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Numerical Analysis of Fractional-Order Diffusion Equation

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Abstract: *Fractional diffusion equations serve as fundamental tools for addressing the non-local properties and long range memory effects that observed in diffusion processes within complex media. This works focuses on solving non-integer order (fractional) diffusion equations by employing the natural decomposition approach which gives the solution in series form. Some numerical examples of one dimensional and two dimensional fractional order diffusion equations are presented to demonstrate its application and obtained solutions are interpreted with the help of the computational software. Compared to other analytical and numerical techniques, the fractional natural decomposition method demonstrates advantages such as reduced computational complexity and faster convergence. Additionally, it can also be readily applied to address linear as well as non-linear problems. The application of natural decomposition approach to solve non-integer order (fractional) diffusion equations provides the most comprehensive understanding of the anomalous diffusion process occurring within complex media, as the fractional model accurately captures the non-local properties and long-range memory effects associated with such processes. To support the technique, we have taken into account a few problems and analyzed their solution by fractional natural decomposition method (FNDM) with solutions for the classical diffusion equations. Keywords: Fractional derivative, Riemann-Liouville (R-L) derivative, Caputo derivative, natural transform, Adomian decomposition, Fractional diffusion.*

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1 Introduction:

Diffusion is a common process that is essential to many fields of science, including physics, chemistry, biology, and engineering [15]. It describes the process by which the particles, energy, or other quantities spread and mix in a medium due to random thermal motion. Fick's law, which bears the name of the German scientist Adolf Fick and was created in the middle of the 19th century, serves as the foundation for the traditional definition of diffusion [14]. The traditional explanation of diffusion, based on Fick's law, offers a fundamental framework for understanding the spreading of substances in homogeneous systems [14]. However, in recent years, researchers have uncovered anomalous diffusion phenomena that deviate from the classical diffusion behavior, exhibiting peculiar characteristics, such as non-local behavior, memory effects, and long-range interaction phenomena [15, 20]. To capture and describe these anomalous diffusion processes, a more generalized mathematical framework is required. This is where fractional calculus and the fractional diffusion process come into play [3, 15]. By extending the idea of differentiation and integration to non-integer orders, fractional calculus makes it possible to describe non-local and long-range interaction phenomena [3, 13, 15]. The fractional diffusion equation arises from the incorporation of fractional derivatives into the classical diffusion equation, providing a mathematical framework to describe and analyze complex diffusion phenomena in fractal media, heterogeneous environments, and systems with memory effect. This capability attracted significant interest from a variety of scientific disciplines [15, 19]. Within the realm of anomalous diffusion, the fractional diffusion equations have been independently developed by considering various non-integer orders (fractional) derivatives in time, space, and both time-space domains [19]. The time-fractional diffusion equation, inspired by studies by Metzler et al. [10], introduces a fractional derivative in the time domain by considering the continuous time random walk. This equation extends the traditional diffusion equation incorporating time derivative of fractional order, enabling the description of memory effects and long-range correlations observed in time-dependent diffusion processes. Similarly, the space fractional diffusion equation, as described in works by Meerschaert et al. (2006), incorporates a fractional order spatial derivative. It allows for the characterization of diffusion processes in non-homogeneous media and fractal geometries. The equation captures sub-diffusion or super-diffusion phenomena, where the spreading behavior is slower or faster than classical diffusion, respectively, in spatial domains [8]. Furthermore, the space-time non-integer (fractional) order diffusion equation, studied by Gorenflo et al. [4], and Meerschaert et al. [9] combines fractional derivatives in both time and space domains. This is particularly relevant for describing anomalous diffusion in highly heterogeneous environments, where temporal and spatial correlations play significant roles.

