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On a New Application of Almost Increasing Sequence to Laguerre Series Associated with Strong Summability of Ultrapherical Series

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Abstract: *The concept of summability of infinite series has been utilized in virtually every field of scientific application, including the enhancement of signals in filters, the acceleration of the rate of convergence, orthogonal series, and approximation theory, to name just a few. In addition, by making use of the main theorem, a collection of new well-known arbitrary findings have been obtained. Taking into account the appropriate conditions of a prior result, which result was produced, verifies the conclusions of the current study.*

Keywords: *Matrix summability, Laguerre series, Ultraspherical series*.

AMS Subject Classification (2020): 40C05, 40F05

1. Introduction

Cauchy's monumental "course d'analyses algebriaic" published in 1821 and Abel's investigations (see [8, 10-18]) into the binomial series (published in 1826) provided a solid foundation for the antiquated and esoteric concept of convergence infinite series. In Dynamical Astronomy, in particular, there were a few non-convergent series that gave close to the right answers. In 1890, a theory of divergent series was formulated for the first time by Cesàro, who wrote a paper on the multiplication of a series. From there, the theory of series emerged as the hub of mathematical analysts' ingenuity, explaining why the sequence of partial sums of a function varies in a periodic fashion. Mathematicians like Holder, Hausdorff, Riesz, Nörlund, etc. worked tirelessly to develop effective solutions, and only in the last decade of the twentieth century and the first year of the twenty-first century have they succeeded. Cauchy's idea of convergence is intended to have a tight relationship with this method through process. In a fair fashion, we can refer to these values as their sums. The process of linking generalized sums, known as summability (Szasz 1946; Hardy 1949), offers a natural generalization of the classical notion of convergence (Hobson 1909).

2. Definitions

Before proceeding with the main work, we now give some notations and definitions that are used in the paper.

2.1 Regular triangular matrix (see [17])

The matrix of triangles (Λ) = ($r_{a,b}$) where a = 0, 1, 2, 3, … and b = 0, 1, 2, 3, …and $r_{a,b}$ = 0 for 0 < a < 1,(defines a regular sequence-to-sequence transformation) b, is regular) if

$\lim_{a\to\infty} r_{a,b} = 0$, for every fixed b; … (1)

$\sum_{b=0}^{a} | r_{a,b} | \leq K$, independent of a; … (2)

and $\lim_{a\to\infty} \sum_{b=0}^{a} r_{a,b} = 1$. … (3)

2.2 Strongly summable (see [5,17]);

An infinite series $\sum v_a$ with the sequence of partial sums $\{ S_a \}$ is said to be strongly summable (Λ) to a fixed finite sums S, if $\sum_{b=0}^{a} r_{a,b} | S_b - S |$ = o(1), as $a \to \infty$. … (4)

2.3 We have the following three cases (see [5,20,21]);

(a) $r_{a,b} = \frac{1}{(a+1)} (b \leq a)$

(b) $r_{a,b} = \frac{1}{\{(b+1) \sum_{m=0}^{a} \frac{1}{m+1}\}} \quad (b \leq a)$

(c) $r_{a,b} = \frac{1}{(1 - b + a) \sum_{m=0}^{a} \frac{1}{m+1}} \quad (b \leq a)$

Summability (Λ) becomes respectively Cesàro summability, Riesz summability and Nörlund summability.

2.4 Ultraspherical series (see [5,20,21]);

Let $f(\theta, \phi)$ be a function defined in range $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$. The ultraspherical series corresponding to $f(\theta, \phi)$ on the sphere S is given by

$$f(\theta, \phi) \sim \frac{1}{2\pi} \sum_{a=0}^{\infty} (a + \alpha) \int_S\int \frac{f(\theta', \phi') Q_n^{(\alpha)} (\cos\omega) \sin\theta' \, d\theta' \, d\phi'}{[sin^2\theta' \; sin^2 \; (\phi - \phi')]^{\frac{1-2\alpha}{2}}} \qquad \ldots (5)$$

Where, $\cos \omega = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos (\phi - \phi')$. … (6)

2.5 Ultraspherical polynomial (see [18,20]);

The ultraspherical polynomial $Q_a^{(\alpha)}(x)$ is defined by generating function

$$(1 - 2xt + t^2)^{-\alpha} = \sum_{a=0}^{\infty} t^a Q_a^{(\alpha)}(x), \; \alpha > 0 \qquad \ldots (7)$$

