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Some Fixed Point Theorems and Applications to Noise Reduction in Signal Processing

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Abstract: In this paper, we introduce the notion of a quasi-convex p-metric space and study some fixed-point theorems for self-mappings satisfying certain contractive conditions in quasi-convex p metric space. Additionally, we establish fixed point theorems for generalized contractions in a quasi-convex p-metric space. This result generalizes previous related work in the literature. An application supports the theoretical analysis by demonstrating the use of quasi convex p-metric spaces in signal processing for noise reduction and signal smoothing. This method introduces an adaptive technique that needs to be modeled and minimized to achieve noise reduction.

Keywords: Convex metric spaces; Fixed point; Quasi convexity; Quasi convex p -metric spaces, signal processing.

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1. Introduction and Preliminaries

The notion of metric spaces is fundamental in analysis, and many authors have generalized metric spaces to derive other spaces or abridged concepts. These include 2-metric spaces b-metric spaces [1–2,18, 22]. In 1922, S. Banach [5] proved a theorem guaranteeing the existence and uniqueness of a fixed point on a complete metric space under specific conditions. This theorem spurred extensive research, extending and generalizing variants of metric space fixed point theory due to its diverse applications in metric spaces (see e.g. [1-4, 8,9, 12-14,16, 18, 20]).

However, the increasing applications of fixed point theorems in diverse fields have prompted researchers to explore generalized variants of metric spaces, notably considering the study of convexity within such spaces. Takahashi [23] initiated the concept of convexity structure in metric spaces and introduced convex structures in metric spaces. Convex structures play a fundamental role in fixed point theorems, serving as a crucial tool in ensuring the existence of fixed points for continuous functions. In 2020, Adewale *et al.* [1], introduced quasi convexity on metric spaces and proved some fixed point theorems in quasi convex metric spaces. Motivated by the work of Takahashi [23] and Adewale *et al.* [1], we introduce the concept of quasi convex p-metric spaces by replacing the notion of a convex function with a quasi convex function and the metric space with a p-metric space. We give examples of quasi convex p-metric spaces not convex metric spaces. Our results generalize convex metric spaces in literature. Here are some basic definitions needed for this study.

Definition 1.1 [17]. Suppose X is a non-empty set. A mapping $\tilde{d} : X \times X \to [0, \infty)$ is a p-metric if there exists a strictly increasing continuous function $\Omega: [0, \infty) \to [0, \infty)$ with $t \leq \Omega(t)$ for all $t \geq 0$

such that for all $x, y, z \in X$:
(i) $\tilde{d}(x,y) = 0$; if x = y,
(ii) $\tilde{d}(x,y) = \tilde{d}(y,x)$,
(iii) $\tilde{d}(x,y) \leq \Omega\left(\tilde{d}(x,z) + \tilde{d}(z,y)\right)$.
The pair $(X, \tilde{d})$ is called a p-metric space.

Remark 1.2 [20]. The class of p-metric spaces is larger than the class of b -metric spaces.
(i) A b-metric space is a p-metric when $\Omega(x) = sx$, where s is a constant with $s \geq 1$.
(ii) while if it is metric then $\Omega(x) = x$.

Example 1.3 [20]. Let $(X, \tilde{d})$ be a metric space and $\phi(x, y) = sinh\ \tilde{d}(x,y)$. It is easy to show that ϕ is a p-metric space with $\Omega(t) = sinh\ t$ for all $t \geq 0$.

Definition 1.4 [20]. Let $(X, \tilde{d})$ be a p-metric space. Then a sequence $\{x_n\}$ in X is said to be:
(i) p-convergent if and only if there exists $x \in X$ such that $\tilde{d}(x_n, x) \to 0$ as $n \to \infty$ i.e., $\lim_{n\to\infty} x_n = x$.
(ii) p-Cauchy if and only if $\tilde{d}(x_n, x_m) \to 0$ as $n, m \to \infty$.
(iii) p-complete if every p -Cauchy sequence in X is p-convergent.

Definition 1.5 [2]. Let $(X, \tilde{d})$ be a metric space and I = [0, 1]. A continuous function $W: X \times X \times [0, 1] \to X$ is said to be a convex structure on X if for each $x, y, u \in X$ and $\lambda \in I = [0, 1]$,

$$d(u, W(x, y; \lambda)) \leq \lambda\, d(u, x) + (1 - \lambda)\, d(u, y).$$

A metric space together with a convex structure W is known as convex metric space and is denoted by $(X, \tilde{d},\ W)$

Definition 1.6 [11]. A function $f : K \to \mathbb{R}$ is quasi convex if

$$f(\lambda x + (1 - \lambda) y) \leq \max\{f(x), f(y)\},\ \forall\ x, y \in K, \lambda \in [0, 1].$$

Definition 1.7 [1]. Let $(X, \tilde{d})$ be a metric space, A mapping $\gamma: X \times X \times [0,1] \to X$ is said to have quasi convex structure on X if for each $(x, y, \lambda) \in X \times X \times [0, 1]$ and $u \in X$,

$$\tilde{d}\ (u, \gamma(x, y; \lambda)) \leq \max\ \{\tilde{d}\ (u, x), \tilde{d}\ (u, y)\}.$$

A metric space $(X, \tilde{d})$ having quasi convex structure γ is called a quasi convex metric space.