Several fundamental methods have been developed by renowned mathematicians for solving non-integer order diffusion equations. Abbasbandy et al. [1] proposed the variational iteration method (VIM) to construct an approximation solution. Lin et al. [7] employed the finite difference scheme method (FDSM) for constructing approximations of fractional diffusion. Additionally, other approaches like homotopy analysis method (HAM) [5], homotopy perturbation transform method (HPTM) [6], natural decomposition method (NDM), [11], Adomian decomposition method (ADM) [18], and so on have been utilized in this context [15]. In this this work, we utilize the natural decomposition method to solve the non-integer order diffusion equations. The natural transform with Adomian decomposition approach for non-linear partial differential equations was first used by Rawashdesh and Matima [16]. Through our investigation, we reveal the numerical solution to the time-fractional diffusion equation, which is a critical step toward developing a general framework to model anomalous diffusion phenomena. This framework captures the intricacies of long-range correlations, memory effects, and time-dependent system dynamics[19, 20].

1.1 Riemann-Liouville (R-L) Derivative

The Riemann-Liouville (R-L) derivative for non-integer order is defined in terms of the fractional integration called R-L fractional integral [3]. The R-L integral of $\phi(\xi),\ \ \xi \geq -1$ of non-integer order $\alpha > 0$ is formulated as [3, 15]

$$
{}_aI_\xi^\alpha(\phi(\xi)) = \frac{1}{\Gamma(\alpha)} \int_a^\xi \frac{\phi(\tau)}{(\xi-\tau)^{1-\alpha}} d\tau, \quad \alpha > 0, \quad \xi > a \tag{1}
$$

where Γ is gamma function. With fractional integral, R-L derivative is given by

$$
{}_a^RD_\xi^\alpha\left(\phi(\xi)\right) = \begin{cases} \dfrac{1}{\Gamma(p-\alpha)} \dfrac{d^p}{d\xi^p} \int_a^\xi \dfrac{\phi(\tau)}{(\xi-\tau)^{1-p+\alpha}} d\tau, & \text{if } \ \alpha \in \mathbb{R}^+, \ \ p-1 < \alpha < p \\ \dfrac{d^p}{d\xi^p}\phi(\xi), & \text{if } \quad \alpha = p \end{cases}
$$

1.2 Caputo Derivative

The Caputo definition is defined by interchanging the order of derivative and fractional integration [3].

$$
{}_a^CD_\xi^\alpha\left(\phi(\xi)\right) = \begin{cases} \dfrac{1}{\Gamma(p-\alpha)} \int_a^\xi \dfrac{\phi^{(p)}(u)}{(\xi-u)^{1-p+\alpha}} du & \text{if} \quad \alpha \in \mathbb{R}^+, \ \ p-1 < \alpha < p \\ \phi^{(p)}(\xi) & \text{if} \quad \alpha = p \in \mathbb{N} \end{cases}
$$

1.3 Natural Transform

The fractional natural transform of a function $\phi(\tau)$ is given by [2, 17]

$$\mathcal{N}^{+}[\phi(\tau)] = \psi(s,u) = \int_0^{\infty} e^{-s\tau}\phi(u\tau)d\tau, \quad s,u \in \mathbb{R} \tag{2}$$

where the variables s and u represent the transformation variable. The definition of the inverse of natural transform for a function is [17];

$$\mathcal{N}^{-}[\psi(s,u)] = \phi(\tau) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{s\tau}\psi(s,u)ds \tag{3}$$

where the variables s and u represent the transformation variable, a is a real constant, and the integration is taken along line $Re(P) = a$ in a complex plane $P = \xi + i\tau$.

1.4 Adomian Decomposition Method (ADM)

Consider a non-linear ordinary fractional differential equation [11],

$${}_a^cD_\tau^\alpha\phi(\tau) + R\phi(\tau) + G\phi(\tau) = \psi(\tau),\ p-1 < \alpha \le p,\ \ p \in \mathbb{N} \tag{4}$$

with initial conditions

$$\phi^{(j)}(0) = \frac{d^j\phi(0)}{d\tau^j},\ j = 0,1,...,p-1.$$

${}_a^cD_\tau^\alpha$ denote the fractional derivative with respect to τ in Caputo sense and it is an invertible linear operator, R is the operator for liner remainders, G represent the non-linear operator that is considered as analytic, and $\psi(\tau)$ is a known function. As per ADM algorithm, the solution of (4) is an infinite series

$$\phi(\tau) = \sum_{i=0}^{\infty}\phi_i(\tau). \tag{5}$$

Taking the fractional integral (inverted operator of ${}_a^cD_\tau^\alpha$) on both side of (5),

$$I_\tau^\alpha {}_a^cD_\tau^\alpha\phi(\tau) + I_\tau^\alpha R\phi(\tau) + I_\tau^\alpha G\phi(\tau) = I_\tau^\alpha\psi(\tau) \tag{6}$$

