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Connection Formulas on Kummer's Solutions and their Extension on Hypergeometric Function

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Abstract: *Hypergeometric functions are transcendental functions that are applicable in various branches of mathematics, physics, and engineering. They are solutions to a class of differential equations called hypergeometric differential equations. Kummer obtained six solutions for the hypergeometric differential equation and twenty connection formulae. This research work has extended those connection formulas to other six solutions* $y_1(x), y_2(x), y_3(x), y_4(x), y_5(x), and\, y_6(x)$ *to show that each solution can be expressed in terms of linear relationship among three of the other solutions.*

Keywords: *Hypergeometric function, Kummer's formula, Connection formula*

1. Introduction and Motivation

Before Proceeding with the main work, we shall now introduce some basic notations,definitions and preliminaries that are used in this paper.

1.1 Hypergeometric Function[12]

The Gaussian hypergeometric function ${}_2F_1(a,b;c;x)$ is a special function represented by the hypergeometric series,

$$ {}_2F_1(a,b;c;x) = {}_2F_1\begin{bmatrix} a & b; & \\ & c; & x \end{bmatrix} = 1 + \sum_{n=1}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!} \qquad ...(1.1.1)$$

Where $(a)_n$ is called the Pochhammer symbol and is defined as

(i) $(a)_n = a(a+1)(a+2)(a+3)....a+(n-1) = \prod_{k=1}^{n}(a+k-1) = \frac{\Gamma(a+k)}{\Gamma(a)}$ …(1.1.2)

(ii) $(a)_0 = 1, \text{for}\, a \neq o$

If the value of $a,b,c \in Z^+$ in (1.1.1) then it is convergent for $|z| < 1$ [13] and if a, and b are the positive integers, $c \in \{0,-1,...,a+1\}$ and $c \in \{0,-1,...,b+1\}$, then the hypergeometric series (1.1.1) is a polynomial of degree $|a|$ or $|b|$, If $c = a$ and $b = c$ then it is not possible to define ${}_2F_1(a,b;c;z)$ [3]. In this case,

$$ {}_2F_1(a,b;c;x) = \sum_{k=0}^{\infty} \frac{(b)_k}{k!} x^k = (1-x)^{-b} \qquad \ldots(1.1.3)$$

If $\mathrm{Re}(c-a-b) > 0$, $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$ in (1.1.1) then it can be expressed in the form of gamma function through the Guass Kummer identity;

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \qquad \ldots (1.1.4)$$

The series (1.1.1) is a solution of a second-order linear ordinary hypergeometric differential equation known as Guass Hypergeometric differential equation[12],

$$x(1-x)y'' + (c-(a+b+1)x)y' - aby = 0 \qquad \ldots(1.1.5)$$

1.2 The solutions at the singularities

The equation (1.1.5) has a regular singularity at $x = 0, 1$, and infinity [5, 16]. The table given below, commonly known as Riemann Scheme table, shows the of local exponents of the hypergeometric differential equations at the variate values of x

$x=0$	$x=1$	$x=\infty$
0	0	a
$1-c$	$c-a-b$	b

According to Riemann scheme, the difference of the local exponent is not an integer. This condition is called the generic condition. In this condition the fundamental system of solutions are defined at each singular points.[4]. The fundamental solutions of this differential equation in different singular points are as below.[2]

(i) For singularity at $x = 0$,

$$y_1(x) = {}_2F_1(a,b;c;x) \qquad \ldots (1.2.1)$$

and $$y_2(x) = x^{1-c}\,{}_2F_1(a-c+1, b-c+1; 2-c; x) \qquad \ldots(1.2.2)$$

(ii) For singularity at $x = 1$,

$$y_3(x) = {}_2F_1(a,b;a+b+1-c;1-x) \qquad \ldots(1.2.3)$$

and $$y_4(z) = x^{c-a-b}\,{}_2F_1(c-a, c-b; c-a-b+1; 1-x) \qquad \ldots(1.2.4)$$

(iii) For singularity at $x = \infty$,

$$y_5(x) = x^{-a}\,{}_2F_1(a, a-c+1; a-b+1; \frac{1}{x}) \qquad \ldots(1.2.5)$$

and $$y_6(x) = x^{-b}\,{}_2F_1(b, b-c+1; b-a+1; \frac{1}{x}) \qquad \ldots(1.2.6)$$

1.3 Local Solutions and Connection formula

These six solutions published by Kummer, has four forms related to one another by Euler transformation giving twenty four forms in total [11]. These twenty four solutions are known as Kummer's solution of hypergeometric differential equation. For details, we refer [2, 8].