Definition 1.8 [1]. A function $\Psi: \mathbb{R}^+ \to \mathbb{R}^+$ is called a comparison function if Ψ is monotone increasing and

$$\lim_{n\to\infty} \Psi^n(x) = 0\,, \quad x \in \mathbb{R}^+.$$

However, the research on the fixed-point theorem in p-metric spaces had been studied by numerous researchers, but in this study, we introduced a new metric spaces called quasi convex p – metric space and consider the generalized quasi contraction mapping . Also, using adaptive method to reduce noise in signal processing

Motivated by the results of Adewale *et al.* [1], and we introduce the concept of quasi convex p-metric space and prove some fixed point theorems in quasi convex p-metric space.

2. Main Results

In this section, we introduce the notion of quasi convex p-metric space and prove some fixed point theorems in this new space. Throughout this paper, we denote a metric space as (X, d) or X.

Definition 2.1. Let $(X, \tilde{d})$ be a p -metric space with $\Omega: [0,\infty) \to [0,\infty)$. Let a mapping $\rho : X \times X \times [0, 1] \to X$, be convex structure on a p−metric space defined by

$$\tilde{d}\,(u, \rho(x, y; \lambda)) \le \Omega[\max\{\tilde{d}\,(u, x), \tilde{d}\,(u, y)\}]. \qquad (1)$$

Then $(X, \tilde{d}, \ \rho)$ is called a quasi convex p−metric space.

Definition 2.2. Let $(X, \tilde{d}, \ \rho)$ be a quasi convex p -metric space. A nonempty subset C of X is said to be quasi convex p-metric space if $\rho(x, y, \lambda) \in C$ for all $(x, y; \lambda) \in C \times C \times I$ and $\lambda \in [0, 1]$.

Example 2.3. An ultrametric space is a quasi convex p-metric space such that $\Omega(x) = x$. However, there exist quasi convex p-metric spaces that are not ultrametric space.

Definition 2.4. Let $(X, \tilde{d}, \ \rho)$ be a quasi convex p-metric space. An open ball $B(z, r)$ in $(X, \tilde{d}, \ \rho)$ is defined by

$$B(z, r) = \{(x, y) \in X^2 : \ \tilde{d}\,(z, \rho(x, y; \lambda)) < r\}. \qquad (2)$$

Definition 2.5. Let $(X, \tilde{d}, \ \rho)$ be a quasi convex p-metric space. A closed ball $B(z, r)$ in $(X, \tilde{d}, \ \rho)$ is defined by $B(z, r) = \{(x, y) \in X^2 : \tilde{d}(z, \rho(x, y; \lambda)) \le r\}$. (3)

Definition 2.6 [1]. Let (X, d, γ) be a complete quasi convex metric space and E a nonempty closed convex subset X. A mapping $T: E \to E$ is said to be (k, L)-Lipschitzian if there exists $k \in [1, \infty)$, and $L \in [0,1)$ such that

$$d(Tx, Ty) \le L\, d(x, Tx) + k\, d\,(x, y) \qquad \forall\, x, y \in E \qquad (4)$$

The above mapping generalizes many known mappings in literature**.**

Theorem 2.7. Let $(X, \tilde{d}, \ \rho)$ be a complete quasi convex p-metric space, and $T: X \to X$ a (k, L) be Lipschitzian mapping. Suppose $\Psi: \mathbb{R}^+ \to \mathbb{R}^+$ is a comparison function such that for arbitrary $x \in X$ there exists $q \in X$ and $\Omega(t) = st$, such that $s \ge 1$

$$\text{and} \quad \tilde{d}(Tq, q) \le \Psi(\tilde{d}(Tx, x)) \qquad (5)$$

Then T has a unique fixed point in X.

Proof.

