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On New Space of Vector-Valued Generalized Bounded Sequences Defined on Product Normed Space

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Abstract: *In this paper, we introduce and study a new vector valued sequence space* $\ell_\infty(X \times Y, \bar{\gamma}, \bar{u}, \|\,.\,\|)$ *with its terms from a product normed space* $X \times Y$. *Beside investigating the linear space structure of* $\ell_\infty(X \times Y, \bar{\gamma}, \bar{u}, \|\,.\,\|)$ *with respect to co-ordinatewise vector operations, our primarily interest is to explore the conditions in terms of* $\bar{u}$ *and* $\bar{\gamma}$ *so that a class* $\ell_\infty(X \times Y, \bar{\gamma}, \bar{u}, \|\,.\,\|)$ *is contained in or equal to another class of same kind .*

Keywords: *Sequence space, Generalized sequence space, Product-normed space.*

1. Introduction and Preliminaries

So far, a large number of research projects have been carried out in mathematical structures built with real or complex numbers. In recent years, many researchers have investigated many results on vector valued sequence space defined on normed space. Many researchers are motivated towards further investigation and application on product-normed space.

In this section, we give some definitions regarding to the product-normed linear space.

Let X be a normed space over $\mathbb{C}$, the field of complex numbers and let $\omega(X)$ denote the linear space of all sequences $\bar{x} = (x_k)$ with $x_k \in X$, $k \geq 1$ with usual coordinate-wise operations .We shall denote $\omega(\mathbb{C})$ by ω . Any subspace S of ω is then called a sequence space. A vector valued sequence space or a generalized sequence space is a linear space consisting of sequences with their terms from a vector space.

The various types of vector and scalar valued single sequence spaces has been significantly developed by several workers for instances, Köthe (1970), Kamthan and Gupta (1980), Maddox (1980), Ruckle (1981), Malkowski and Rakocevic (2004), Khan (2008), Kolk (2011), Pahari (2012), (2014), Srivastava and Pahari (2012) etc. Recently, Ghimire and Pahari (2022),(2023) studied various types of vector valued sequence spaces defined by Orlicz function. Paudel and Pahari (2021),(2022) extended the work related to scalar valued single sequences in fuzzy metric space.

Let $(X, \|\,.\,\|_X)$ and $(Y, \|\,.\,\|_Y)$ be Banach spaces over the field $\mathbb{C}$ of complex numbers. Clearly the linear space structure of X and Y provides the Cartesian product of X and Y given by

$$X \times Y = \{ < x, y > : x \in X, y \in Y \}$$

forms a normed linear space over $\mathbb{C}$ under the algebraic operations

$$< x_1, y_1 > + < x_2, y_2 > = < x_1 + x_2, y_1 + y_2 > \text{ and } \alpha < x, y > = < \alpha x, \alpha y >$$

with the norm

$$\| < x, y > \| = \max \{\| x \|_X, \| y \|_Y \},$$

where $< x_1, y_1 >, < x_2, y_2 >, < x, y > \in X \times Y$ and $\alpha \in \mathbb{C}$.

Moreover since $(X, \| \,.\, \|_X)$ and $(Y, \| \,.\, \|_Y)$ are Banach spaces therefore $(X \times Y, \| < . , . > \|)$ is also a Banach space.

Sanchezl *et al*(2000), Castillo *et al* (2001) and Yilmaz *et al*(2004) and many others have introduced and examined some properties of bilinear vector valued sequence spaces defined on product normed space which generalize many sequence spaces.

2. The Space $\ell_\infty (X \times Y, \bar{\gamma}, \bar{u}, \| \,.\, \|)$

Let $\bar{u} = (u_k)$ and $\bar{v} = (v_k)$ be any sequences of strictly positive real numbers and $\bar{\gamma} = (\gamma_k)$ and $\bar{\mu} = (\mu_k)$ be sequences of non-zero complex numbers.

We now introduce and study the following class of Normed space $X \times Y$ -valued sequences:

$$\ell_\infty (X \times Y, \bar{\gamma}, \bar{u}, \| \,.\, \|) = \{\bar{u} = (< x_k , y_k >) : < x_k , y_k > \in X \times Y, \ \sup_k \| \gamma_k < x_k , y_k > \|^{u_k} < \infty\}.$$

Further, when $\gamma_k = 1$ for all k, then $\ell_\infty (X \times Y, \bar{\gamma}, \bar{u}, \| \,.\, \|)$ will be denoted by $\ell_\infty (X \times Y, \bar{u}, \| \,.\, \|)$ and when $u_k = 1$ for all k then $\ell_\infty (X \times Y, \bar{\gamma}, \bar{u}, \| \,.\, \|)$ will be denoted by $\ell_\infty (X \times Y, \bar{\gamma}, \| \,.\, \|)$.