Using the initial condition,

$$\phi(\tau) = \sum_{j=0}^{p-1}\frac{\tau^j}{j!}\phi^{(j)}(0) + I_\tau^\alpha\psi(\tau) - I_\tau^\alpha R\phi(t) - I_t^\alpha G\phi(\tau) \tag{7}$$

and the expression for the non linear expression $G\phi(\tau)$ is given by

$$G\phi(\tau) = \sum_{k=0}^{\infty}A_k(\tau) \tag{8}$$

where $A_k(\tau)$,depending on $\phi_0, \phi_1...$, are Adomian polynomials and can be calculated for non-linearity $G\phi = f(\phi(\tau))$ as,

$$A_k(\tau) = \frac{1}{k!}\left[\frac{d^k}{d\lambda^k}f\left(\sum_{i=0}^{k}\phi_i(\tau)\lambda^i\right)\right]_{\lambda=0} \tag{9}$$

From (5), (8) and (9), equation (7) becomes;

$$\sum_{n=0}^{\infty}\phi_n(\tau) = \sum_{j=0}^{p-1}\frac{\tau^j}{j!}\phi^{(j)}(0) + I_\tau^\alpha[\phi(\tau)] - I_\tau^\alpha\left[R\sum_{n=0}^{\infty}\phi_n(\tau)\right] - I_\tau^\alpha\left[\sum_{k=0}^{\infty}A_k(\tau)\right] \tag{10}$$

Then, from (10), we find the iterative scheme and then the approximate solution to equation (5) is the sum of thus obtained term.

2 Natural Decomposition Method (NDM)

The fractional natural transform method (FNTM) and Adomian decomposition method (ADM) are combined to create a new method natural decomposition method (NDM) [12]. Let $\Omega = S \times I$, where $S = [0, L]$ be spatial domain and $I = [0, T]$ be time domain. Then an equation of one-dimensional time-fractional diffusion is [7, 11];

$$ {}^{c}D_{\tau}^{\alpha}\mathscr{U}(\xi,\tau) = K\frac{\partial^2 \mathscr{U}(\xi,\tau)}{\partial \xi^2} + \psi(\xi,\tau), \quad (\xi,\tau)\in\Omega, \;\; 0<\alpha\le 1 \tag{11} $$

with initial and boundary conditions

$$ \mathscr{U}(\xi,0) = h(\xi), \;\; 0\le\xi\le L \tag{12} $$

$$ \mathscr{U}(0,\tau) = \mathscr{U}(L,\tau) = 0, \;\; \tau>0 \tag{13} $$

where ${}^{c}D_{\tau}^{\alpha} = \dfrac{\partial^{\alpha}}{\partial\tau^{\alpha}}$ is non-integer order (fractional) derivative in Caputo sense, $\mathscr{U}(\xi,\tau)$ is solute concentration , $\psi(\xi,\tau)$ is the source function, and K represents the diffusion coefficient (constant or function of ξ) which controls the anomalous diffusion in complex medium.
The solution of non-integer order diffusion equation by NDM, taking natural transform of (11)

$$ \mathscr{N}^{+}\left[{}^{c}D_{\tau}^{\alpha}\mathscr{U}(\xi,\tau)\right] = \mathscr{N}^{+}\left[K\frac{\partial^2 \mathscr{U}(\xi,\tau)}{\partial \xi^2} + \psi(\xi,\tau)\right] \tag{14} $$