Consider the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$ with some initial point $x_0 \in X$. Suppose $x_n \in X$ is an arbitrary point. By (k, L) Lipschitzian mapping (4) we obtain

$$\tilde{d}(Tx_{n+1}, x_{n+1}) \le \Psi\,(\,\tilde{d}(Tx_n, x_n)) \qquad \forall\ n = 0, 1, 2, \ldots \qquad (6)$$

By induction in (6) we obtain

$$\tilde{d}(Tx_1, x_1) \le \Psi^0\ (\,\tilde{d}(Tx_0, x_0)),$$
$$\tilde{d}(Tx_2, x_2) \le \psi(\,\tilde{d}(Tx_1, x_1)) = \Psi^2\ (\tilde{d}(Tx_0, x_0)),$$
$$\vdots$$
$$\tilde{d}(Tx_{n+1}, x_{n+1}) \le \ \Psi^{n+1}(\tilde{d}(Tx_0, x_0)). \qquad (7)$$

Thus $\tilde{d}\,(Tx_{n+1}, x_{n+1}) \to 0$ as $n \to \infty$ i.e., $\tilde{d}\,(Tx_n, x_n) \to 0$ as $n \to \infty$.

To show that $\{x_n\}$ is a Cauchy sequence for $n, m \in \mathbb{N}$ with $n > m$,

$$\tilde{d}(\,x_n, x_m) \le \tilde{d}(x_n, x_{n-1}) + \tilde{d}(x_{n-1}, x_m).$$

By triangle inequality, and $\Omega\,(t) = st$

$$\begin{aligned}
\tilde{d}(\,x_n, x_m) &\le \Omega(\,\tilde{d}\,(x_n, x_{n-1}) + \tilde{d}\ (x_{n-1}, x_m)) \\
&= s(\,\tilde{d}(x_n, x_{n-1}) + \tilde{d}(x_{n-1}\,,\ x_m)) \\
&= s\,(\tilde{d}(\,x_n\,, x_{n-1}\,) + s[\,\tilde{d}(x_{n-1}x_{n-2}) + s\,(\,\tilde{d}(x_{n-2}x_{n-3}) + \cdots + s(\,\tilde{d}(x_{m-1}\,,\ x_m))] \\
&\ \ \vdots \\
&= s\,(\tilde{d}(\,x_n\,, x_{n-1}\,) + s^2\tilde{d}(x_{n-1}x_{n-2}) + s^3(\,\tilde{d}(x_{n-2}x_{n-3}) + \cdots + s^{n-m}(\tilde{d}(x_{m+1}\,,\ x_m))]
\end{aligned}$$

$\leq \sum_{k=0}^{n-m-1} s^{k+1}\, \tilde{d}\,(x_{k+1},\ x_k)$
$\leq \sum_{k=m}^{n-1} s^{n-1-k} \quad \Psi^k(\,\tilde{d}\,(x_{k+1},\ x_k))\,.$ (8)

By (7), $\tilde{d}T\,(x_{k+1}, x_k) = \ \tilde{d}\,(Tx_k, x_k) \leq \Psi^k(r)$ where $r = \tilde{d}\,(Tx_0, x_0)$,
Thus, $\tilde{d}\,(x_n\,, x_m) \leq \Psi^k(r)$. As $k \to \infty$, $\tilde{d}\,(x_n,\ x_m) \to 0$.
Hence, $\{x_n\}$ is a Cauchy sequence in X. By completeness, $x^* \in X$ such that $\lim_{n\to\infty} x_n = x^*$.
We need to show that x^* is a fixed point of X. By (4), (5), and triangle inequality, we have

$\tilde{d}(Tx^*, x^*) \leq s\,(\tilde{d}(Tx^*, Tx_n) + \tilde{d}(Tx_n, x_n) + \tilde{d}(x_n, x^*)\,,$
$\leq s\,\{\,L(\tilde{d}(\,Tx_n, x_n) + k\,\tilde{d}(x^*, x_n) + \tilde{d}(Tx_n, x_n) \ + \tilde{d}(x_n, x^*)\}\,.$
$\leq$ s{(1+L) $\tilde{d}(Tx_n, x_n) + (1+k)\tilde{d}(x^*, x_n)$,.

By continuity of T, $\tilde{d}(Tx^*, x^*) \to 0$ as n $\to \infty$ which implies that $Tx^* = x^*$. Hence, x^* is a fixed point of T.

To show the uniqueness of the fixed point. Suppose x^* and y^* are fixed points such that $Tx^* = x^*$ and $Ty^* = y^*$. Assume that $x^* \neq y^*$

$\tilde{d}(Tx^*, Ty^*) \leq$ L $\tilde{d}\,(x^*, Tx^*)$ + k $\tilde{d}\,(x^*, y^*)$,
$= \tilde{d}(x^*, x^*) + k\,\tilde{d}(x^*, y^*)$,
$=$ k$\tilde{d}\,(x^*, y^*)$.