In fact, this class is the generalization of the space introduced and studied by Srivastava and Pahari (2012) to the product normed space.

3. Main Results

In this section we shall derive the linear space structure of the class $\ell_\infty (X \times Y, \bar{\gamma}, \bar{u}, \| \,.\, \|)$ over the field $\mathbb{C}$ of complex numbers and thereby investigate conditions in terms of $\bar{u}$, $\bar{v}$, $\bar{\gamma}$ and $\bar{\mu}$ so that a class is contained in or equal to another class of same kind .

As far as the linear space structure of $\ell_\infty (X \times Y, \bar{\gamma}, \bar{u}, \| \,.\, \|)$ over $\mathbb{C}$ is concerned we throughout take the co-ordinatewise vector operations i.e., for $\bar{w} = (< x_k , y_k >)$, $\bar{z} = (< x'_k , y'_k >)$ in $\ell_\infty (X \times Y, \bar{\gamma}, \bar{u}, \| \,.\, \|)$ and scalar α, we have

$$\bar{w} + \bar{z} = (< x_k , y_k >) + (< x'_k , y'_k >) = (< x_k + x'_k, y_k + y'_k >)$$

$$\text{and } \alpha\bar{u} = (\alpha < x_k , y_k >) = (< \alpha x_k , \alpha y_k >).$$

The zero element of the space will be denoted by

$$\bar{\theta} = (< 0, 0 >, < 0, 0 >, < 0, 0 >, ...).$$

Further, by $\bar{u} = (u_k) \in \ell_\infty$, we mean $\sup_k u_k < \infty$.

We see below that $\sup_k u_k < \infty$ is the necessary condition for linearity of the space. Moreover, we shall denote $M = \max (1, \sup_k u_k)$ and $A(\alpha) = \max(1, |\alpha|)$.

Theorem 3.1: $\ell_\infty (X \times Y, \bar{\gamma}, \bar{u}, \| \,.\, \|)$ **forms a linear space over** $\mathbb{C}$ **if and only if** $\bar{u} = (u_k) \in \ell_\infty$.

Proof:

For the sufficiency, assume that $\bar{u} = (u_k) \in \ell_\infty$ and $\bar{w} = (< x_k , y_k >)$ and

$\bar{z} = (< x'_k , y'_k >) \in \ell_\infty (X \times Y, \bar{\gamma}, \bar{u}, \| \,.\, \|)$.

So that we have

$$\sup_k \| \gamma_k < x_k , y_k > \|^{uk} < \infty \text{ and } \sup_k \| \gamma_k < x'_k , y'_k > \|^{uk} < \infty.$$

Thus considering

$$\sup_k \| \gamma_k (< x_k , y_k > + < x'_k , y'_k >)\|^{uk/M} \le \sup_k \| \gamma_k < x_k , y_k > \|^{uk/M} + \sup_k \| \gamma_k < x'_k , y'_k > \|^{uk/M}$$

and we see that

$$\sup_k \| \gamma_k (< x_k , y_k > + < x'_k , y'_k >)\|^{uk/M} < \infty.$$

and hence $\bar{w} + \bar{z} \in \ell_\infty (X \times Y, \bar{\gamma}, \bar{u}, \| \,.\, \|)$.

Similarly for any scalar α, $\alpha\bar{w} \in \ell_\infty (X \times Y, \bar{\gamma}, \bar{u}, \| \,.\, \|)$ since

$$\sup_k \| \alpha\gamma_k < x_k , y_k > \|^{uk/M} = \sup_k |\alpha|^{uk/M} \| \gamma_k < x_k , y_k > \|^{uk/M}$$
$$\le A(\alpha) \sup_k \| \gamma_k < x_k , y_k > \|^{uk/M} < \infty.$$

Conversely if $\bar{u} = (u_k) \notin \ell_\infty (X \times Y, \bar{\gamma}, \bar{u}, \| \,.\, \|)$ then we can find a sequence $(k(n))$ of positive integers with $k(n) < k(n + 1)$, $n \ge 1$ such that $u_{k(n)} > n$ for each $n \ge 1$.

