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Collocation Computational Technique For Fractional Integro-Differential Equations

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Abstract: *In this study, the collocation method and first-kind Chebyshev polynomials are used to investigate the solution of fractional integral-differential equations. In order to solve the problem, we first convert it to a set of linear algebraic equations, which are then solved by using matrix inversion to get the unknown constants. To demonstrate the theoretical findings, a few numerical examples are given and compared with other results obtained by other numerical techniques. Tables and figures are utilized to demonstrate the accuracy and effectiveness of the method. The outcomes demonstrate that the method improved accuracy more effectively while requiring less labor-intensive tasks.*

Keywords: *First-kind Chebyshev polynomials, Fractional integro-differential equations, Numerical technique, Matrix inversion.*

Introduction

The utilization of fractional integro-differential equations (IDEs) has significantly enhanced the modeling of real-world physical problems. Fractional calculus stands out as the most effective approach for capturing unusual phenomena. Illustrative instances encompass the dispersion of heat within a furnace, the spread of viruses, the positioning of satellites in space, and the memory characteristics of a system, among others. As commonly acknowledged, the collocation method hinges on the notion of approximating the precise solution of a given functional equation using an appropriate approximant selected from a finite-dimensional space, typically a piecewise algebraic polynomial. This approximant precisely satisfies the equation within a specific subset of the integration interval, known as the set of collocation points. [1–8]. According to [9–12], the concept of fractional calculus originated from a question over whether the definition of a derivative to an integer order n could be expanded to still hold true when is not an integer. This question was later forgotten because the formula for fractional derivatives is complex, making it difficult to work with ordinary pencil and paper. However, because we have computers and machines, complexity is no longer an issue.

The majority of fractional integro-diffrential equations (FIDEs) cannot be solved analytically; hence, extensive research has been done to find approximations and numerical methods of solving FIDEs.

Fractional Fredholm IDEs are solved using Laguerre polynomials in [13] and Bernstein polynomials as the basis function in [14, 15] to approximate the solution of FIDEs. In [16–18], collocation techniques were used to solve FIDEs using various basis functions. In [19], the Sumudu transform method and the Hermite spectral collocation method are used to solve FIDEs; When solving Volterra fractional IDEs, [20] used Bernstein modified homotopy perturbation approach; and in [21], approximate solutions of Volterra-Fredholm IDEs of fractional order are introduced. Using Galerkin method and Taylor series expansion, as well as a quick numerical algorithm based on the second kind of Chebyshev polynomials, [22, 23] investigated the numerical solution of fractional singular IDEs. [24, 25] used the least-squares method to solve FIDEs. [26] investigated the solution of linear fractional Fredholm integro-differential equation by using second kind Chebyshev wavelet and [27] employed numerical techiques for the solution of nonlinear integro-differential equations. [28] proposed and investigate a spectral approximation for numerical solutions of fractional integro-differential equations with weakly kernels. In order to eliminate the solution's singularity, the original equations are changed into an equivalent weakly singular Volterra integral equation by incorporating some relevant smoothing transformations. The above work serves as the motivation for the present study.

In this study, we present an innovative and precise numerical method for addressing fractional integro-differential equation systems. Our approach employs the collocation computational technique, utilizing first-kind Chebyshev polynomials as the basis functions for solving these fractional IDEs. This method results in less demanding work in terms of computational cost and better accuracy.

The rest of the paper is structured as follows: Section 2 deals with some relevant basic definitions, section 3 deals with the demonstration of the suggested method. Numerical examples which demonstrate the method's applicability and validity is given in section 4, section 5 deals with results and discussion of results. Finally, the conclusion of the study is presented in section 6. The general form of the class of problem considered in this work is given as:

$$D^{\alpha}\omega(s) = p(s)\omega(s)f(s) + \int_0^s K(s,t)\omega(t)dt; 0 \leq s,t \leq 1, \tag{1}$$

with the following supplementary conditions:

$$\omega^{(j)}(0) = \omega_j; j = 0,1,2,...n-1; n-1 < \alpha \leq n, n \in N. \tag{2}$$

Where $D^{\alpha}\omega(s)$ is the α^{th} Caputo fractional derivative of $\omega(s)$; $p(s)$, $f(s)$ and $K(s,t)$ are given smooth functions, ω_j are real constant, and s are real variables varying [0, 1] and $\omega(s)$ is the unknown function to be determined.