Since $Tx^* = x^*$ and $Ty^* = y^*$ and $k \in [0, 1)$,

$(1-k)\,\tilde{d}(x^*, y^*) \leq 0\,,$

This implies $\tilde{d}\,(x^*, y^*) = 0$
Hence $x^* = y^*$.

Therefore, x^* is a unique fixed point of mapping T.

Corollary 2.8. Let (X, d, γ) be a complete quasi convex metric space, C, a nonempty closed quasi convex subset of X and T: $C \to C$, a (k, L)- Lipschitzian mapping. Suppose Ψ: $\mathbb{R} \to \mathbb{R}$ is a comparison function such that for arbitrary $x \in C$ there exists $q \in C$ with $\tilde{d}\,(Tq, q) \leq \Psi\,\left(\tilde{d}\,(Tx, x)\right)$.
Then T has a fixed point in C.

Theorem 2.9. $(X, \tilde{d},\ \rho)$ be a complete quasi convex p-metric space and T: $X \to X$, a (k,L)-Lipschitzian mapping. Suppose $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ is a comparison function such that for arbitrary $x \in X$ there exists $q \in X$ and $\Omega(t) = st$ such that $s \geq 1$
(i) $\tilde{d}(Tq, q) \leq \Psi\,(\tilde{d}(Tx, x))$, and
(ii) $\tilde{d}(Tq, Tx) \leq$ c $\tilde{d}(Tx, x)$ for all $c > 0$.
Then T has a unique fixed point in X.

Proof.
Suppose $x_0 \in X$ is an arbitrary point. By condition (i) and (ii), we obtain

$\tilde{d}(Tx_{n+1}, x_{n+1}) \leq \Psi\,(\tilde{d}(Tx_n, x_n)),\quad \forall\, n = 0, 1, 2, \ldots$ (9)

and

$d(Tx_n, x_n) \leq c\ \tilde{d}\,(x_n, x_{n-1})$ (10)

By induction in (9) we obtain

$\tilde{d}\,(Tx_1, x_1) \leq \Psi^0\ (\check{d}(Tx_0, x_0)),$
$\tilde{d}\,(Tx_2, x_2) \leq \Psi\,(\tilde{d}\,(Tx_1, x_1)) = \Psi^2\,(\tilde{d}\,(Tx_0, x_0)),$
$\vdots$
$\tilde{d}(Tx_{n+1}, x_{n+1}) \leq \Psi^n\,(\tilde{d}(Tx_n, x_n))$ (11)

Thus $\tilde{d}\,(Tx_{n+1}\,, x_{n+1}\,) \to 0$ as $n \to \infty$ i.e., $\tilde{d}\,(Tx_n\,, x_n) \to 0$ as $n \to \infty$.
To show that $\{x_n\}$ is a Cauchy sequence for $n,m \in \mathbb{N}$ with $n > m$,

$\tilde{d}(x_n, x_m) \le \tilde{d}(x_n, x_{n-1}) + \tilde{d}(x_{n-1}, x_m)$.

By using triangle inequality, and Ω (t) = s

$$\begin{aligned}
\tilde{d}(x_n, x_m) &\le \Omega(\tilde{d}(x_n, x_{n-1}) + \tilde{d}(x_{n-1}, x_m)),\\
&= s(\tilde{d}(x_n, x_{n-1}) + \tilde{d}(x_{n-1}, x_m))\\
&= s(\tilde{d}(x_n, x_{n-1}) + s[\tilde{d}(x_{n-1}x_{n-2}) + s(\tilde{d}(x_{n-2}x_{n-3}) + \cdots + s(\tilde{d}(x_{m-1}, x_m))].\\
&\vdots\\
&= s(\tilde{d}(x_n, x_{n-1}) + s^2\tilde{d}(x_{n-1}x_{n-2}) + s^3(\tilde{d}(x_{n-2}x_{n-3}) + \cdots + s^{n-m}(\tilde{d}(x_{m+1}, x_m))].\\
&\le \sum_{k=0}^{n-m-1} s^{k+1}\tilde{d}(x_{k+1}, x_k)\\
&\le \sum_{k=m}^{n-1} s^{n-1-k}\ \Psi^k(r)(\tilde{d}(x_{k+1}, x_k)) \qquad (12)
\end{aligned}$$

By (7), $\tilde{d}T(x_{k+1}, x_k) = \tilde{d}(Tx_k, x_k) \le \Psi^k(r)$ where $r = \tilde{d}(Tx_0, x_0)$,

Thus $\tilde{d}(x_n, x_m) \le \Psi^k(r)$. As $k \to \infty$, $\tilde{d}(x_n, x_m) \to 0$.

Hence, $\{x_n\}$ is a Cauchy sequence in X. By completeness, there $x^* \in X$ such that $\lim_{n\to\infty} x_n = x^*$.