Now taking $< r, t > \in X \times Y$, $\| < r, t > \| = 1$ we define a sequence $\bar{w} = (< x_k , y_k >)$ by

$$< x_k , y_k > = \begin{cases} \lambda_{k(n)}^{-1} n^{-rk(n)} < r, \; t >, \text{ for } k = k(n), \, n \ge 1 \text{ , and} \\ < 0, \, 0 >, \text{ otherwise.} \end{cases}$$

where $< r, t > \in X \times Y$ with $\| < r, t > \| = 1$, then we have

$$\sup_k \| \gamma_k < x_k , y_k > \|^{uk} = \sup_n \| \gamma_{k(n)} < x_{k(n)}, y_{k(n)} > \|^{uk(n)}$$
$$= \sup_n \| n^{-rk(n)} < r, t > \|^{uk(n)}$$
$$= \sup_n \frac{1}{n} = 1.$$

Thus we easily see that $\bar{w} \in \ell_\infty (X \times Y, \bar{\gamma}, \bar{u}, \| \,.\, \|)$ but on the other hand for $k = k(n)$, $n \ge 1$ and for the scalar $\alpha = 2$,we have

$$\sup_k \| \gamma_k (\alpha < x_k , y_k >) \|^{uk} = \sup_k \| \gamma_{k(n)} (\alpha < x_{k(n)} , y_{k(n)} >)\|^{uk(n)}$$
$$= \sup_n |2|^{uk(n)} \| n^{-rk(n)} < r, t > \|^{uk(n)}$$
$$= \sup_n |2|^{uk(n)} . \frac{1}{n}$$
$$> \sup_n \frac{2^n}{n} \ge 1$$

This shows that $\alpha \bar{w} \notin \ell_\infty (X \times Y, \bar{\gamma}, \bar{u}, \| \,.\, \|)$. Hence $\ell_\infty (X \times Y, \bar{\gamma}, \bar{u}, \| \,.\, \|)$ forms a linear space if and only if $\bar{u} = (u_k) \in \ell_\infty$.

Theorem 3.2: For any $\bar{u} = (u_k)$, $\ell_\infty (X \times Y, \bar{\gamma}, \bar{u}, \| \,.\, \|) \subset \ell_\infty (X \times Y, \bar{\mu}, \bar{u}, \| \,.\, \|)$ if and only if

$$\liminf_k \left| \frac{\gamma_k}{\mu_k} \right|^{uk} > 0.$$

Proof : Suppose $\liminf_k \left| \frac{\gamma_k}{\mu_k} \right|^{uk} > 0$, and $\bar{w} = (< x_k , y_k >) \in \ell_\infty (X \times Y, \bar{\gamma}, \bar{u}, \| \,.\, \|)$. Then there exists $m > 0$, such that $m|\mu_k|^{uk} < |\gamma_k|^{uk}$ for all sufficiently large values of k. Thus

$$\sup_k \| \mu_k < x_k , y_k > \|^{uk} \leq \sup_k \frac{1}{m} \| \gamma_k < x_k , y_k > \|^{uk} < \infty$$

for all sufficiently large values of k, implies that $\overline{w} \in \ell_\infty (X \times Y, \overline{\mu}, \overline{u} ,\| . \|)$. Hence

$$\ell_\infty (X \times Y, \overline{\gamma}, \overline{u} ,\| . \|) \subset \ell_\infty (X \times Y, \overline{\mu}, \overline{u} ,\| . \|).$$

Conversely, let $\ell_\infty (X \times Y, \overline{\gamma}, \overline{u} ,\| . \|) \subset \ell_\infty (X \times Y, \overline{\mu}, \overline{u},\| . \|)$ but $\liminf_k \left| \frac{\gamma_k}{\mu_k} \right|^{uk} = 0$.