1.4 Connection Formulas

The six formulas as mentioned by Kummer [2, 8] for three parameters a, b, c and combination of three solutions,[14,15] with the property (1.1.4) will give ${}_6C_3 = 20$ connection formulae as the principle branches of Kummer's solution[1, 2, 10]. They are listed as follows;

$$y_3(x) = \frac{\Gamma(1-c)\Gamma(a+b-c+1)}{\Gamma(a-c+1)\Gamma(b-c+1)} y_1(x) + \frac{\Gamma(c-1)\Gamma(a+b-c+1)}{\Gamma(a)\Gamma(b)} y_2(x) \quad \dots (1.4.1)$$

$$y_4(x) = \frac{\Gamma(1-c)\Gamma(c-a-b+1)}{\Gamma(1-a)\Gamma(1-b)} y_1(x) + \frac{\Gamma(c-1)\Gamma(c-a-b+1)}{\Gamma(c-a)\Gamma(c-b)} y_2(x) \quad \dots (1.4.2)$$

$$y_5(x) = \frac{\Gamma(1-c)\Gamma(a-b+1)}{\Gamma(a-c+1)\Gamma(1-b)} y_1(x) + e^{(c-1)\pi i} \frac{\Gamma(c-1)\Gamma(a-b+1)}{\Gamma(a)\Gamma(c-b)} y_2(x) \quad \dots (1.4.3)$$

$$y_6(x) = \frac{\Gamma(1-c)\Gamma(b-a+1)}{\Gamma(b-c+1)\Gamma(1-a)} y_1(x) + e^{(c-1)\pi i} \frac{\Gamma(c-1)\Gamma(b-a+1)}{\Gamma(b)\Gamma(c-a)} y_2(x) \quad \dots (1.4.4)$$

$$y_1(x) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} y_3(x) + \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} y_4(x) \quad \dots (1.4.5)$$

$$y_2(x) = \frac{\Gamma(2-c)\Gamma(c-a-b)}{\Gamma(1-a)\Gamma(1-b)} y_3(x) + \frac{\Gamma(2-c)\Gamma(a+b-c)}{\Gamma(a-c+1)\Gamma(b-c+1)} y_4(x) \quad \dots (1.4.6)$$

$$y_5(x) = e^{a\pi i} \frac{\Gamma(a-b+1)\Gamma(c-a-b)}{\Gamma(1-b)\Gamma(c-b)} y_3(x) + e^{(c-b)\pi i} \frac{\Gamma(a-b+1)\Gamma(a+b-1)}{\Gamma(a)\Gamma(a-c+1)} y_4(x) \quad \dots (1.4.7)$$

$$y_6(x) = e^{b\pi i} \frac{\Gamma(b-a+1)\Gamma(c-a-b)}{\Gamma(1-a)\Gamma(c-a)} y_3(x) + e^{(c-a)\pi i} \frac{\Gamma(b-a+1)\Gamma(a+b-c)}{\Gamma(b)\Gamma(b-c+1)} y_4(x) \quad \dots (1.4.8)$$

$$y_1(x) = \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)} y_5(x) + \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)} y_6(x) \quad \dots (1.4.9)$$

$$y_2(x) = e^{(1-c)\pi i} \frac{\Gamma(2-c)\Gamma(b-a)}{\Gamma(1-a)\Gamma(b-c+1)} y_5(x) + e^{(1-c)\pi i} \frac{\Gamma(2-c)\Gamma(a-b)}{\Gamma(1-b)\Gamma(a-c+1)} y_6(x) \quad \dots (1.4.10)$$