Some relevant basic definitions

Definition 1. Fractional integro-differential equation is an equation in which the unknown $\omega(s)$ appears under the integral sign and contain fractional derivatives $D^{\alpha}\omega(s)$ as well. According to [29], a standard fractional integro-differential equations is defined as:

$$D^{\alpha}\omega(s) = f(s) + \lambda \int_{g(s)}^{h(s)} K(s,t)\omega(s)dt,$$

where $K(s,t)$ is a function of two variables s and t known as the kernel or the nucleus of the integral equation, $g(s)$ and $h(s)$ are the limits of integration, λ is a constant parameter.

Definition 2. The Caputo Factional Derivative is defined as [30]

$$D^{\alpha}\omega(s) = \frac{1}{\Gamma(r-\alpha)} \int_0^s (s-t)^{r-\alpha-1}\omega^{r}(t)dt \tag{3}$$

n is non-negative integer such that, $r-1 < \alpha < n$. For example, if $0 < \alpha < 1$, the Caputo fractional derivative is

$$D^{\alpha}\omega(s) = \frac{1}{\Gamma(1-\alpha)} \int_0^s (s-t)^{r-\alpha-1}\omega^{'}(t)dt. \tag{4}$$

Definition 3. The Chebyshev polynomials [31] of degree r over $[0,1]$ is defined by the relation

$$\upsilon_r^*(s) = cos\left\{Cos^{-1}(2s-1)\right\}; n \geq 0.$$

The recurrence relation is given as,

$$\upsilon_{r+1}^*(s) = 2(2s-1)\upsilon_r^*(s) - \upsilon_{r-1}^*(s); r \geq 1,$$

where

$$\upsilon_0^*(s) = 1, \upsilon_1^*(s) = 2s-1.$$

Implementation of the method

The study considered an estimated solution represented in the form of first-kind Chebyshev polynomials:

$$\omega(s) = \sum_{i=0}^{r} \upsilon_i^*(s) a_i, \tag{5}$$

Here, the constants a_i for $i = 0(1)r$ represent the undisclosed coefficients of the shifted Chebyshev polynomials that need to be ascertained. The approach relies on the approximation of the unknown function $\omega(s)$ by employing equation (3) to equation (1). Additionally, substituting equation (5) into (1) yields,

$$\frac{1}{\Gamma(1-\alpha)} \int_0^s (s-t)^{r-\alpha-1} \frac{d^r}{dt^r} (\sum_{i=0}^{r} \upsilon_i^*(t)) a_i dt - p(x)\upsilon_i^*(s) - \int_0^s k(s,t)\upsilon_i^*(t)dt = f(s) \tag{6}$$

$$Let \quad \zeta(s) = \frac{1}{\Gamma(1-\alpha)} \int_0^s (s-t)^{r-\alpha-1} \frac{d^r}{dt^r} (\sum_{i=0}^{r} \upsilon_i^*(t)) a_i dt,$$

$$\eta(s) = \int_0^s k(s,t)\upsilon_i^*(t).$$

Substituting $\zeta(s)$ and $\eta(s)$ in equation (6), gives

$$\zeta(s) - p(s)\upsilon_i^*(x) - \eta(s) = f(s). \tag{7}$$

Collocating (7) at equally spaced point $s_i = a + \frac{(b-a)i}{r}$, $(i = 0(1)(r))$ results into linear system algebraic of equations in (r+1) unknown constants $a_i's$. Also, additional equations are also derived from (2) and are represented in matrix form:

$$\begin{pmatrix} Q_{11} & Q_{12} & Q_{13} & \cdots & \cdots & \cdots & Q_{1r} \\ Q_{21} & Q_{22} & Q_{23} & \cdots & \cdots & \cdots & Q_{2r} \\ \vdots & \vdots & \vdots & & \vdots & & \\ \vdots & \vdots & \vdots & & \vdots & & \\ Q_{m1} & Q_{m2} & Q_{m3} & \cdots & \cdots & \cdots & Q_{mr} \\ Q_{11}^0 & Q_{12}^0 & Q_{13}^0 & \cdots & \cdots & \cdots & Q_{1r}^0 \\ Q_{21}^1 & Q_{22}^1 & Q_{23}^1 & \cdots & \cdots & \cdots & Q_{2r}^1 \\ \vdots & \vdots & \vdots & & \vdots & & \\ \vdots & \vdots & \vdots & & \vdots & & \\ Q_{m1}^{r-1} & Q_{m2}^{r-1} & Q_{m3}^{r-1} & \cdots & \cdots & \cdots & Q_{mr}^{r-1} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ a_r \end{pmatrix} = \begin{pmatrix} R_{11} \\ R_{22} \\ \vdots \\ \vdots \\ Rmr \\ R_{11}^0 \\ R_{22}^1 \vdots \\ \vdots \\ R_{mr}^{r-1} \end{pmatrix} \tag{8}$$

where Q_{is} and Q_{is}^{r-1} are the coefficients of a_{is} and R_{is} are values of $f(s_i)$
The matrix inversion approach is then used to solve the system of equations in order to obtain the unknown constants.