A generalized mean value of $f(\theta, \phi)$ on the sphere S has been defined Gupta (see [4]) as follows:

$$f(\omega) = \frac{1}{2\pi(\sin\omega)^{2\alpha}} \int_{C_\omega} \frac{f(\theta', \phi') \, ds'}{[sin^2\theta' \, sin^2 (\phi - \phi')]^{\frac{1-2\alpha}{2}}} \qquad \ldots (8)$$

where the integral is taken along the small circle C whose center is (θ, ϕ) on the sphere S and whose curvilinear radius is ω.

The series (5) reduces to

$$f(\theta, \phi) \sim \frac{\Gamma(\alpha)}{\Gamma(\frac{1}{2}) \; \Gamma(\frac{1}{2}+\alpha)} \sum_{a=0}^{\infty} (a + \alpha) \int_0^{\pi} f(\omega) \, sin^{2\alpha} \omega Q_a^{(\alpha)} (\cos \omega) \, d\omega . \qquad \ldots (9)$$

Also, we write

$\phi(\omega) = \{ f(\omega) - A \} (\sin \omega)^{2\alpha - 1}$ … (10)

where A is a fixed constant.

Various researchers have explored various intriguing generalizations; here we list just a handful (see [1-9, 19-21]). The following theorem on the summability of Laguerre series in matrices was proved by them (see [6]):

2.6 Known Theorem:

Let the non- negative real sequence $\{r_{a,b}\}$ be none decreasing with respect to b

and

$$\lim_{a\to\infty}\sum_{b=0}^{a}\lambda_{a,b} = 1 \qquad \ldots \qquad (11)$$

If $|\ \phi\ |$ is integrable in the sense of Lebesgue integral in any bounded interval (0, ω)

and if,

$$\int_1^\infty e^z z^{\frac{-1}{5}}\ |\ \phi\,(z)|\ dz\ <\ \infty\,. \qquad \ldots \qquad (12)$$

Then for $-2<2\alpha<-1$, the Leguerre series corresponding to the function $f\ \in L\,[0,\infty]$ given by

$$f(x) \sim \sum_{a=0}^{\infty} v_n\, L_a^{(\alpha)}(x) \qquad \ldots \qquad (13)$$

where

$$v_a = \left\{\Gamma(\alpha+1)\begin{pmatrix} a+\alpha \\ \alpha \end{pmatrix}\right\}^{-1}\int_0^\infty e^{-z}z^{\alpha}f(z)L_a^{(\alpha)}(z)\,dz \qquad \ldots \qquad (14)$$

and $L_a^{(\alpha)}(z)$ denotes the a^{th} Laguerre polynomial of order $-\alpha<1$ which is defined as

$$\sum_{a=0}^{\infty} L_a^{(\alpha)}(x)\,\omega^n = (1-\omega)^{-\alpha-1}e^{\frac{-x\omega}{1-\omega}} \qquad \ldots \qquad (15)$$

and

$$\phi\,(z) = (\Gamma(\alpha+1))^{-1}e^{-z}z^{\alpha}\{f(z)-f(0)\} \qquad \ldots \qquad (16)$$

is summable (Λ) at $x = 0$ to the sum $f(0)$.

3. Main Theorem

We prove the following theorem

3.1 Theorem:

Let $\{r_{a,b}\}$ be non-negative, monotone increasing sequence with respect to b, and

$$\lim_{a\to\infty}\sum_{b=0}^{a} r_{a,b} = 1. \qquad \ldots \qquad (17)$$

Let μ be a large constant and δ be such that

$$1 > \alpha(1+\delta) > \alpha, \quad \alpha \in (0, 1). \qquad \ldots \qquad (18)$$

Let ϕ be a function ω which is bounded variation in open interval (ξ,π) i.e.$|\phi(\omega)| \in (\xi,\pi)$where ξ is defined as follows:

$$\xi = \mu\, a^{-\delta},\ (0 < \xi < \pi) \qquad \ldots \qquad (19)$$

and, if

$$\int_0^t |\phi(\omega)|\,d\omega = O(t^{\,\varepsilon+1}),\ \text{ as } t\to 0 \qquad \ldots \qquad (20)$$

$$\text{and } \varepsilon = \frac{2\alpha+1}{\delta} - 1. \qquad \ldots \qquad (21)$$

Then the given ultraspherical series corresponding$f(\theta,\phi)$ on the sphere S is strongly summable (Λ) to the sum A.