We need to show that x^* is a fixed point of X. By (4), (5) and triangle inequality we have

By (4), (9) and triangle inequality, we have

$$\begin{aligned}
\tilde{d}(Tx^*, x^*) &\le s(\tilde{d}(Tx^*, Tx_n) + \tilde{d}(Tx_n, x_n) + \tilde{d}(x_n, x^*)),\\
&\le s\{L(\tilde{d}(Tx_n, x_n) + k\,\tilde{d}(x^*, x_n) + \tilde{d}(Tx_n, x_n) + \tilde{d}(x_n, x^*))\},\\
&\le s\{(1+L)\,\tilde{d}(Tx_n, x_n) + (1+k)\tilde{d}(x^*, x_n)\},\\
&\le s(1+L)\ \psi^n(\tilde{d}(Tx_n, x_n) + (1+k)\tilde{d}(x^*, x_n)).
\end{aligned}$$

By continuity of T, d $(Tx^*, x^*) \to 0$ as n $\to \infty$ which implies that $Tx^* = x^*$ .Hence, x^* is a fixed point of T.

To show the uniqueness of the fixed point. Suppose x^* and y^* are fixed points such that $Tx^* = x^*$and $Ty^* = y^*$. Assume that $x^* \ne y^*$

By contractive condition (1) in the theorem, and $Ty^* = y^*$.

$$\tilde{d}(Tx^*, x^*) \le \Psi(\tilde{d}(Ty^*, y^*)),$$

$$\tilde{d}(Tx^*, x^*) \le \Psi(0) = 0$$

Applying condition (ii) in the Theorem.

$$\tilde{d}(Tx^*, Ty^*) \le c\,\tilde{d}(Ty^*, y^*)$$

$$\tilde{d}(x^*, y^*) \le c\,\tilde{d}(y^*, y^*)$$

since $\tilde{d}(y^*, y^*) \le 0$, $x^* = y^*$

which contradicts the assumption that the fixed points are distinct. The fixed point $x^* \in X$ of mapping T is unique.

Corollary 2.10. Let $(X, \tilde{d}, \gamma)$ be a complete quasi convex metric space, C, a nonempty closed quasi convex subset of X and T: C → C, a (k,L)-Lipschitzian mapping. Suppose $\Psi: \mathbb{R} \to \mathbb{R}$ is a comparison function such that for arbitrary $x \in C$ there exists $q \in C$ such that $\Psi(t) = st$ and

i) $\tilde{d}(Tq, q) \le \Psi(\tilde{d}(Tx, x))$,

ii) $\tilde{d}(Tq, Tx) \le$ c $\tilde{d}(Tx, x)$, Then T has a fixed point in C, for all c > 0.

Theorem 2.11. Let $(X, \tilde{d}; \rho)$ be a complete quasi convex p-metric space, X, where Ω: [0, ∞) → [0, ∞) and T: $X \to X$, a k−Lipschitz involution with k ∈ [0, 1), $\tilde{d}(Tx, Ty) \le k\,\tilde{d}(x, y)$,

Then T has a unique fixed point in X.

Proof.

For any $x \in X$, Let $u = \rho(x, Tx, \alpha_n)$. Then, $\tilde{d}(u, x) \le \tilde{d}(Tx, Ty) \le k$.

$\tilde{d}$ $(u, x) =$ $\tilde{d}$ $(\rho(x, Tx, \alpha_n), x)$,
$\leq$ Ω (max$\{\tilde{d}$ (x, x), $\tilde{d}$ $(Tx, x)\}$),
$=$ $\Omega(\tilde{d}(Tx, x))$

Also, $\tilde{d}(u, Tu)=$ $\tilde{d}$ $(\rho(x, Tx, \alpha_n), Tu))$ $\leq \Omega(\max(\tilde{d}\ (x, Tu), \tilde{d}\ (Tx, Tu))$.

$\tilde{d}(u, Tu) \leq \Omega\big(\max(\tilde{d}\ (T^2x\ , Tu), \tilde{d}\ (Tx, Tu)\big)$,
$\leq \Omega(\max(k\ \tilde{d}\ (Tx\ , u), k\ (x, u))$,
$\leq k\Omega(\max\big(k\tilde{d}\ (x\ , Tx)\big\}$, (13)

Using (13) repeatedly, we obtain
$\tilde{d}(u, Tu) \leq k^n\Omega^n(\max\{k\ \tilde{d}\ (x_0\ , Tx_0)\})$

Therefore, $d(Tx_n, x_n) \to 0$ as $n \to \infty$.

