu [9]).

iii. If $\delta_n = \delta$ for all $n \in$ N and ρ is an identity function in Definition 1.3, we obtain a definition for Henstock integral (see Iluebe and Mogbademu [10]).

Definition 1.5. [1] For any $r \in [a, b]$, we define the right-graininess and left-graininess function on $[a, b]$ by $\mu(r) = \sigma(r) - r$ and $v(r) - \eta(r)$ respectively.

The following new definitions are analog of Definition 2.9 in Yang and Yabin [25].

Definition 1.6 For any $r \in [a, b]$, we say $\delta = (\delta_K, \delta_L)$ *is* a δ-gauge on $[a, b]$.

if $\delta_L(r) > 0$, $\delta_L(b) \geq 0$, $\delta_L(r) \geq \mu(r)$ and $\delta_K(r) > 0$ on $[a, b]$, then for any δ -gauge on $[a ,b]$, we say a partition θ is a δ -fine partition if

$$t_i - \delta_K(t_i) \leq r_{i-1} < r_i \leq t_i - \delta_L(t_i), \ i = 1,2,...,n.$$

Remark 1.7 i. If $[a, b]$ = R, then $\delta_K = \delta_L$.

ii. If δ_K is a δ -gauge for $[a, b]$, there exists a δ -fine partition *P* for $[a, b]$.

We define newly the following definitions.

Definition 1.8 . Let $\rho : [a,b]_T \to R$ be non decreasing function. A function. $h : [a,b]_T \to R$ is Sequential Henstock Stieltjes delta integrable with respect to ρ to a number $\alpha \in$ R if for any $\varepsilon > 0$ there exists a sequence of δ_n-gauge functions $(\{\delta_n(x)\}_{n=1}^{\infty} = \{\delta_K^n(x), \delta_L^n(x)\}_{n=1}^{\infty})$ such that

$$| S(h, \rho, \theta_n) - \alpha | < \varepsilon$$

for each $\delta_n(x)$ -fine tagged partition $\theta_n = \{[w_{(i-1)_n}, w_{i_n}], t_i\}_{i=1}^{m_n}$ on $[a,b]_T$, where $S(h, \rho.\theta_n) = \sum_{i=1}^{m_n} h(t_{i_n})(\rho(w_{i_n}) - \rho(w_{(i-1)_n}))$. We say that α is a Sequential Henstock Stieltjes delta integral of *h* on $[a,b]_T$. i.e $\alpha = (SHSD) \int_{[a,b]_T} h \Delta \rho$

and use *SHSDh* $[a,b]_T$ to denote the set of all Sequential Henstock Stieltjes delta integrable functions defined on $[a,b]_T$.

Remark 1.9 i. Suppose $[a, b]$ = R and $\delta_n = \delta_K = \delta_L$ in Definition 1.8, then we have a definition for Sequential Henstock integral.

ii If ρ is an identity function in Definition 1.8, we obtain a definition for Sequential Henstock delta integral (see Yang and Yabin [25]).

iii. If $\delta_n = \delta$ for all *n* in Definition 1.8, we obtain a definition for Henstock Stieltjes delta integral (see Iluebe and Mogbademu [7]).

2. The Main Results

In this section, we prove some of the fundamental properties of Sequential Henstock Stieltjes delta integral.

Theorem 2.1 Let $\rho:[a,b]_T \to R$ be non decreasing function. If $h \in SHSD[a,b]_T$, then the integral is unique.

Proof. Suppose the integral value are not unique. Let $\alpha_1 = (SHSD)\int\limits_{[a,b]_T} h\,\Delta\rho$ and

$\alpha_2 = (SHSD)\int\limits_{[a,b]_T} h\,\Delta\rho$ where $\alpha_1 \neq \alpha_2$. Let ε >0, then there exists a