Then we can find a sequence $(k(n))$ of positive integers with $k(n) < k(n + 1)$, $n \geq 1$ such that

$$\left| \frac{\gamma_k}{\mu_k} \right|^{uk} < \frac{1}{n} \quad \text{i.e., } |\mu_{k(n)}|^{uk(n)} > n| \gamma_{k(n)}|^{uk(n)}.$$

So, if we take the sequence $\overline{w} = (< x_k , y_k >)$ defined by

$$< x_k , y_k > = \begin{cases} \gamma_{k(n)}^{-1} < r, \ t >, \text{ for } k = k(n), \ n \geq 1 \ , \text{ and} \\ < 0, 0 >, \text{ otherwise.} \end{cases}$$

where $< r, t > \in X \times Y$ with $\| < r, t >\| = 1$, then we easily see that

$$\sup_k \| \gamma_k < x_k , y_k > \|^{uk} = \sup_n \| \gamma_{k(n)} < x_{k(n)}, y_{k(n)} > \|^{uk(n)}$$

$$= \sup_n \| < r, t > \|^{uk(n)} = 1$$

and , $\sup_k \|\mu_k < x_k , y_k > \|^{uk} = \sup_n \|\mu_{k(n)} < x_{k(n)}, y_{k(n)} >\|^{uk(n)}$

$$= \sup_n \left\{ \left| \frac{\mu_{k(n)}}{\gamma_{(n)}} \right|^{uk(n)} \| < r, t > \|^{uk(n)} \right\} > \sup_n n = \infty.$$

Hence $\overline{w} \in \ell_\infty (X \times Y, \overline{\gamma}, \overline{u},\| . \|)$ but $\overline{w} \notin \ell_\infty (X \times Y, \overline{\mu}, \overline{u},\| . \|)$, a contradiction. This completes the proof.

Theorem 3.3: For any $\overline{u} = (u_k)$, $\ell_\infty (X \times Y, \overline{\mu}, \overline{u},\| . \|) \subset \ell_\infty (X \times Y, \overline{\gamma}, \overline{u},\| . \|)$

if and only if $\limsup_k \left| \frac{\gamma_k}{\mu_k} \right|^{uk} < \infty$.

Proof :

For the sufficiency, suppose $\limsup_k \left| \frac{\gamma_k}{\mu_k} \right|^{uk} < \infty$, and $\overline{w} = (< x_k , y_k >) \in \ell_\infty (X \times Y, \overline{\mu}, \overline{u},\| . \|)$.

Then there exists $L > 0$, such that $\left| \frac{\gamma_k}{\mu_k} \right|^{uk} < L$ i.e., $L|\mu_k|^{uk} > |\gamma_k|^{uk}$

for all sufficiently large values of k.

Thus $\sup_k \| \gamma_k < x_k , y_k > \|^{uk} \leq \sup_k L \| \mu_k < x_k , y_k > \|^{uk} < \infty$,

for all sufficiently large values of k, implies that $\overline{w} \in \ell_\infty (X \times Y, \overline{\gamma}, \overline{u},\| . \|)$. Hence

$$\ell_\infty (X \times Y, \overline{\mu}, \overline{u},\| . \|) \subset \ell_\infty (X \times Y, \overline{\gamma}, \overline{u},\| . \|).$$

For the necessity, suppose that $\ell_\infty (X \times Y, \overline{\mu}, \overline{u} ,\| . \|) \subset \ell_\infty (X \times Y, \overline{\gamma}, \overline{u} ,\| . \|)$

but $\limsup_k \left| \frac{\gamma_k}{\mu_k} \right|^{uk} = \infty$. Then we can find a sequence $(k(n))$ of positive integers $k(n) < k(n + 1)$, $n \geq 1$ such that

$$n|\mu_{k(n)}|^{uk(n)} < | \gamma_{k(n)}|^{uk(n)}, \text{ for each } n \geq 1$$

For $< r, t > \in X \times Y$ with $\| < r, t > \| = 1$ we define sequence $\bar{w} = (< x_k , y_k >)$ such that

$$< x_k , y_k > = \begin{cases} \mu_{k(n)}^{-1} < r, \ t >, \text{ for } k = k(n), n \geq 1 & \text{, and} \\ < 0, 0 >, \text{ otherwise.} \end{cases}$$

Then we easily see that

$$\sup_k \| \mu_k < x_k , y_k > \|^{uk} = \sup_n \| \mu_{k(n)} < x_{k(n)}, y_{k(n)} > \|^{uk(n)}$$

$$= \sup_n \| < r, t > \|^{uk(n)} \quad = 1$$

and $\quad \sup_k \|\gamma_k < x_k , y_k > \|^{uk} \quad = \sup_n \|\gamma_{k(n)} < x_{k(n)}, y_{k(n)} >\|^{uk(n)}$

$$= \sup_n \left\{ \left| \frac{\gamma_{k(n)}}{\mu_{(n)}} \right|^{uk(n)} \| < r, t > \|^{uk(n)} \right\}$$

$$> \sup_n n = \infty.$$

Hence $\bar{w} \in \ell_\infty(X \times Y, \bar{\mu}, \bar{u}, \| . \|)$ but $\bar{w} \notin \ell_\infty(X \times Y, \bar{\gamma}, \bar{u}, \| . \|)$, , which leads to a contradiction.