To show that $\{x_n\}$ is a Cauchy sequence for $n,m \in$ N with $n > m$,

$\tilde{d}(x_n, x_m) \leq \tilde{d}(x_n, x_{n-1}) + \tilde{d}(x_{n-1}, x_m)$.

By using triangle inequality, and Ω (t) = s

$\tilde{d}(x_n, x_m) \leq \Omega(\tilde{d}\ (x_n, x_{n-1}) + \tilde{d}\ (x_{n-1}, x_m))$

$\tilde{d}(x_n, x_m) \leq s(\tilde{d}(x_n, x_{n-1}) + \tilde{d}(x_{n-1}\ , x_m))$

$= s\ (\tilde{d}(x_n\ , x_{n-1}\ ) + s[\ \tilde{d}(x_{n-1}x_{n-2}) + s\ (\ \tilde{d}(x_{n-2}x_{n-3}) + \cdots + s(\ \tilde{d}(x_{m-1}\ , x_m))]$

$\vdots$

$= s\ (\tilde{d}(x_n\ , x_{n-1}\ ) + s^2\tilde{d}(x_{n-1}x_{n-2}) + s^3(\ \tilde{d}(x_{n-2}x_{n-3}) + \cdots + s^{n-m}(\tilde{d}(x_{m+1}\ , x_m))]$

$\leq \sum_{k=0}^{n-m-1} s^{k+1}\ \tilde{d}\ (x_{k+1},\ x_k)$

$\leq \sum_{k=m}^{n-1} s^{n-1-k}\ \ \Psi^k(r)\ \tilde{d}\ (x_{k+1},\ x_k)$ (14)

By (7), $\tilde{d}T\ (x_{k+1}, x_k) =$ $\tilde{d}\ (Tx_k, x_k) \leq \Psi^k(r)$ where $r = \tilde{d}\ (Tx_0, x_0)$,

Thus $\tilde{d}\ (x_n\ , x_m) \leq \Psi^k(r)$. As $k \to \infty$, $\tilde{d}\ (x_n,\ x_m) \to 0$.

Hence, $\{x_n\}$ is a Cauchy sequence in X. By completeness, there $x^* \in X$ such that $\lim_{n\to\infty} x_n = x^*$.

We need to show that x^* is a fixed point of X. By (4), (5) and triangle inequality we have

Applying triangle inequality, we get

$\tilde{d}(Tx^*, x^*) \leq \Omega(\ \tilde{d}(Tx^*, Tx_n) + \tilde{d}(Tx_n, x_n) + \tilde{d}(x_n, x^*)$,
$\leq \Omega\ \{k\tilde{d}(\ Tx_n, x_n) + \tilde{d}(x^*, x_n) + \tilde{d}(Tx_n, x_n) + \tilde{d}(x_n, x^*)\}$,
$\leq \Omega\{\ \tilde{d}(Tx_n, x_n) + (1 + k)\tilde{d}(x^*, x_n)$,,
$\leq \Omega\{k^n\tilde{d}(Tx_0, x_0) + (1 + k)\tilde{d}(x^*, x_n)$.

So, $\tilde{d}$ $(Tx^*, x^*) \to 0$ as $n \to \infty$ which implies that $Tx^*= x^*$and $x^*$is a fixed point. To show the uniqueness of the fixed point. Suppose $x^*$and $y^*$are fixed points such that $Tx^* = x^*$and $Ty^*= y^*$. Assume that $x^* \neq y^*$, we have

$$\tilde{d}\ \ (Tx^*, Ty^*) \leq k\tilde{d}\ (x^*, y^*) \quad \text{i.e.,} \quad \tilde{d}\ \ (x^*, y^*) \leq k\ \tilde{d}\ (x^*, y^*)$$

Since $k \in [0,1)$ and $\tilde{d}\ (x^*, y^*) \leq 0$, then $x^* = y^*$.

which contradicts the assumption that the fixed points are not distinct. Therefore, the fixed point $x^* \in X$ of mapping T is unique.

Remark 2.12. Theorem 2.16 generalizes Theorem 2.14 in [16] and extends the result of Beg [6] as well as the result of Goebel [14].

Corollary 2.13. Let (X, d, γ) be a complete quasi convex metric space, C, a nonempty closed quasi convex subset of X and T: $C \to C$, a (k)-Lipschitzian involution with $k \in [0, 1)$. Then T has a unique fixed point in C.

3. An Application of Quasi Convex p−metric Spaces to Noise Reduction and Smoothing in Signals

Noise reduction and smoothing often involve filtering techniques applied to signals. These techniques are used in various domains such as audio processing, vibration analysis, image processing, and electromagnetic signal processing. This problem traditionally is tackled by the use of Kalman filters which provide optimal solution”. In 2014, Oppenheim et al.[19] introduced the quantization error of a digital filter employing fixed-point arithmetic with sign-magnitude truncation is analyzed.