$(\{\delta_n(x)\}_{n=1}^{\infty} = \{\delta_K^n(x), \delta_L^n(x)\}_{n=1}^{\infty})$ such that for each Sequential Henstock Stieltjes $\delta_n^1(x)$ -fine tagged partitions $\theta_n^1(x)$ on $[a,b]_T$ and $\delta_n^2(x)$ -fine tagged partition $\theta_n^2(x)$ on $[a,b]_T$, we have that

$$|\sum_{i=1}^{m_n\in N} h(t_{i_n})[\rho(w_{i_n}) - \rho(w_{(i-1)_n})] - \alpha_1 | < \frac{\varepsilon}{2}$$

and

$$|\sum_{i=1}^{m_n\in N} h(t_{i_n})[\rho(w_{i_n}) - \rho(w_{(i-1)_n})] - \alpha_2 | < \frac{\varepsilon}{2}.$$

respectively. Define a Sequential Henstock Stieltjes δ_n -gauge function $\delta_n(x)$ on $[a,b]_T$ $(\{\delta_n^1(x)\}_{n=1}^{\infty} = \min\{\delta_K^{n^1}(x), \delta_L^{n^1}(x)\}_{n=1}^{\infty})$ and $(\{\delta_n^2(x)\}_{n=1}^{\infty} = \min\{\delta_K^{n^2}(x), \delta_L^{n^2}(x)\}_{n=1}^{\infty})$.
Let θ_n be any Sequential Henstock Stieltjes $\delta_n(x)$ -fine partition of $[a,b]_T$. Then by triangular inequality, we have

$$|\alpha_1 - \alpha_2| \leq | \sum_{i=1}^{m_n\in N} h(t_{i_n})(\rho(w_{i_n}) - \rho(w_{(i-1)_n})) - \alpha_1 | + | \sum_{i=1}^{m_n\in N} h(t_{i_n})(\rho(w_{i_n}) - \rho(w_{(i-1)_n})) - \alpha_2 |$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

since for all ε >0, there exists a gauge $\delta_n(x)$ >0 on $[a,b]_T$, thus $\alpha_1 = \alpha_2$. This completes the proof.

Theorem 2.2 If h_1, h_2, $\in SHSD[a,b]_T$ and k_1, k_2 ∈R, then $(k_1h_1 + k_2h_2) \in SHSD[a,b]_T$ and

$$(SHSD)\int\limits_{[a,b]_T} (k_1h_1 + k_2h_2)\,\Delta\rho = k_1(SHSD)\int\limits_{[a,b]_T} h_1\,\Delta g + k_2(SHSD)\int\limits_{[a,b]_T} h_2\Delta\rho$$

Proof. Let $\alpha_1 = \int\limits_{[a,b]_T} h_1$ and $\alpha_2 = \int\limits_{[a,b]_T} h_2$. Choose ε >0. There is ε' >0 such that $(k_1 + k_2)\frac{\varepsilon^{;}}{2} \leq \varepsilon$. Then for any ε >0, there is a sequence of δ_n -gauge $(\{\delta_n^1(x)\}_{n=1}^{\infty} = \{\delta_K^{n^1}(x), \delta_L^{n^1}(x)\}_{n=1}^{\infty})$ on $[a,b]_T$ such that for any $\delta_n^1(x)$-fine tagged partition $\theta_n^1(x)$, we have

$$.| \sum_{i=1}^{m_n\in N} h(t_{i_n})(\rho(w_{i_n}) - \rho(w_{(i-1)_n})) - \alpha_1 | < \frac{\varepsilon^{'}}{2}$$

Similarly, for any ε >0, there is a sequence of δ_n -gauge $(\{\delta_n^2(x)\}_{n=1}^{\infty} = \{\delta_K^{n^2}(x), \delta_L^{n^2}(x)\}_{n=1}^{\infty})$ on

$[a,b]_T$ such that for any δ_n^2 (x)-fine tagged partition $\theta_n^2(x)$ we have

$$\left|\sum_{i=1}^{m_n\in N} h(t_{i_n})(\rho(w_{i_n})-\rho(w_{(i-1)_n}))-\alpha_2\right|<\frac{\varepsilon'}{2}.$$

Define a sequence positive functionδ_n-gauge function $\delta_n(x)$ on $[a,b]_T$ by $(\{\delta_n^1(x)\}_{n=1}^\infty=\min\{\delta_K^{n^1}(x),\delta_L^{n^1}(x)\}_{n=1}^\infty)$ and $(\{\delta_n^2(x)\}_{n=1}^\infty=\min\{\delta_K^{n^2}(x),\delta_L^{n^2}(x)\}_{n=1}^\infty)$. Therefore for any $\delta_n(x)$-fine tagged partition $\theta_n=\{\theta_n^1\cup\theta_n^2\}$ P_n of $[a,b]_T$, we have