Using the natural transform's differentiation property

$$ \begin{aligned} \left(\frac{s}{u}\right)^{\alpha}\mathscr{N}^{+}\left[\mathscr{U}(\xi,\tau)\right] - \frac{s^{\alpha-1}}{u^{\alpha}}\mathscr{U}(\xi,0) &= \mathscr{N}^{+}\left[K\frac{\partial^2\mathscr{U}}{\partial\xi^2} + \psi(\xi,\tau)\right] \\ \Longrightarrow \mathscr{N}^{+}\left[\mathscr{U}(\xi,\tau)\right] &= \frac{1}{s}h(\xi) + \frac{u^{\alpha}}{s^{\alpha}}\mathscr{N}^{+}\left[K\frac{\partial^2\mathscr{U}}{\partial\xi^2} + \psi(\xi,\tau)\right] \end{aligned} $$

$\mathscr{U}(\xi,\tau)$ can be written as an infinite series by using the ADM technique.

$$ \mathscr{U}(\xi,\tau) = \sum_{k=0}^{\infty}\mathscr{U}_k(\xi,\tau) = \sum_{k=0}^{\infty}\mathscr{U}_k \tag{15} $$

The Adomian polynomials infinite series is used in this problem to represent any existent non-linear components

$$ G\mathscr{U}(\xi,\tau) = \sum_{k=0}^{\infty} A_k \tag{16} $$

where $A_k = \dfrac{1}{k!}\left[\dfrac{d^k}{d\lambda^k}G\left[\sum_{k=0}^{\infty}(\lambda^k\mathscr{U}_k)\right]\right]_{\lambda=0}$ $k = 0, 1, 2...$, are adomian polynomials
From equation (15) and (16)

$$ \mathscr{N}^{+}\left[\sum_{k=0}^{\infty}\mathscr{U}_k\right] = \frac{1}{s}h(\xi) + \frac{u^{\alpha}}{s^{\alpha}}\mathscr{N}^{+}\left[K\sum_{k=0}^{\infty}\frac{\partial^2\mathscr{U}_k}{\partial\xi^2} + \psi(\xi,\tau)\right] $$

Using the Adomian decomposition and inverse natural transform,
$\mathscr{U}_0(\xi,\tau) = \mathscr{N}^{-}\left[\dfrac{1}{s}h(\xi)\right] + \mathscr{N}^{-}\left[\dfrac{u^{\alpha}}{s^{\alpha}}\mathscr{N}^{+}\left[\psi(\xi,\tau)\right]\right]$ and $\mathscr{U}_{k+1}(\xi,\tau) = \mathscr{N}^{-}\left[\dfrac{u^{\alpha}}{s^{\alpha}}\mathscr{N}^{+}\left[K\dfrac{\partial^2\mathscr{U}_k}{\partial\xi^2}\right]\right]$
for k=0,1,2,..., the NDM method's solution is derived by substituting the values of $\mathscr{U}_k(\xi,\tau)$ in (15)

3 Result and Discussion

In this section, we use the NDM approach to illustrate a few time-fractional diffusion equations.

Ex.1. Consider the following fractional diffusion equation in one dimension [11]

$$\frac{\partial^\alpha \mathscr{U}(\xi,\tau)}{\partial \tau^\alpha} = \frac{\xi^2}{2}\frac{\partial^2 \mathscr{U}(\xi,\tau)}{\partial \xi^2}, \quad (\xi,\tau)\in\Omega, \quad 0<\alpha\le 1 \tag{17}$$

with initial condition

$$\mathscr{U}(\xi,0)=\xi^2, \;\; 0\le\xi\le 2 \tag{18}$$

By employing natural transform on (18)

$$\mathscr{N}^+\left[\frac{\partial^\alpha \mathscr{U}(\xi,\tau)}{\partial\tau^\alpha}\right] = \mathscr{N}^+\left[\frac{\xi^2}{2}\frac{\partial^2\mathscr{U}}{\partial\xi^2}\right]$$