In 2022, Hazem, et.al.[15] introduced a fixed point iteration-based adaptive controller which is applicable for various strongly nonlinear models.

Many researchers (see [21, 24, 25]) have studied fixed points and signal processing. In this study we study fixed points in a convex p metric space to introduce an adaptive method using convexity structure of quasi convex p metric space as ρ (s_i, s_j ; λ) adaptive interpolation to smoothed values.

3.1. Moving Average Filtering

A moving average filter smooths the signals by averaging a set of number of consecutive samples. It is defined to calculate the moving average for each point in the signal. For a discrete-time signal x[n], the moving average filtered signal y[n] is given by:

$$y[n] = \frac{1}{M} \sum_{k=0}^{M-1} x|n-k|,$$

where M is the number of samples in the moving average with a window size.

3.2 Application of Quasi convex p-metric space to Noise Reduction and Smoothing Signal

Steps Involve in the Adaptive Method Used
1. Apply a sliding window.
2. Calculate the differences between the central points.
3. Apply Ω(t) = t^2 to the differences obtained.
4. Take the max of the squared differences.
5. Apply the adaptive interpolation using ρ (s_i, s_j ; λ) which can also be referred to smoothed values.

Define a p-metric $\tilde{d}(x, y)$ to calculate the distance between two signal points, that accounts for both the amplitude and frequency components of the signal. The convex structure ρ (x, y; λ) can represent the weighted average of two signal points (calculate the convex combination of two signal) and the filter signal function applies the quasi convex p-metric space for noise reduction by providing a smooth transition that minimizes noise.

Original Signal: S = {6, 7, 9, 12, 15, 14, 13, 17}.

Step- by -Step Calculation

Apply the quasi convex p-metric space with Ω(t) = t^2.

1. For s_2'

Sliding Window for s_2: {s_1, s_2 , s_3} = {6, 7, 9}

$\tilde{d}(s_1, s_2)$ = |6 − 7| = 1 and $\tilde{d}(s_3, s_2)$ = |9 − 7| = 2

max{ $\tilde{d}(s_1, s_2)$, $\tilde{d}(s_3, s_2)$}) = max {1, 2}=2

Ω (max{1, 2}) = 2^2 = 4.

Adaptive interpolating s_2 : ρ(s_1, s_3; 0.5) = 0.5(6) + 0.5(9) = 3 + 4.5 = 7.5

$$s_2' = 7.5$$

Note: s_i'; $i = 1, 2, \ . \ . \ .$ stand for new filtered or interpolated value of the original signal point.

Computing $s_1, s_2 , s_3 , s_4 \ \cdots \ s_8$, we generate the result is as shown in the table below:

TABLE 3.3: COMPARISON OF FILTERED SIGNALS TO ORGINAL SIGNAL

Signal	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8
$S_{original}$	6	7	9	12	15	14	13	17
$S_{filtered\ QCP}$	6	7.5	9.75	12.75	13.1815	13.09375	15.0465	17
$S_{filtered\ MA}$	7.33	9.33	12	13.67	14	14.69	NIL	NIL

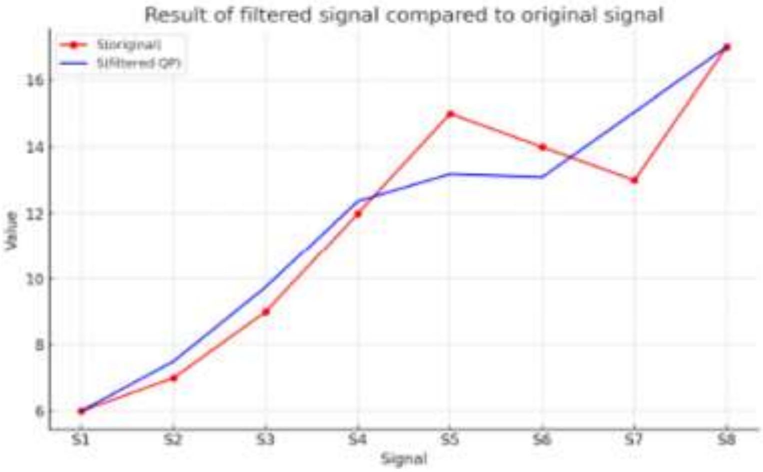

Figure 1 : Graphical 2D Representation of Signal Original and Quasi convex p-Filtered Signal .