$$\begin{aligned}
&\left|\sum_{i=1}^{m_n}(k_1h_1+k_2h_2)(t_{i_n})(\rho(w_{i_n})-\rho(w_{(i-1)_n}))-(k_1\alpha_1+k_2\alpha_2)\right|\\
&\qquad\le\left|\sum_{i=1}^{m_n}k_1h_1(t_{i_n})(\rho(w_{i_n})-\rho(w_{(i-1)_n}))-k_1\alpha_1\right|\\
&\qquad+\left|\sum_{i=1}^{m_n}k_2h_2(t_{i_n})(\rho(w_{i_n})-\rho(w_{(i-1)_n}))-k_2\alpha_2\right|\\
&\qquad\le k_1\frac{\varepsilon^*}{2}+k_2\frac{\varepsilon^*}{2}\\
&\qquad\le(k_1+k_2)\frac{\varepsilon^*}{2}\\
&\qquad<\varepsilon
\end{aligned}$$

Since $\varepsilon>0$ is arbitrary, this gives

$$(SHSD)\int_{[a,b]_T}(k_1h_1+k_2h_2)\,\Delta\rho=k_1(SHSD)\int_{[a,b]_T}h_1\,\Delta\rho+k_2(SHSD)\int_{[a,b]_T}h_2\,\Delta\rho$$

Theorem 2.3(Cauchy Criterion) Suppose $\rho:[a,b]_T\to R$ is an increasing function. $h\in SHSD([a,b]_T)$ if and only if for any $\varepsilon>0$, there is a sequence of δ_n-guages

$(\{\delta_n(x)\}_{n=1}^\infty=\{\delta_K^n(x),\delta_L^n(x)\}_{n=1}^\infty)$ on $[a,b]_T$ such that

$$|S(h,\rho,.\theta_n)-S(h,\rho,\vartheta_n)|<\varepsilon$$

for all $\delta_n(x)$-fine tagged partitions θ_n and ϑ_n on $[a,b]_T$.

Proof.

Suppose $f\in SHSD([a,b]_T)$ and $\varepsilon>0$, there exists a $(\{\delta_n(x)\}_{n=1}^\infty=\{\delta_K^n(x),\delta_L^n(x)\}_{n=1}^\infty)$ on $[a,b]_T$ such that for $\theta_n\ll\delta_n(x)$ and $\vartheta_n\ll\delta_n(x)$, we have $|S(h,\rho,.\theta_n)-\alpha|<\frac{\varepsilon}{2}$ and $|S(h,\rho,.\vartheta_n)-\alpha|<\frac{\varepsilon}{2}$ for all $\delta_n(x)$-fine tagged partitions θ_n and ϑ_n on $[a,b]_T$.

Now, if $\theta_n\ll\delta_n(x)=(\delta_K^n(x),\delta_L^n(x))$ and $\vartheta_n\ll\delta_n(x)=(\delta_K^n(x),\delta_L^n(x))$, then

$$|S(h,\rho,.\theta_n)-\alpha+\alpha-S(h,\rho,\vartheta_n)|<|S(h,\rho,.\theta_n)-\alpha|+|S(h,\rho,\vartheta_n)-\alpha|<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$$

Conversely, let ε >0, there is a $(\{\delta_n(x)\}_{n=1}^{\infty} = \{\delta_K^n(x), \delta_L^n(x)\}_{n=1}^{\infty})$ on $[a,b]_T$ such that $\theta_n \ll \delta_n(x)$ and $\vartheta_n \ll \delta_n(x)$ we have

$$|S(f,g,P_n) - S(f,g,Q_n)| < \frac{1}{n}.$$

We create an α -convergent Cauchy sequence of Sequential Henstock Stieltjes delta sums. Without giving up generality condition, we assume $(\{\delta_n(x)\}_{n=1}^{\infty} = \{\delta_K^n(x), \delta_L^n(x)\}_{n=1}^{\infty})$ is a decreasing sequence for every $x \in [a,b]_T$. Thus, for any *r* > *n*, P_w is $\delta_n(x)$ -fine and letting $n \to \infty$, then

$$|S(h,\rho,.\theta_n) - S(h,\rho,\vartheta_n)| < \frac{1}{n}$$

is a sequence that is Cauchy. Hence $\{S(h,\rho,.\theta_n)\}_{n=1}^{\infty} \to \alpha$ as $n \to \infty$ for any ε >0 and for all $n \geq$ N, we have

$$|S(h,\rho,.\theta_n) - \alpha| < \frac{1}{n}$$

Let ε >0, there is a $(\{\delta_n(x)\}_{n=1}^{\infty} = \{\delta_K^n(x), \delta_L^n(x)\}_{n=1}^{\infty})$ on $[a,b]_T$ where $\frac{1}{n} < \frac{\varepsilon}{2}$ and for