Using the differentiation property of natural transform,

$$\mathscr{N}^+\left[\mathscr{U}(\xi,\tau)\right] = \frac{1}{s}\mathscr{U}(\xi,0) + \frac{u^\alpha}{s^\alpha}\mathscr{N}^+\left[\frac{\xi^2}{2}\frac{\partial^2\mathscr{U}}{\partial\xi^2}\right] = \frac{\xi^2}{s} + \frac{u^\alpha}{s^\alpha}\mathscr{N}^+\left[\frac{\xi^2}{2}\frac{\partial^2\mathscr{U}}{\partial\xi^2}\right] \tag{19}$$

Then by ADM algorithm, the solution $\mathscr{U}(\xi,\tau)$ can be expressed in infinite series as

$$\mathscr{U}(\xi,\tau) = \sum_{k=0}^{\infty}\mathscr{U}_k(\xi,\tau) = \sum_{k=0}^{\infty}\mathscr{U}_k \tag{20}$$

From equation (19) and (20)

$$\mathscr{N}^+\left[\sum_{k=0}^{\infty}\mathscr{U}_k(\xi,\tau)\right] = \frac{\xi^2}{s} + \frac{u^\alpha}{s^\alpha}\mathscr{N}^+\left[\frac{\xi^2}{2}\sum_{k=0}^{\infty}\frac{\partial^2\mathscr{U}_k}{\partial\xi^2}\right]$$

Taking inverse natural transform

$$\sum_{k=0}^{\infty}\mathscr{U}_k(\xi,\tau) = \mathscr{N}^-\left[\frac{\xi^2}{s}\right] + \mathscr{N}^-\left[\frac{u^\alpha}{s^\alpha}\mathscr{N}^+\left[\frac{\xi^2}{2}\sum_{k=0}^{\infty}\frac{\partial^2\mathscr{U}_k}{\partial\xi^2}\right]\right]$$

By ADM algorithm

$$\mathscr{U}_0(\xi,\tau) = \mathscr{N}^-\left[\frac{\xi^2}{s}\right] = \xi^2$$

and

$$\mathscr{U}_{k+1}(\xi,\tau) = \mathscr{N}^-\left[\frac{u^\alpha}{s^\alpha}\mathscr{N}^+\left[\frac{\xi^2}{2}\frac{\partial^2\mathscr{U}_k}{\partial\xi^2}\right]\right], \quad for \quad k=0,1,2,...$$

For all values of $k=0,1,2,...$, equation (20) becomes;

$$\mathscr{U}(\xi,\tau) \;=\; \xi^2\left(1+\frac{\tau^\alpha}{\Gamma(\alpha+1)}+\frac{\tau^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{\tau^{3\alpha}}{\Gamma(3\alpha+1)}+\frac{\tau^{4\alpha}}{\Gamma(4\alpha+1)}+...\right) \tag{21}$$

Using computational software, figure 1 shows that the three-dimensional plot that visually represents the NDM solution for different values of the variable α. On the other hand, figure 2 presents a two-dimensional plot illustrating the solution for various values of α specifically when τ is fixed at 1. Notably, by observing both figures, it becomes evident that as the value of α progressively approaches 1, the solution curve increasingly converges towards the curve corresponding to $\alpha = 1$. The figures provide clear evidence that the non-integer (fractional) order diffusion equation effectively captures the diffusive