$$\begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ a_r \end{pmatrix} = \begin{pmatrix} Q_{11} & Q_{12} & Q_{13} & \cdots & \cdots & \cdots & Q_{1r} \\ Q_{21} & Q_{22} & Q_{23} & \cdots & \cdots & \cdots & Q_{2r} \\ \vdots & \vdots & \vdots & & \vdots & & \\ \vdots & \vdots & \vdots & & \vdots & & \\ Q_{m1} & Q_{m2} & Q_{m3} & \cdots & \cdots & \cdots & Q_{mr} \\ Q_{11}^0 & Q_{12}^0 & Q_{13}^0 & \cdots & \cdots & \cdots & Q_{1r}^0 \\ Q_{21}^1 & Q_{22}^1 & Q_{23}^1 & \cdots & \cdots & \cdots & Q_{2r}^1 \\ \vdots & \vdots & \vdots & & \vdots & & \\ \vdots & \vdots & \vdots & & \vdots & & \\ Q_{m1}^{r-1} & Q_{m2}^{r-1} & Q_{m3}^{r-1} & \cdots & \cdots & \cdots & Q_{mn}^{n-1} \end{pmatrix}^{-1} \begin{pmatrix} R_{11} \\ R_{22} \\ \vdots \\ \vdots \\ Rmr \\ R_{11}^0 \\ R_{22}^1 \vdots \\ \vdots \\ R_{mr}^{r-1} \end{pmatrix} \tag{9}$$

The sought-after approximate solution is derived through the solution of equation (9), followed by the insertion of the determined constant values into the assumed approximate solution.

Numerical examples with results and discussion

In this section, three numerical problems are presented to test the efficiency and simplicity of the suggested method. We perform the computation with the help of Maple 18 software.

Example 1. Consider the fractional Volterra integro-differential equation [32]

$$D^{\alpha}\omega(s) = \frac{s^2 e^s}{5}\omega(s) + \frac{6x^{2.25}}{\Gamma(3.25)} + \int_0^s t\,\omega(t)dt, \tag{10}$$

subject to $\omega(0) = 0$, for $\alpha = \frac{3}{4}$, the exact solution is $\omega(s) = s^3$. Applying the proposed technique for different values $\alpha = 0.65, 0.75, 0.85, 0.95$ respectively, we have the following approximate solutions.

$$\begin{aligned}
\omega(s) &= -3\times 10^{-11} + 3\times 10^{-9}s - 3\times 10^{-9}s^2 + 1.0000000001s^3\\
\omega(s) &= 2\times 10^{-11} - 0.027653591s + 0.153378821s^2 + 1.008595542s^3\\
\omega(s) &= -4\times 10^{-11} + 0.034398374s - 0.1426233144s^2 + 0.9903261389s^3\\
\omega(s) &= 1\times 10^{-10} + 0.07693657s - 0.277005541s^2 + 0.9802676637s^3
\end{aligned}$$

Example 2. Consider the fractional Fredholm Integro-differential equation [24]

$$D^{\alpha}\omega(s) = \omega(s) + \frac{8}{3\Gamma(0.5)}s^{1.5} - s^2 - \frac{1}{3}s^3 + \int_0^s \omega(t)dt \tag{11}$$

Subject to $\omega(0) = 0$, for $\alpha = \frac{1}{2}$, the exact solution is $\omega(s) = s^2$. Applying the proposed technique for different values $\alpha = 0.5, 0.65, 0.75, 0.85$ respectively, we have the following approximate solutions.

$$\begin{aligned}
\omega(s) &= 1.16826354\times 10^{-11} - 2.83\times 10^{-9}s + 1.0000000043s^2 - 3.573844333s\times 10^{-9}s^3\\
\omega(s) &= -7.2\times 10^{-11} - 0.0928995883s + 0.8741581798s^2 - 0.1738219395s^3\\
\omega(s) &= -2.7\times 10^{-11} - 0.1234325707s + 0.7798780867s^2 - 0.1944653643s^3\\
\omega(s) &= -1.9\times 10^{-11} - 0.1453990287s + 0.6972345157s^2 - 0.193248671s^3
\end{aligned}$$

Example 3. Consider the fractional Volterra Integro-differential equation [33].