$\theta_n \ll \delta_n(x) = (\delta_K^n(x), \delta_L^n(x))$ and $\vartheta_n \ll \delta_n(x) = (\delta_K^n(x), \delta_L^n(x))$, then

$$|S(h,\rho,.\vartheta_n) - \alpha| < |S(h,\rho,\vartheta_n) - S(h,\rho,\theta_n)| + |S(h,\rho,\theta_n) - \alpha| < \frac{1}{n} + \frac{1}{n} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Thus, $h \in SHSD([a,b]_T)$ and $\int_{[a,b]_T} h = \alpha$.

Theorem 2.5 must be proved in order to use the new definition that follows.

Definition 2.4 Suppose $\rho : [a,b]_T \to R$ is an increasing function and $h : [a,b]_T \to R.$ a real valued function. if the interval $(w_{i_n} - w_{(i-1)_n}) \subset [a,b]_T$ is divided into *i* subintervals of equal width and from the interval, choose a sequence of tag point $t_{i_n} \in [w_{(i-1)_n}, w_{i_n}]$, the definite integral of *h(x)* from *a* to *b* for $x \in [a,b]_T$ is defined by

$$\int_{[a,b]_T} h \,\Delta \rho = \lim_{n\to\infty} \sum_{i=1}^{m_n} h(t_{i_n})(\rho(w_{i_n}) - \rho(w_{(i-1)_n})).$$

With regard to ρ. It is the definite delta integral. A condition for the integral value of the function to exist is provided by the limit sum of the integral function.

Theorem 2.5 Suppose $\rho : [a,b]_T \to R$ is an increasing function. If $h \in SHSD[a,c]_T$ and $h \in SHSD$ $[c,b]_T$, then $h \in SHSD[a,b]_T$ and

$$(SHSD) \int_{[a,b]_T} h \,\Delta \rho = (SHSD) \int_{[a,c]_T} h \,\Delta \rho + (SHSD) \int_{[c,b]_T} h \,\Delta \rho$$

i.e

$$\sum_{i=1}^{m_n} h(t_{i_n})(\rho(w_{i_n}) - \rho(w_{(i-1)_n})) = \sum_{i=1}^{m_k} h(t_{i_n})(\rho(w_{i_n}) - \rho(w_{(i-1)_n})) + \sum_{i=1}^{m_{n-k}} h(t_{i_n})(\rho(w_{i_n}) - \rho(w_{(i-1)_n}))$$

Proof.

Since $h \in SHSD[a,c]_T$, Let ε >0 be arbitrary and α_1 = (*SHSD*) $\int\limits_{[a,c]_T} h$ Δg then there is a δ_n - gauge $(\{\delta_n^1(x)\}_{n=1}^{\infty} = \{\delta_K^{n^1}(x), \delta_L^{n^1}(x)\}_{n=1}^{\infty})$ on $[a,c]_T$ such that for any δ_n^1 (*x*)-fine tagged partition $\theta_n^1(x)$, we have

$$|\sum_{i=1}^{m_n \in N} h(t_{i_n})(\rho(w_{i_n}) - \rho(w_{(i-1)_n})) - \alpha_1 | < \frac{\varepsilon'}{2}$$

Similarly, for any ε >0, there is a sequence of δ_n -gauge $(\{\delta_n^2(x)\}_{n=1}^{\infty} = \{\delta_K^{n^2}(x), \delta_L^{n^2}(x)\}_{n=1}^{\infty})$ on $[a,b]_T$ such that for any δ_n^2 (*x*)-fine tagged partition $\theta_n^2(x)$ we have

$$|\sum_{i=1}^{m_n \in N} h(t_{i_n})(\rho(w_{i_n}) - \rho(w_{(i-1)_n})) - \alpha_2 | < \frac{\varepsilon'}{2} .$$

We define $\delta_n = \min(\delta_n^1(x), \delta_n^2(x))$ and $\delta_n(x)$ -fine tagged partition