This completes the proof.

When Theorems 3.2 and 3.3 are combined, we get

Theorem 3.4: For any $\bar{u} = (u_k)$, $\ell_\infty(X \times Y, \bar{\gamma}, \bar{u}, \| . \|) = \ell_\infty(X \times Y, \bar{\mu}, \bar{u}, \| . \|)$

if and only if $\quad 0 < \liminf_k \left| \frac{\gamma_k}{\mu_k} \right|^{uk} \leq \limsup_k \left| \frac{\gamma_k}{\mu_k} \right|^{uk} < \infty.$

Corollary 3.5: For any $\bar{u} = (u_k)$,

(i) $\ell_\infty(X \times Y, \bar{\gamma}, \bar{u}, \| . \|) \subset \ell_\infty(X \times Y, \bar{u}, \| . \|)$ if and only if $\liminf_k |\gamma_k|^{uk} > 0$;

(ii) $\ell_\infty(X \times Y, \bar{u}, \| . \|) \subset \ell_\infty(X \times Y, \bar{\gamma}, \bar{u}, \| . \|)$ if and only if $\limsup_k |\gamma_k|^{uk} < \infty$;

(iii) $\ell_\infty(X \times Y, \bar{\gamma}, \bar{u}, \| . \|) = \ell_\infty(X \times Y, \bar{u}, \| . \|)$ if and only if

$$0 < \liminf_k |\gamma_k|^{uk} \leq \limsup_k |\gamma_k|^{uk} < \infty.$$

Proof :

Proof follows if we take $\mu_k = 1$ for all k in Theorems 3.2, 3.3 and 3.4.

Theorem 3.6: For any $\bar{\gamma} = (\gamma_k)$, $\ell_\infty(X \times Y, \bar{\gamma}, \bar{u}, \| . \|) \subset \ell_\infty(X \times Y, \bar{\gamma}, \bar{v}, \| . \|)$

if and only if $\limsup_k \frac{v_k}{u_k} < \infty$.

Proof: Let the condition hold. Then there exists $L > 0$ such that $\frac{v_k}{u_k} < L$ for all sufficiently large values of k. Thus $\sup_k \| \gamma_k < x_k , y_k > \|^{uk} \leq N$ for some $N > 1$ implies that

$$\sup_k \| \gamma_k < x_k , y_k >\|^{vk} \leq N^L,$$

and hence $\quad \ell_\infty(X \times Y, \bar{\gamma}, \bar{u}, \| . \|) \subset \ell_\infty(X \times Y, \bar{\gamma}, \bar{v}, \| . \|)$.

Conversely, let the inclusion hold but $\limsup_k \frac{v_k}{u_k} = \infty$.

Then there exists a sequence $(k(n))$ of positive integers with $k(n) < k(n + 1)$, $n \geq 1$ such that

$$\frac{v_{k(n)}}{u_{k(n)}} > n \quad \text{i.e., } v_{k(n)} > n\, u_{k(n)} , n \geq 1.$$

We now define a sequence $\bar{w} = (< x_k , y_k >)$ as follows:

$$< x_k\,, y_k > \;= \begin{cases} \gamma_{k(n)}^{-1}\, 2^{\,1/uk(n)} < r,\; t >, \text{ for } k = k(n),\, n \geq 1 \;\;, \text{ and} \\ < 0,\, 0 >, \text{ otherwise.} \end{cases}$$

where $< r, t > \in X \times Y$ with $\| < r, t > \| = 1$.

Then for $k = k(n)$, $n \geq 1$, we easily see that

$$\begin{aligned} \sup_k \| \gamma_k < x_k\,, y_k > \|^{\,uk} &= \sup_n \| \gamma_{k(n)} < x_{k(n)}, y_{k(n)} > \|^{uk(n)} \\ &= 2 \sup_n \| \;< r, t > \|^{uk(n)} \quad = 2 \end{aligned}$$

$$\begin{aligned} \text{and },\quad \sup_k \|\gamma_k < x_k\,, y_k > \|^{vk} &= \sup_n \|\gamma_{k(n)} < x_{\,k(n)}, y_{\,k(n)} >\|^{vk(n)} \\ &= \sup_n \| 2^{\,1/uk(n)} < r,\;\; t > \|^{vk(n)} \\ &> \sup_n 2^{\,n} = \infty. \end{aligned}$$

Hence $\overline{w} \in \ell_\infty(X \times Y, \overline{\gamma}, \overline{u}, \| \,.\, \|)$ but $\overline{w} \notin \ell_\infty(X \times Y, \overline{\gamma}, \overline{v}, \| \,.\, \|)$, a contradiction.