$$y_3(x) = e^{-a\pi i} \frac{\Gamma(a+b-c+1)\Gamma(b-a)}{\Gamma(b)\Gamma(b-c+1)} y_5(x) + e^{-b\pi i} \frac{\Gamma(a+b-c+1)\Gamma(a-b)}{\Gamma(a)\Gamma(a-c+1)} y_6(x) \quad \dots (1.4.11)$$

$$y_4(x) = e^{(b-c)\pi i} \frac{\Gamma(c-a-b+1)\Gamma(b-a)}{\Gamma(1-a)\Gamma(c-a)} y_5(x) + e^{(a-c)\pi i} \frac{\Gamma(c-a-b+1)\Gamma(a-b)}{\Gamma(1-b)\Gamma(c-b)} y_6(x) \dots (1.4.12)$$

$$y_1(x) = e^{b\pi i} \frac{\Gamma(c)\Gamma(a-c+1)}{\Gamma(a+b-c+1)\Gamma(c-b)} y_3(x) + e^{(b-c)\pi i} \frac{\Gamma(c)\Gamma(a-c+1)}{\Gamma(b)\Gamma(a-b+1)} y_5(x) \quad \dots (1.4.13)$$

$$y_1(x) = e^{a\pi i} \frac{\Gamma(c)\Gamma(b-c+1)}{\Gamma(a+b-c+1)\Gamma(c-a)} y_3(x) + e^{(a-c)\pi i} \frac{\Gamma(c)\Gamma(b-c+1)}{\Gamma(a)\Gamma(b-a+1)} y_6(x) \quad \dots (1.4.14)$$

$$y_2(x) = e^{(b-c+1)\pi i} \frac{\Gamma(2-c)\Gamma(a)}{\Gamma(a+b-c+1)\Gamma(1-b)} y_3(x) + e^{(b-c)\pi i} \frac{\Gamma(2-c)\Gamma(a)}{\Gamma(a-b+1)\Gamma(b-c+1)} y_5(x) \dots (1.4.15)$$

$$y_2(x) = e^{(a-c+1)\pi i} \frac{\Gamma(2-c)\Gamma(b)}{\Gamma(a+b-c+1)\Gamma(1-a)} y_3(x) + e^{(a-c)\pi i} \frac{\Gamma(2-c)\Gamma(b)}{\Gamma(b-a+1)\Gamma(a-c+1)} y_6(x) \dots (1.4.16)$$

$$y_1(x) = e^{(c-a)\pi i}\frac{\Gamma(c)\Gamma(1-b)}{\Gamma(a)\Gamma(c-a-b+1)}y_4(x) + e^{-a\pi i}\frac{\Gamma(c)\Gamma(1-b)}{\Gamma(a-b+1)\Gamma(c-a)}y_5(x) \qquad \ldots (1.4.17)$$

$$y_1(x) = e^{(c-b)\pi i}\frac{\Gamma(c)\Gamma(1-a)}{\Gamma(b)\Gamma(c-a-b+1)}y_4(x) + e^{-b\pi i}\frac{\Gamma(c)\Gamma(1-a)}{\Gamma(b-a+1)\Gamma(c-b)}y_6(x) \qquad \ldots (1.4.18)$$

$$y_2(x) = e^{(1-a)\pi i}\frac{\Gamma(2-c)\Gamma(c-b)}{\Gamma(a-c+1)\Gamma(c-a-b+1)}y_4(x) + e^{-a\pi i}\frac{\Gamma(2-c)\Gamma(c-b)}{\Gamma(a-b+1)\Gamma(a-1)}y_5(x) \qquad \ldots (1.4.19)$$

$$y_2(x) = e^{(1-b)\pi i}\frac{\Gamma(2-c)\Gamma(c-a)}{\Gamma(b-c+1)\Gamma(c-a-b+1)}y_4(x) + e^{-b\pi i}\frac{\Gamma(2-c)\Gamma(c-a)}{\Gamma(b-a+1)\Gamma(1-b)}y_6(x) \qquad \ldots (1.4.20)$$

2. Research Objective

In 1837, Kummer introduced the solution to the Kummer differential equation which is known as Confluent hypergeometric function. In the meantime he had discovered twenty four solutions for the same, which subsequently formed the twenty formulas as the branches of Kummer solution [13]. They are listed in relations (1.4.1-1.4.20). In this paper, our objective is to find the relations between any four sets of solutions and also to express any one of them as the linear combination of the other three solutions.