$\theta_n = \{\theta_n^1 \cup \theta_n^2\}$ in order to force a point *c* to be a tag of each $\theta_n \ll \delta_n(x)$. Using the right-left procedure, we split each partition θ_n at the tag *c* so that it becomes a partition point of each θ_n

$$\delta_n(t) = \begin{cases} min\{\delta_K^{n^1}(x), \delta_L^{n^1}(x), \frac{1}{2}(c-x)\}, & \text{if } x \in [a,c)_T \\ min\{\delta_K^{n^1}(x), \delta_L^{n^1}(x), \delta_K^{n^2}(x), \delta_L^{n^2}(x)(c)\} & \text{if } x = c \\ min\{\delta_K^{n^2}(x), \delta_L^{n^2}(x), \frac{1}{2}(t-c)\}, & \text{if } x \in [c, b]_T \end{cases}$$

Let $\theta_n \ll \delta_n(x)$. Let $\theta_n^1 \in [a,c]_T$ consisting $\theta_n \cap [a,c]_T$ and $\theta_n^2 \in [c,b]_T$ consisting $\theta_n \cap [c, b]_T$. Then the right-left procedures provides that

$$\sum_{i=1}^{m_n} h(t_{i_n})(\rho(w_{i_n}) - \rho(w_{(i-1)_n})) = \sum_{i=1}^{m_k} h(t_{i_n})(\rho(w_{i_n}) - \rho(w_{(i-1)_n})) + \sum_{i=1}^{m_{n-k}} h(t_{i_n})(\rho(w_{i_n}) - \rho(w_{(i-1)_n}))$$

Given ε >0, there exists a $\delta_n(x)$ -gauge $(\{\delta_n^2(x)\}_{n=1}^{\infty} = \{\delta_K^{n^2}(x), \delta_L^{n^2}(x)\}_{n=1}^{\infty})$ such that for $\theta_n \ll \delta_n(x)$, we have

$$| \sum_{i=1}^{m_n} h(t_{i_n})(\rho(w_{i_n}) - \rho(w_{(i-1)_n})) - (\alpha_1 + \alpha_2) |$$

$$= | \sum_{i=1}^{m_n} h(t_{i_n})(\rho(w_{i_n}) - \rho(w_{(i-1)_n})) - \sum h(t_{i_n})(\rho(w_{i_n}) - \rho(w_{(i-1)_n})) - (\alpha_1 + \alpha_2) |$$

$$= | \sum_{i=1}^{m_n} h(t_{i_n})(\rho(w_{i_n}) - \rho(w_{(i-1)_n})) - \alpha_1 | + | \sum h(t_{i_n})(\rho(w_{i_n}) - \rho(w_{(i-1)_n})) - \alpha_2 |$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Hence by Theorem 2.2, $h \in SHSD[a,b]_T$ and

$$(SHSD)\int_{[a,b]_T} h\,\Delta\rho = (SHSD)\int_{[a,c]_T} h\,\Delta\rho + (SHSD)\int_{[c,b]_T} h\,\Delta\rho$$

Example 2.6 Let $T = Z = \{\ldots -2,-1,0,1,2,\ldots\}$. Suppose $g:[a,b]_T \to R$ is an increasing function defined by $g(x) = 2x$ and the function

$$h(x) = \begin{cases} 1 & t \in E \\ 0 & t \notin E \end{cases},$$

,

where *E* is a countable dense subset of $[0,1]_T$. It will be proved that *h* is *SHSD* integrable on $[0,1]_T$. Let $E = \{(r_1,q_1),(r_2,q_2),\ldots\}$ be the enumeration of t_{i_n}, q_{i_n} is a rational number in $[0,1]_T$ since *E* is countable. So for any *ε* >0, let *n* be large enough so that $\varepsilon > \dfrac{1}{2n}$. We define a sequence of δ_n -gauge function $\delta_K^n(q_i)$, $\delta_L^n(q_i) = \max(1,\mu(r_i))$ by

$$\delta_n(x) = \begin{cases} 1 & t \in [0,1]/E \\ \dfrac{\varepsilon}{4^{k+1}} & t = r_k \in E \end{cases},$$

Let $\theta_n = \{[w_{(i-1)_n}, w_{i_n}], t_i\}_{i=1}^{m_n}$ be a $\delta_{n\varepsilon}(x)$ -fine tagged partition of $[0,1]_T$. Since *h*= *0* on $[0,1]_T \setminus E$, the tags in $[0,1]_T \setminus E$ contribute nothing to the Riemann sum of *h* induced by θ_n which is $\delta_{n\varepsilon}(x)$ -fine, we only consider the sum

$$|S(h,\rho.\theta_n)| = |\sum_{i=1}^{m_n} h(t_{i_n})(\rho(w_{i_n}) - \rho(w_{(i-1)_n}))| \le |\sum_{n=1}^{n} \delta_K^n(q_i) + \delta_L^n(q_i)| = \varepsilon.$$

Therefore, *h* is Sequential Henstock integrable on $[0,1]_T$ and

$$SHSD\int_{[0.1]_T} h\,\Delta\rho = 0.$$

Example 2.7 Let $T = Z = \{\ldots -2,-1,0,1,2,\ldots\}$.