This completes the proof.

Theorem 3.7: For any $\overline{\gamma} = (\gamma_k)$, $\ell_\infty(X \times Y, \overline{\gamma}, \overline{v}, \| \,.\, \|) \subset \ell_\infty(X \times Y, \overline{\gamma}, \overline{u}, \| \,.\, \|)$

if and only if $\liminf_k \dfrac{v_k}{u_k} > 0$.

Proof : Let the condition hold and $\overline{w} = (< x_k\,, y_k >) \in \ell_\infty(X \times Y, \overline{\gamma}, \overline{v}, \| \,.\, \|)$.Then there exists $m > 0$ such that $v_k < m\, u_k$ for all sufficiently large values of k and

$$\sup_k \| \gamma_k < x_k\,, y_k >\|^{vk} \leq N \text{ for some } N > 1.$$

This implies that

$$\sup_k \| \gamma_k < x_k\,, y_k >\|^{uk} \leq N^{1/m} \;\text{ i.e, } \overline{w} = (< x_k\,, y_k >) \notin \ell_\infty(X \times Y, \overline{\gamma}, \overline{u}\,)$$

and hence $\ell_\infty(X \times Y, \overline{\gamma}, \overline{v}, \| \,.\, \|) \subset \ell_\infty(X \times Y, \overline{\gamma}, \overline{u}\,, \| \,.\, \|)$.

Conversely let the inclusion hold but $\liminf_k \dfrac{v_k}{u_k} = 0$. Then we can find a sequence $(k(n))$ of positive integers with $k(n) < k(n + 1)$, $n \geq 1$ such that $\dfrac{v_{k(n)}}{u_{k(n)}} < \quad n$ i.e., $v_{k(n)} < u_{k(n)}$, $n \geq 1$.

Now taking $< r, t > \in X \times Y$ with $\| < r, t > \| = 1$, we define the sequence $\overline{w} = (< x_k\,, y_k >)$ by

$$< x_k\,, y_k > \;= \begin{cases} \gamma_{k(n)}^{-1}\, 2^{1/vk(n)} < r,\; t >, \text{ for } k = k(n),\, n \geq 1 \;\;, \text{ and} \\ < 0,\, 0 >, \text{ otherwise.} \end{cases}$$

Then for $k = k(n)$, $n \geq 1$, we easily see that

$$\begin{aligned} \sup_k \| \gamma_k < x_k\,, y_k > \|^{\,vk} &= \sup_n \| \gamma_{k(n)} < x_{k(n)}, y_{k(n)} > \|^{vk(n)} \\ &= 2 \sup_n \| \;< r, t > \|^{vk(n)} \\ &= 2 \end{aligned}$$

$$\begin{aligned} \text{and}\quad \sup_k \|\gamma_k < x_k\,, y_k > \|^{uk} &= \sup_n \|\gamma_{k(n)} < x_{\,k(n)}, y_{\,k(n)} >\|^{uk(n)} \\ &= \sup_n \| 2^{\,1/vk(n)} < r,\;\; t > \|^{uk(n)} \\ &> \sup_n 2^{\,n} = \infty. \end{aligned}$$

Hence $\overline{w} \in \ell_\infty(X \times Y, \overline{\gamma}, \overline{v}\,, \| \,.\, \|)$ but $\overline{w} \notin \ell_\infty(X \times Y, \overline{\gamma}, \overline{u}\,, \| \,.\, \|)$, a contradiction.

This completes the proof.

On combining Theorems 3.6 and 3.7,we get the following theorem:

Theorem 3.8: For any $\bar{\gamma} = (\gamma_k)$, $\ell_\infty(X \times Y, \bar{\gamma}, \bar{u}, \|\,.\,\|) = \ell_\infty(X \times Y, \bar{\gamma}, \bar{v}, \|\,.\,\|)$

if and only if $0 < \liminf_k \frac{v_k}{u_k} \le \limsup_k \frac{v_k}{u_k} < \infty$.