$$D^{1/3}\omega(s) = \frac{3\sqrt{\pi}}{4\Gamma(\frac{13}{6})}s^{\frac{4}{3}} - \frac{2}{63}s^{\frac{9}{2}}(9+7s^2) + \int_0^s (st - s^2t^2)\omega(t)dt, \tag{12}$$

subject to initial conditions $\omega(0) = 0$ with the non-polynomial exact solution $\omega(s) = s^{\frac{3}{2}}$. Applying the proposed technique for different values $\alpha = 0.333333, 0.35, 0.45, 0.55, 0.65$ respectively, we have the following approximate solutions.

$$\begin{aligned}
\omega(s) &= 0.1593261367s - 0.5714144863s^7 + 2.492317064s^6 - 4.596356361s^5\\
&\quad + 4.735599489s^4 - 3.136374607s^3 + 1.917377053s^2 + 3.601\times 10^{-11}\\
\omega(s) &= 0.1468793779s + 2.342527711s^6 - 4.324730514s^5 + 4.464678275s^4\\
&\quad - 0.5389237087s^7 + 1.872627517s^2 - 2.979559309s^3 + 5.184\times 10^{-11}\\
\omega(s) &= 0.08760750183s + 1.540065996s^6 - 2.408\times 10^{11} - 2.864694840s^5\\
&\quad + 2.987408235s^4 - 0.3625998682s^7 + 1.590190322s^2 - 2.088567904s^3\\
\omega(s) &= 0.04859546975s + 0.9314622181s^6 - 1.745512022s^5 + 1.822787227s^4\\
&\quad - 0.2276405006s^7 + 1.310079152s^2 - 1.333448084s^3 - 5.874\times 10^{-11}\\
\omega(s) &= 0.02341269350s + 0.5076386080s^6 - 0.9551665749s^5 + 0.9749333984s^4\\
&\quad - 0.1336620560s^7 + 1.053811826s^2 - .7390996908s^3 + 2.425\times 10^{-11}
\end{aligned}$$

In this section, we present the results and discussion of the study. Tables 1-3 shows comparison of the absolute errors for examples 1-3, while figures 1-3 shows the graphical behaviour of the approximation solutions of example 1-3.

Table 1. Comparison of the absolute errors for example 1

s	Exact	Appro. Solu. n=3	Absol. Error at n =3	Absol. Error n=4 [32]
0.0	0.000	−0.00000000003000	$3.000E-11$	$3.000E-5$
0.2	0.008	0.00800000025800	$2.580E-10$	$3.710E-5$
0.4	0.064	0.06400000035000	$3.540E-10$	$2.400E-5$
0.6	0.216	0.21600000030000	$3.060E-10$	$8.400E-5$
0.8	0.512	0.51200000020000	$1.620E-10$	$4.300E-5$
0.1	1.000	1.00000000000000	$0.000E+00$	$2.800E-5$

Table 2. comparison of the absolute errors for example 2

s	Exact	Appro. Solu. at n=3	Absol. Error n=3	Absol. Error [24]
0.0	0.000	0.00000000001168	$1.168E-11$	$0.000E+00$
0.2	0.040	0.03999999958000	$4.169E-10$	$1.557E-04$
0.4	0.160	0.15999999930000	$6.970E-10$	$2.887E-03$
0.6	0.360	0.35999999890000	$1.000E-09$	$1.681E-02$
0.8	0.640	0.63999999860000	$1.498E-09$	$6.069E-02$
0.1	1.000	0.99999999740000	$2.362E-09$	$1.683E-01$

Table 3. comparison of the absolute errors for example 3

s	Exact	Appro. Solu. n=3	Absol. Error n=3	Absol. Error [33]
0.0	0.0000000000	0.00000000003601	$3.601E-11$	–
0.2	0.0894427191	0.08972763196000	$2.849E-04$	$9.8E-03$
0.4	0.2529822128	0.25321979130000	$2.376E-04$	$4.9E-03$
0.6	0.4647580015	0.46500112560000	$2.431E-04$	$3.2E-03$
0.8	0.7155417528	0.71583758500000	$2.958E-04$	$3.5E-03$
1.0	1.00000000000	1.00047428800000	$4.743E-04$	$3.5E-03$