Suppose $\rho:[a,b]_T \to R$ is an increasing function defined by $\rho(x) = x$.

Consider the function $\rho:[0,1]_T \to R$ is defined by

$$h(x) = \begin{cases} x^3 \sin\left(\dfrac{\pi}{x^3}\right), if & x \in [0,1]_T \\ 0, & x = 0 \end{cases},$$

This function has a derivative given by

$$h^*(x) = \begin{cases} x^3 \sin\left(\dfrac{\pi}{x^3}\right) + \dfrac{3\pi}{x}\cos\left(\dfrac{\pi}{x^3}\right), if & x \in [0,1]_T \\ 0, & x = 0 \end{cases},$$

Suppose we define our guage function $\delta_n(x) = \begin{cases} \dfrac{1}{\sqrt{k}} & t \in Q \\ 1 & t \notin Q \end{cases},$

So, we have

$$|\sum_{i=1}^{m_n} h(t_{i_n})(\rho(w_{i_n}) - \rho(w_{(i-1)_n}))| = \sum_{i=1}^{N} |\frac{1}{k}\cos k\pi + \frac{1}{k+1}\cos(k+1)\pi$$

$$= \sum_{i=1}^{N}\left(\frac{1}{k} + \frac{1}{k+1}\right) = \left(1 + \frac{1}{2}\right) + \left(\frac{1}{2} + \frac{1}{3}\right) + \ldots + \left(\frac{1}{N} + \frac{1}{N+1}\right)$$

$$= 2(1) + 2\left(\frac{1}{2}\right) + 2\left(\frac{1}{3}\right) ... + 2\left(\frac{1}{N}\right) - 1 + 2\left(\frac{1}{N+1}\right) = 2\sum_{i=1}^{N}\left(\frac{1}{k} + \frac{1}{k+1}\right)$$

Since the harmonic series diverges, we see that this series goes to ∞ as $N \to +\infty$. Thus , *h* is not of bounded variation on [0,1] and so *h* is of Sequential Henstock-delta integrable function on [0,1]. Hence, *h*(*x*) on [0,1] satisfies

$$\sum_{i=1}^{m_n} h(t_{i_n})[\rho(w_{i_n}) - \rho(w_{(i-1)_n})] - \int_0^1 h(x)d\rho$$

and $\int_0^1 hd\rho = h(1) - h(0) = 1$.

Remark 2.8 The aforementioned conclusions also apply to a broader category of *S*-Henstock delta-type integrals defined as follows:

Suppose $\rho : [a,b]_T \to R$ is an increasing function. For all *i* ∈N, consider $h_i : [a,b]_T \to R$ with a sequence of positive δ_n-guage functions $(\{\delta_n^2(x)\}_{n=1}^{\infty} = \{\delta_K^{n^2}(x), \delta_L^{n^2}(x)\}_{n=1}^{\infty})$ on [*a*, *b*]$_T$ such that for every $\delta_n(x)$-fine tagged partitions $\theta_n = \{[w_{(i-1)_n}, w_{i_n}], t_i\}_{i=1}^{m_n}$, we have

$$\sum_{i=1}^{m_n} h(t_{i_n})(\rho(w_{i_n}) - \rho(w_{(i-1)_n})) \to \alpha \text{ as } n \to \infty$$

We say $\alpha = (SHSD)^S \int_{[a,b]_T} h\Delta g$ The set of all functions f_i for all *i* ∈N which are *S*-Sequential Henstock Stieltjes delta integrable on $[a,b]_T$ is denoted by $SHSD^S[a,b]_T$.

3. Conclusion

This study establishes the theory of the Sequential Henstock–Stieltjes delta integral on time scales, proves its fundamental properties, and extends existing results, opening new directions for research particularly its applicability to broader classes of functions and more abstract spaces.

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