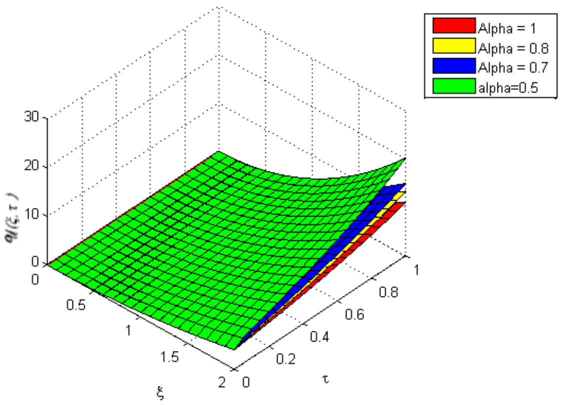

Figure 1: 3D plot of numerical solution of example 1 for different values α

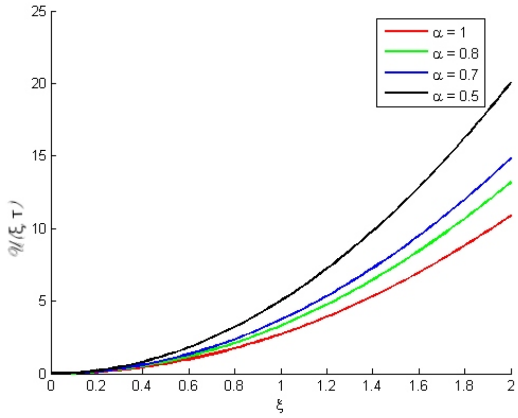

Figure 2: 2D plot of solution of example 1 at $\tau = 1$

behavior in continuous time. This property enables it to accurately represent the non-local nature and long-range memory effects observed in anomalous diffusion processes occurring within complex medium.

When $\alpha = 1$, (21) gives;

$$\mathscr{U}(\xi,\tau) = \xi^2\left(1+\tau+\frac{\tau^2}{2!}+\frac{\tau^3}{3!}+\frac{\tau^4}{4!}+...\right)$$

This is the somewhat like exact solution in closed form

$$\mathscr{U}(\xi,\tau) = \xi^2 e^{\tau}$$

By computational software,
Figure 3 shows that three-dimensional plot illustrating the error of the solution across different values of the variable α. This plot visually demonstrates how the error changes with varying α. On the other hand, Figure 4 presents a two-dimensional graph that specifically focuses on the error of the solution for different α values when τ is fixed at 1. Notably, both figures provide clear evidence that as α approaches 1, the corresponding error consistently decreases. This observation suggests a strong correlation between the proximity of α to 1 and the reduction of error in the solution.

Ex.3. Consider following two dimensional fractional diffusion equation [11]

$$\frac{\partial^\alpha \mathscr{U}(\xi,y,\tau)}{\partial\tau^\alpha} = \frac{y^2}{2}\frac{\partial^2\mathscr{U}(\xi,y,\tau)}{\partial\xi^2} + \frac{\xi^2}{2}\frac{\partial^2\mathscr{U}(\xi,y,\tau)}{\partial y^2}, \quad (\xi,y,\tau)\in\Omega, \quad 0<\alpha\le 1 \tag{22}$$

with initial

$$\mathscr{U}(\xi,y,0) = y^2, \ \ 0\le y\le 1 \tag{23}$$

Applying the natural transform on both side of (23)

$$\mathscr{N}^+\left[\frac{\partial^\alpha\mathscr{U}}{\partial\tau^\alpha}\right] = \mathscr{N}^+\left[\frac{y^2}{2}\frac{\partial^2\mathscr{U}}{\partial\xi^2}+\frac{\xi^2}{2}\frac{\partial^2\mathscr{U}}{\partial y^2}\right]$$

By differentiation property of natural transform and using initial condition

$$\mathscr{N}^+\left[\mathscr{U}(\xi,y,\tau)\right] = \frac{y^2}{s} + \frac{u^\alpha}{s^\alpha}\mathscr{N}^+\left[\frac{y^2}{2}\frac{\partial^2\mathscr{U}}{\partial\xi^2}+\frac{\xi^2}{2}\frac{\partial^2\mathscr{U}}{\partial y^2}\right] \tag{24}$$

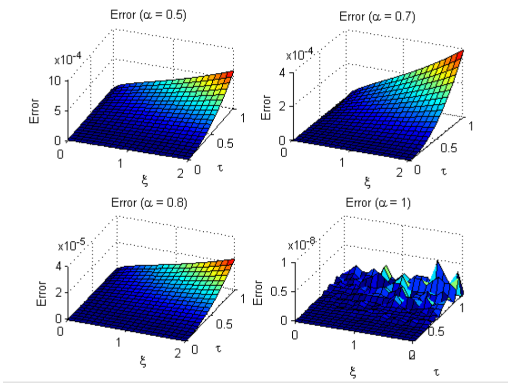

Figure 3: Error plots of the solution by NDM for different α in 3D.