3. Main Result

In section 1.4, the connection formulas for six different solutions, each consisting of two different solutions are presented. The extension of connection formula refers to the combination of any three solutions for a given solution. Each extension formulas is obtained as the combination of three different solutions. The combination of six formulas taken four at a time constitute of ${}^6C_4 = 12$ solutions. The six extension of connection formula are already evaluated by Poudel et.al [9] The remaining six connection formulas will be obtained in this research paper. Those results are presented as follows.

2.1 Extension formula

1. $$y_1(x) = \frac{\Gamma(a-c+1)\Gamma(b-c+1)}{\Gamma(1-c)}\left[\begin{array}{l} e^{-a\pi i}\dfrac{\Gamma(b-a)}{\Gamma(b)\Gamma(b-c+1)}y_5(x) \\ +e^{-b\pi i}\dfrac{\Gamma(a-b)}{\Gamma(a)\Gamma(a-c+1)}y_6(x) - \dfrac{\Gamma(c-1)}{\Gamma(a)\Gamma(b)}y_2(x) \end{array}\right] \qquad \ldots (2.1.1)$$

2. $$y_2(x) = \frac{1}{\Gamma(c-1)}\left[\begin{array}{l} e^{-a\pi i}\dfrac{\Gamma(b-a)\Gamma(a)}{\Gamma(b-c+1)}y_5(x) + e^{-b\pi i}\dfrac{\Gamma(a-b)\Gamma(b)}{\Gamma(a-c+1)}y_6(x) \\ -\dfrac{\Gamma(1-c)\Gamma(a)\Gamma(b)}{\Gamma(a-c+1)\Gamma(b-c+1)}y_1(x) \end{array}\right] \qquad \ldots (2.1.2)$$

3. $$y_3(x) = \frac{\Gamma(a+b-c+1)\Gamma(c-b)}{\Gamma(a-c+1)}\left[\begin{array}{l} e^{(c-2b)\pi i}\dfrac{\Gamma(1-a)y_4(x)}{\Gamma(b)\Gamma(c-a-b+1)} + e^{-2b\pi i}\dfrac{\Gamma(1-a)y_6(x)}{\Gamma(b-a+1)\Gamma(c-b)} \\ -e^{-c\pi i}\dfrac{\Gamma(a-c+1)y_5(x)}{\Gamma(b)\Gamma(a-b+1)} \end{array}\right] \qquad \ldots (2.1.3)$$

4. $$y_4(x)=\frac{\Gamma(a)\Gamma(c-a-b+1)}{\Gamma(1-b)}\left[\begin{array}{l} e^{(2a-c)\pi i}\dfrac{\Gamma(b-c+1)y_3(x)}{\Gamma(a+b-c+1)\Gamma(c-a)}+e^{2(a-c)\pi i}\dfrac{\Gamma(b-c+1)y_6(x)}{\Gamma(a)\Gamma(b-a+1)} \\ -e^{-c\pi i}\dfrac{\Gamma(1-b)y_5(x)}{\Gamma(a-b+1)\Gamma(c-a)} \end{array}\right] \quad \ldots(2.1.4)$$

5. $$y_5(x)=e^{(c-b)\pi i}\frac{\Gamma(a-b+1)\Gamma(b-c+1)}{\Gamma(2-c)\Gamma(a)}\left[\begin{array}{l} \dfrac{(e^{(1-b)\pi i})\Gamma(c-a)y_4(x)}{\Gamma(b-c+1)\Gamma(c-a-b+1)}+\dfrac{(e^{-b\pi i})\Gamma(c-a)y_6(x)}{\Gamma(b-a+1)\Gamma(1-b)} \\ -\dfrac{(e^{(b-c+1)\pi i})\Gamma(a)y_3(x)}{\Gamma(a+b-c+1)\Gamma(1-b)} \end{array}\right] \quad \ldots(2.1.5)$$