3.2 Lemmas:

The following lemmas are necessary for us to prove our theorem:

Lemma 1 (see [22]) : We have for $\alpha > 0$,

$$Q_a^{(\alpha)}(\cos\theta) = \theta^{-\alpha}\,O\,(a^{\alpha-1}),\ \frac{C}{a} \le \theta \le \frac{\pi}{2} \qquad \ldots \qquad (22)$$

and,

$Q_a^{(\alpha)}$ (cos θ) = O $(\frac{1}{a^{1-2\alpha}})$, $0 \leq \theta \leq \frac{C}{a}$ … (23)

Lemma 2 (see [22]) : For $a \geq 0$, we have

$\frac{d}{dx}$ { $Q_a^{(\alpha)}(x)$ } = 2 $\alpha Q_a^{(\alpha+1)}(x)$, … (24)

Where $Q_{-1}^{(\alpha)}(x) = 0$

Lemma 3 : Under the condition of theorem, we have

$\sum_{b=0}^{a} r_{a,b}\, K^{\alpha-1}$ = O($a^{\alpha-1}$), … (25)

and similarly,

$\sum_{b=0}^{a} r_{a,b}\, K^{2\alpha+1}$ = O($a^{2\alpha+1}$), as $a \to \infty$. … (26)

The proof obviously follows on using (17).

3.3 Proof of the theorem:

Let σ_a denote the a^{th} partial sum of the series (5) .Then we have (see [22]).

$$\sigma_a = \frac{\Gamma(\alpha)}{\sqrt{\pi}\;\Gamma(\alpha+\frac{1}{2})} \int_0^{\pi} f(\omega) \sum_{m=0}^{a} (m+\alpha)\, Q_a^{(\alpha)} (\cos\omega)\, (\sin\omega)^{2\alpha} d\omega$$

$$\sigma_a = \frac{\Gamma(\alpha)}{\sqrt{\pi}\,\Gamma(\alpha+\frac{1}{2})} \int_0^{\pi} f(\omega)\, sin^{2\alpha}\,\omega \Big[\frac{d}{dx}\Big\{Q_{a+1}^{(\alpha)}(x) + Q_a^{(\alpha)}(x)\Big\}\Big]_{x=cos\omega}\, d\omega. \quad \ldots \quad (27)$$

Therefore, with view of (10), we have

$$\sigma_a - A = \frac{\Gamma(\alpha)}{\sqrt{\pi}\;\Gamma\left(\alpha+\frac{1}{2}\right)} \int_0^{\pi} \phi(\omega) \frac{d}{d\omega} \{ Q_{a+1}^{(\alpha)}(\cos\omega) + Q_a^{(\alpha)}(\cos\omega) \}\, d\omega$$

$$\sigma_a - A = O\,[\int_0^{\pi} \phi(\omega) \frac{d}{d\omega} (Q_{a+1}^{(\alpha)}(\cos\omega)\, d\omega\,] + O\,[\,\int_0^{\pi} \phi(\omega) \frac{d}{d\omega} (Q_a^{(\alpha)}(\cos\omega))\, d\omega],$$

$$= I_1 + I_2, \quad \ldots \quad (28)$$

In order to establish our theorem, we must demonstrate that

$$\sum_{b=0}^{a} r_{a,b}\, |\sigma_a - A|^{p} = O(1), \text{ as } a \to \infty. \quad \ldots \quad (29)$$

Now applying Minkowski's inequality, we get

$$\{ \sum_{b=0}^{a} r_{a,b}\, |\sigma_a - A|^{p}\}^{\frac{1}{p}} \leq \{\sum_{b=0}^{a} r_{a,b} |I_1|^{p}\}^{\frac{1}{p}} + \{ \sum_{b=0}^{a} r_{a,b} |I_2|^{p}\}^{\frac{1}{p}}$$

$$= (M)^{\frac{1}{p}} + (N)^{\frac{1}{p}} \quad \text{(say)} \quad \ldots \quad (30)$$