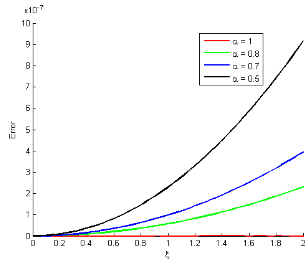

Figure 4: Error plots of the solution by NDM for $\tau = 1$ in 2D.

Using ADM algorithm, solution $\mathscr{U}(\xi, y, \tau)$ is given by infinite series

$$\mathscr{U}(\xi, y, \tau) = \sum_{k=o}^{\infty} \mathscr{U}_k(\xi, y, \tau) \tag{25}$$

From equation (24) and (25)

$$\mathscr{N}^{+}\left[\sum_{k=o}^{\infty} \mathscr{U}_k\right] = \frac{y^2}{s} + \frac{u^\alpha}{s^\alpha}\mathscr{N}^{+}\left[\frac{y^2}{2}\sum_{k=0}^{\infty}\frac{\partial^2 \mathscr{U}_k}{\partial \xi^2} + \frac{\xi^2}{2}\sum_{k=0}^{\infty}\frac{\partial^2 \mathscr{U}_k}{\partial y^2}\right]$$

Taking inverse natural transform

$$\sum_{k=o}^{\infty} \mathscr{U}_k = \mathscr{N}^{-}\left[\frac{y^2}{s}\right] + \mathscr{N}^{-}\left[\frac{u^\alpha}{s^\alpha}\mathscr{N}^{+}\left[\frac{y^2}{2}\sum_{k=0}^{\infty}\frac{\partial^2 \mathscr{U}_k}{\partial \xi^2} + \frac{\xi^2}{2}\sum_{k=0}^{\infty}\frac{\partial^2 \mathscr{U}_k}{\partial y^2}\right]\right]$$

By ADM algorithm

$$\mathscr{U}_0(\xi, y, \tau) = \mathscr{N}^{-}\left[\frac{y^2}{s}\right] = y2$$

and

$$\mathscr{U}_{k+1}(\xi, y, \tau) = \mathscr{N}^{-}\left[\frac{u^\alpha}{s^\alpha}\mathscr{N}^{+}\left[\frac{y^2}{2}\frac{\partial^2 \mathscr{U}_k(\xi, \tau)}{\partial \xi^2} + \frac{\xi^2}{2}\sum_{k=0}^{\infty}\frac{\partial^2 \mathscr{U}_k}{\partial y^2}\right]\right], \quad for \quad k = 0, 1, 2, ...$$

putting different values of k

$$\mathscr{U}_{2k-1}(\xi, y, \tau) = \xi^2 \frac{\tau^{(2k-1)\alpha}}{\Gamma((2k-1)\alpha + 1)}, \quad for \ \ k = 1, 2, ...$$

and

$$\mathscr{U}_{2k-2}(\xi, y, \tau) = y^2 \frac{\tau^{(2k-2)\alpha}}{\Gamma((2k-2)\alpha + 1)}, \quad for \ \ k = 1, 2, ...$$

From all above, equation (25) becomes;

$$\mathscr{U}(\xi, y, \tau) = \xi^2\left(\frac{\tau^\alpha}{\Gamma(\alpha+1)} + \frac{\tau^{3\alpha}}{\Gamma(3\alpha+1)} + ...\right) + y^2\left(1 + \frac{\tau^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{\tau^{4\alpha}}{\Gamma(4\alpha+1)} + ...\right)$$