Corollary 3.9: For any $\bar{\gamma} = (\gamma_k)$,

(i) $\ell_\infty(X \times Y, \bar{\gamma}, \|\,.\,\|) \subset \ell_\infty(X \times Y, \bar{\gamma}, \bar{u}, \|\,.\,\|)$ if and only if $\limsup_k u_k < \infty$;

(ii) $\ell_\infty(X \times Y, \bar{\gamma}, \bar{u}, \|\,.\,\|) \subset \ell_\infty(X \times Y, \bar{\gamma}, \|\,.\,\|)$ if and only if $\liminf_k u_k > 0$;

(iii) $\ell_\infty(X \times Y, \bar{\gamma}, \bar{u}, \|\,.\,\|) = \ell_\infty(X \times Y, \bar{\gamma}, \|\,.\,\|)$ if and only if $0 < \liminf_k u_k \le \limsup_k v_k < \infty$.

Proof :

Proof easily follows when we take $u_k = 1$ and $v_k = u_k$ for all k in theorem 3.6, 3.7 and 3.8.

Theorem 3.10: For any sequences $\bar{\gamma} = (\gamma_k)$, $\bar{\mu} = (\mu_k)$, $\bar{u} = (u_k)$ and $\bar{v} = (v_k)$,

$$\ell_\infty(X \times Y, \bar{\gamma}, \bar{u}, \|\,.\,\|) \subset \ell_\infty(X \times Y, \bar{\mu}, \bar{v}, \|\,.\,\|)$$

if and only if (i) $\liminf_k \left|\frac{\gamma_k}{\mu_k}\right|^{uk} > 0$, and

(ii) $\limsup_k \frac{v_k}{u_k} < \infty$.

Proof : Proof directly follows from Theorems 3.2 and 3.6.

In the following example we show that $\ell_\infty(X \times Y, \bar{\gamma}, \bar{u}, \|\,.\,\|)$ is strictly contained in $\ell_\infty(X \times Y, \bar{\gamma}, \bar{v}, \|\,.\,\|)$ however (i) and (ii) of Theorem 3.10 are satisfied.

Example 3.11:

Let $\bar{w} = (<x_k, y_k>)$ be a sequence in normed space $X \times Y$ such that $\|<x_k, y_k>\| = k^k$.

Take $u_k = \frac{1}{k}$ if k is odd integer and $u_k = \frac{1}{k^2}$, if k is even integer, $v_k = \frac{1}{k^2}$ for all values of k, $\gamma_k = 3^k$ for all values of k ;and $\mu_k = 2^k$, for all values of k. Then

$$\left|\frac{\gamma_k}{\mu_k}\right|^{uk} = \frac{3}{2} \text{ if } k \text{ is odd integer}$$

$$\text{and } \left|\frac{\gamma_k}{\mu_k}\right|^{uk} = \left(\frac{3}{2}\right)^{1/2}, \text{ if } k \text{ is even integer.}$$

Thus $\liminf_k \left|\frac{\gamma_k}{\mu_k}\right|^{uk} = 1$ i.e. condition (i) of Theorem 3.10 is satisfied.

Further since $\frac{v_k}{u_k} = \frac{1}{k}$, if k is odd integer and $\frac{v_k}{u_k} = 1$, if k is even integer, therefore condition (ii) of Theorem 3.10 is also satisfied as $\limsup_k \frac{v_k}{u_k} = 1$.

We now see that $\bar{w} = (<x_k, y_k>) \in \ell_\infty(X \times Y, \bar{\mu}, \bar{v})$ for all $k \ge 1$ as

$$\sup_k \| \mu_k <x_k, y_k> \|^{vk} = \sup_k (2k)^{1/k} < 2,$$

but $\bar{w} = (<x_k, y_k>) \notin \ell_\infty(X \times Y, \bar{\gamma}, \bar{u}, \|\,.\,\|)$, when k is odd integer as

$$\sup_k \| \gamma_k <x_k, y_k> \|^{uk} = \sup_k 3k = \infty.$$

This shows that the condition (i) and (ii) are satisfied but $\ell_\infty(X \times Y, \bar{\gamma}, \bar{u}, \|\,.\,\|)$ is strictly contained in $\ell_\infty(X \times Y, \bar{\gamma}, \bar{v}, \|\,.\,\|)$.

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