Let us first consider I_1,

$$|I_1| = O[\int_0^{\pi} |\phi(\omega)|\, |\frac{d}{d\omega}\{Q_{b+1}^{(\alpha)}(\cos\omega)|\, d\omega]$$

$$= O\,[\,(\int_0^{\xi} + \int_{\xi}^{\pi})\; |\phi(\omega)|\; |\frac{d}{d\omega} \{ Q_{b+1}^{(\alpha)}(\cos\omega)\, |d\omega$$

$$= O(I_{1,1}) + O(I_{1,2}), \quad \text{(say)}. \quad \ldots \quad (31)$$

We have,

$$I_{1,1} = \int_0^{\xi} |\phi(\omega)|\; |\frac{d}{d\omega} \{ Q_{b+1}^{(\alpha)}(\cos\omega)\, |d\omega$$

$$= \int_0^{\xi} |\phi(\omega)|\; 2\alpha |Q_{b+1}^{(\alpha+1)}(\cos\omega)\,|, \text{ using (24)}$$

$$= \int_0^{\xi} |\phi(\omega)|\; 2\alpha\, O\{ (b+1)^{2\alpha+1} \}\, d\omega, \text{ using (23)}$$

$$= O(b^{2\alpha+1}) \int_0^{\xi} |\phi(\omega)|\, d\omega$$

$$= O(b^{2\alpha+1})\, O(\xi^{\varepsilon+1}), \text{ using (20).}$$

Hence,

$\sum_{b=0}^{a} r_{a,b}$ O($I_{1,1}$) = $\sum_{b=0}^{a} r_{a,b}$O($b^{2\alpha+1}$) O($\xi^{\varepsilon+1}$)

= O ($\sum_{b=0}^{a} r_{a,b}\, b^{2\alpha+1}$) O($\xi^{\varepsilon+1}$)

= O($a^{2\alpha+1}$) O($a^{-\delta(\varepsilon+1)}$), using(26) and (19)

= O($a^{2\alpha+1-\varepsilon\delta-\delta}$)

=O(1), as $a\to\infty$, using (21) . … (32)

Again,

$I_{1,2} = \int_{\xi}^{\pi} |\phi(\omega)|\, |\frac{d}{d\omega}\{Q_{b+1}^{(\alpha)}(\cos\omega)\, d\omega$

= $[\phi(\omega)Q_{b+1}^{(\alpha)}(\cos\omega)]_{\xi}^{\pi}$ - $\int_{\xi}^{\pi} d\phi(\omega)\, Q_{k+1}^{(\alpha)}(\cos\omega)\, d\omega$

= O($b^{\alpha-1}$) $\xi^{-\alpha}$, using (22)

and $\phi(\omega) \in$ BV(ξ,π).

Hence,

$\sum_{b=0}^{n} r_{a,b}$ O($I_{1,2}$) = $\sum_{b=0}^{a} r_{a,b}$O($b^{\alpha-1}$) $\xi^{-\alpha}$

= O ($\sum_{b=0}^{a} r_{a,b}\, b^{\alpha-1}$) $\xi^{-\alpha}$

= O ($a^{\alpha-1}$) ($\mu a^{-\delta})^{-\alpha}$, using (25) and (19)

= O ($a^{\alpha-1+\alpha\delta}$)

= O(1), as a→ ∞ by (27) and (19). … (33)

Therefore, $M = \sum_{b=0}^{a} r_{a,b}\, |I_1|^{p}$ = O$\{\sum_{b=0}^{a} r_{a,b}\}$

$(M)^{\frac{1}{p}}$ = O(1), as $a\to\infty$ by (3). … (34)

We can also demonstrate that

$(N)^{\frac{1}{p}}$ = O(1) as $n\to\infty$. … (35)

Combining (34) and (35), we get the required result (29).

Conclusion

In this article, we have used the Generalization procedure to establish advanced systems. Summability methods are instructed to reduce the error. Some new result can be generated by using suitable conditions in the main result. The results [10-22] can be found by applying conditions on the main result.

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Data Availability

All the data used in this study "Degree of approximation of signals by strong summability of ultraspherical series" supports the findings and are cited within the article.

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