6. $$y_1(x)=e^{(c-1)\pi i}\frac{\Gamma(1-b)\Gamma(a-c+1)}{\Gamma(a-b)}\left[\begin{array}{l} \dfrac{\Gamma(c-a-b)y_3(x)}{\Gamma(1-a)\Gamma(1-b)}+\dfrac{\Gamma(a+b-c)y_4(x)}{\Gamma(a-c+1)\Gamma(b-c+1)} \\ -e^{(1-c)\pi i}\dfrac{\Gamma(b-a)y_5(x)}{\Gamma(1-a)\Gamma(b-c+1)} \end{array}\right] \quad \ldots(2.1.6)$$

The proof of the above extension formulas are as follows;

2.1 Derivation of the extension formula for (2.1.1)

From (1.4.1) and (1.4.11), we get

$$\frac{\Gamma(1-c)\Gamma(a+b-c+1)}{\Gamma(a-c+1)\Gamma(b-c+1)}y_1(x)+\frac{\Gamma(c-1)\Gamma(a+b-c+1)}{\Gamma(a)\Gamma(b)}y_2(x)$$

$$=e^{-a\pi i}\frac{\Gamma(a+b-c+1)\Gamma(b-a)}{\Gamma(b)\Gamma(b-c+1)}y_5(x)+e^{-b\pi i}\frac{\Gamma(a+b-c+1)\Gamma(a-b)}{\Gamma(a)\Gamma(a-c+1)}y_6(x)$$

or, $$\frac{\Gamma(1-c)}{\Gamma(a-c+1)\Gamma(b-c+1)}y_1(x)$$

$$=e^{-a\pi i}\frac{\Gamma(b-a)}{\Gamma(b)\Gamma(b-c+1)}y_5(x)+e^{-b\pi i}\frac{\Gamma(a-b)}{\Gamma(a)\Gamma(a-c+1)}y_6(x)-\frac{\Gamma(c-1)}{\Gamma(a)\Gamma(b)}y_2(x)$$

or, $$y_1(x)=\frac{\Gamma(a-c+1)\Gamma(b-c+1)}{\Gamma(1-c)}\left[e^{-a\pi i}\frac{\Gamma(b-a)}{\Gamma(b)\Gamma(b-c+1)}y_5(x)+e^{-b\pi i}\frac{\Gamma(a-b)}{\Gamma(a)\Gamma(a-c+1)}y_6(x)-\frac{\Gamma(c-1)}{\Gamma(a)\Gamma(b)}y_2(x)\right]$$

This proves the extension formula (2.1.1).

Applying the similar derivations from the given relations we obtain the formulae (2.1.2)-(2.1.6). From formulas (1.4.2) and (1.4.12), we get the connection formula for $y_2(x)$ in(2.1.2), Similarly using the formulas (1.4.13) and (1.4.18) we get the connection formula for $y_3(x)$ in (2.1.3),from (1.4.14) and (1.4.17), we get the connection formula for $y_4(x)$ in(2.1.4), from (1.4.15) and (1.4.20), we get the connection formula for $y_5(x)$ in (2.1.5) and finally from (1.4.6) and (1.4.10), we get the formula for $y_6(x)$ in (2.1.6).

3. Conclusion

The hypergeometric function is the solution of the Gaussian hypergeometric differential equation[1]. Kummer has obtained six solutions and twenty connecting formulas for the second-order hypergeometric differential equation. By the help of these formulas listed in (1.3.1-1.7.6) and (1.4.1-1.4.20) respectively, for the hypergeometric differential equation, we have obtained additional six extensions [(2.1.1)-(2.1.6)] of the connecting formulas for $y_1(x)$, $y_2(x)$ $y_3(x)$, $y_4(x)$, $y_5(x)$ and $y_6(x)$. Every solution are expressed as the linear combination of other three solutions. These solutions are highly applicable in various branches of applied sciences.

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