Remark 3.4. The blue line showed the unprocessed signals and the natural fluctuations. From S_6 it rises steadily to S_4 and reaches its peak at S_5, slightly decreases at S_5 and S_6 but sharply increases at S8 indicating noises and fluctuation. Moving average does not exhibit sharp changes but can lag behind significant changes in the signal. Finally, quasi convex p-signal filtering smoothed the transitions. The rise in S_4 is gradual and it is sharp from S_5 to S_6. Also, it preserves the increase at S_7 to S_8 with a smoother, less sharp, rise better at preserving the structure of the original signal while reducing noise of fluctuations.

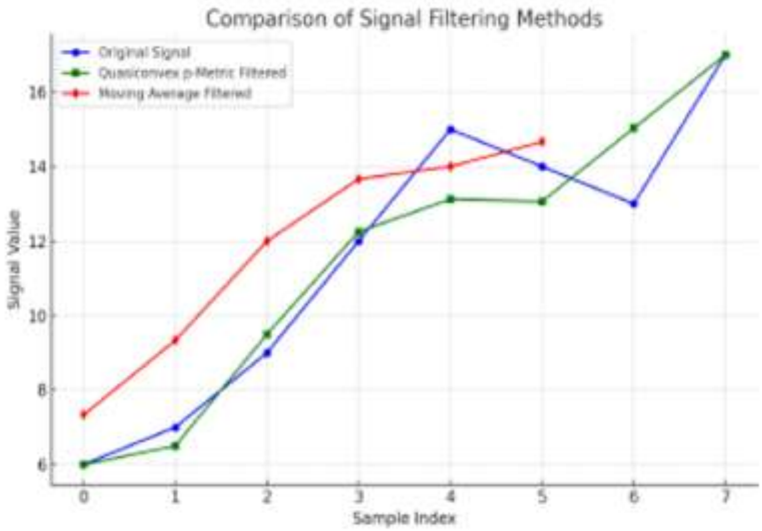

Figure 2 : Graphical 2D Representation of Original Signal, Moving Average Filtered and Quasi convex p Signal Filtered.

Remark 3.5. The original signal maintains sharp transition, detailed changes, but with more noise, while the filtered signal preserves the structure and the original signal, making the change less abrupt.

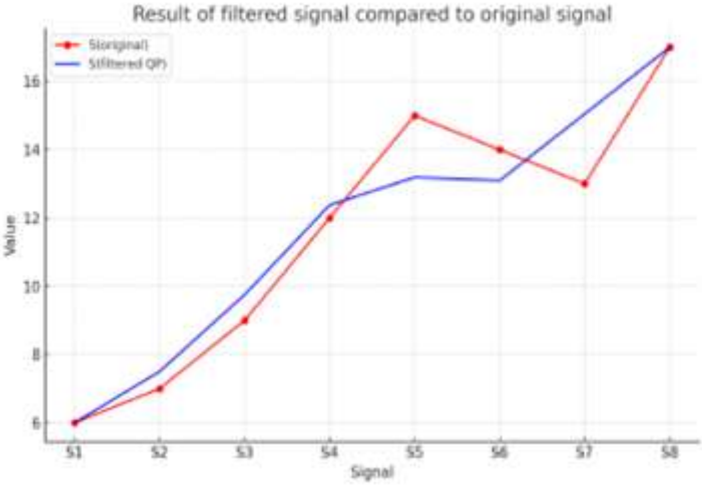

Figure 3 : Graphical 2D Representation of Signal Original and Quasi convex p- Signal p-Filtered Signal.

Remark 3.6. A 3D representation offers a more intuitive and visual clear understanding of the differences between the original and quasi convex p-filtered signals.

Conventional techniques, such as the moving average filter, work by smoothing signals by averaging over a local window. While effective, they often have the following characteristics: can blur sharp features, edges or sudden changes in the signal (like peaks) can be flattened. Cannot adapt to the shape or distribution of signal values (especially with irregular signals or outliers). While our adaptive method called quasi-convex *p*-metric space filter is an adaptive and nonlinear approach that filters sharp random spikes more effectively. Keeps peaks and signal transitions intact i.e., feature preservation. Finally, it is robust, especially in real-life signal processing (ECG, speech, sensor readings), where standard averaging might smooth away meaningful variations.

Conclusion

In this research, we introduced the notion of a quasi-convex p-metric space and proved several fixed point theorems that generalize some existing results in the literature. The adaptive method utilizing $\rho(s_i, s_j; \lambda)$ leads to more effective noise reduction, as it tailors the smoothing process to the local characteristics of the signal. This ensures that noise and outliers are managed more effectively while preserving the signal's integrity. Consequently, the utility of this method lies in its ability to elucidate the stability, periodicity, and long-term behavior of dynamical systems. Furthermore, this method is also applicable in areas such as the semantics of programming languages and the definition of recursive functions.

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