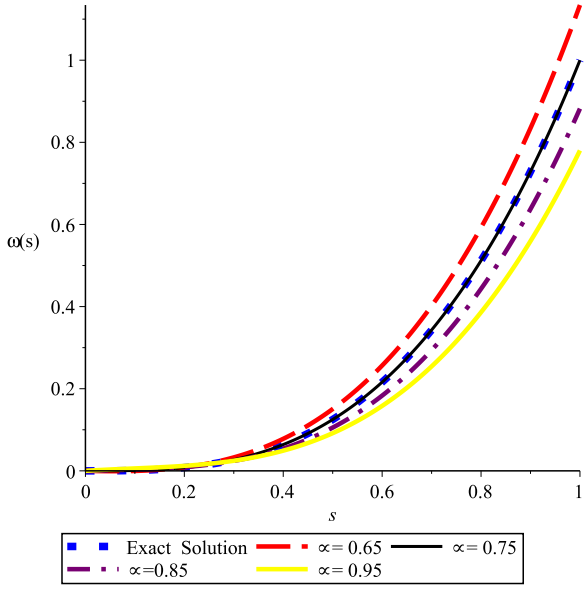

Figure 1. Showing the graphical behaviour of the approximation solutions of example 1

Using the collocation method via cubic B-spline wavelets, example 1 was solved by [31] at $n = 4$, [32] applied the homotopy analysis transform method for solving example 2, and example 3 was solved by [24] using three numerical schemes. By comparing the results, it can be seen from tables 1- 3 that the proposed method performed better when compared with the results obtained by other numerical methods. Also, figures 1-3, demonstrate that the approximate solutions are in excellent agreement with the exact solutions, and as the values of α increase the curve tend to zero.

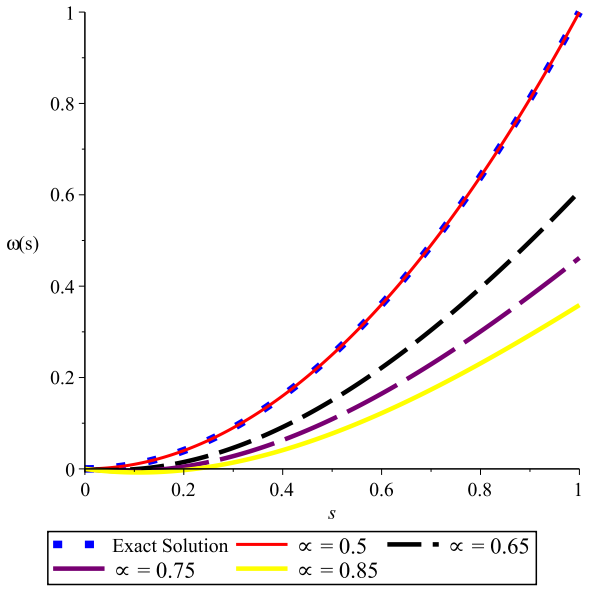

Figure 2. Showing the graphical behaviour of the approximation solutions of example 2

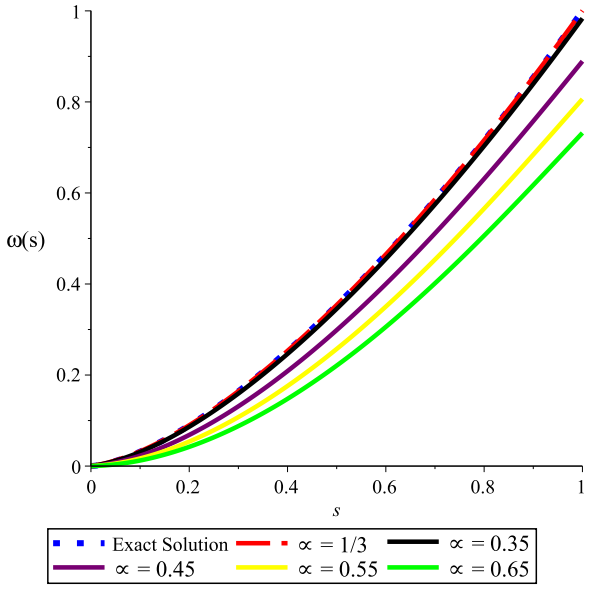

Figure 3. Showing the graphical behaviour of the approximation solutions of example 3

Conclusion

This work demonstrates a numerical solution of fractional integro-differential equations using collocation computational technique. Three examples are used to demonstrate the method's applicability and validity, and it appears that the method produces favourable results. We confirmed that the proposed method is in excellent agreement with the exact solutions, the solution obtained using the proposed method is more accurate than the obtained result in [24, 31, 32]. On the basis of this work, researchers can extend this technique to some other fractional integro-differential equations. The research will be valuable in multidisciplinary fields such as science and engineering, among others. It is helpful because it addresses the challenge of dealing with fractional order integro-differential problems by employing a simple collocation technique. The method has the advantage of being more accurate and requiring less computer time to run. Fractional integro-differential equations can be used to simulate many real life situations.

Data Availability Statement

No data associated in the manuscript.

Conflicts of interest

The authors declare that they have no conflict of interest concerning the publication of this manuscript.

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