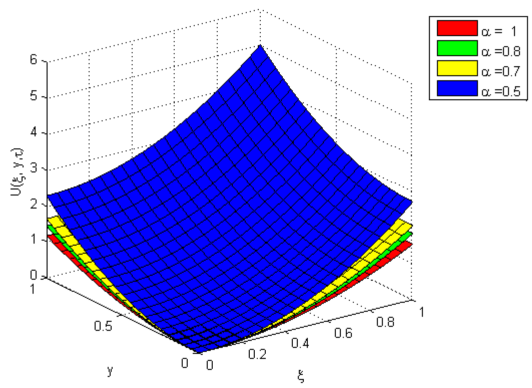

Figure 5: Solution of two dimensional fractional diffusion Equation by NDM

Now, by using computational software,
Figure 5 shows that the three-dimensional plot that visually represents the NDM solution of two dimensional fractional order diffusion equation for different values of the variable α. Notably, by observing figure, it becomes evident that whenever value of α progressively approaches 1, the solution curve increasingly converges towards the curve corresponding to $\alpha = 1$ and at $\alpha = 1$ it coincides with the exact solution in closed form.

When $\alpha = 1$, above gives;

$$\mathscr{U}(\xi, y, \tau) = \xi^2 \left(\frac{\tau}{1!} + \frac{\tau^3}{3!} + ...\right) + y^2 \left(1 + \frac{\tau^2}{2!} + \frac{\tau^4}{4!} + ...\right) = \xi^2 \sinh\tau + y^2 \cosh\tau.$$

This is the somewhat like exact solution in closed form Plotting the error by using computational software
Figure 6 shows that three-dimensional plot illustrating the error of the solution of two dimensional

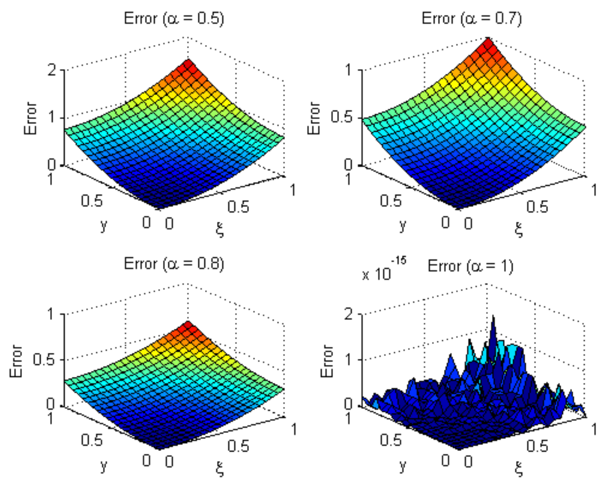

Figure 6: Error plot of the solution by NDM

diffusion equation by NDM across different values of the variable α. This plot visually demonstrates how the error changes with varying α. Notably, figures provide clear evidence that as α approaches 1, the corresponding error consistently decreases. This observation suggests a strong correlation between the proximity of α to 1 and the reduction of error in the solution.

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4 Conclusion

In this work, we have examined the numerical analysis of the non-integer (fractional) order diffusion equation by employing the natural decomposition method (NDM). Proposed method offers a valuable approach for approximating solutions to fractional differential equations, including the fractional diffusion equation, which exhibits anomalous diffusion behavior. The fractional diffusion equations are the best tools to capture the diffusion in complex media where non-local property and and long-range memory effect plays a crucial role. Through the application of the NDM, we have successfully illustrated the numerical solutions for one dimensional and two dimensional fractional diffusion equations and from the result we discovered that whenever on-integer order α tends towards integer order, the non-integer order solutions converge rapidly close to exact solution. Therefore the accuracy and convergence of the NDM have been validated through our numerical experiments. The application of NDM to illustrative instances has further proved that, when comparing the integer-order model with fractional order model, it becomes apparent that the fractional-order mathematical model provides the most effective approach for capturing the non-local property and long-range memory effect that exhibit by anomalous diffusion process. In conclusion, the non integer order diffusion equation offers a best mathematical framework to capture the anomalous diffusion process in complex media and the fractional natural decomposition method (NDM) is regarded as the best tool for solving linear as well as non-linear fractional partial differential equations due to its superior convergence and accuracy compared to other methods.

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