Nepal Journal of Mathematical Sciences (NJMS)
ISSN: 2738-9928 (*online*), 2738-9812 (*print*)
Vol. 4, No. 2, 2023 (August): 17-30
DOI: https://doi.org/10.3126/njmathsci.v4i2.59527

Research Article
Received Date: May 3, 2023
Accepted Date: August 2, 2023
Published Date: August 30, 2023

The Gradshteyn and Ryzhik's Integral and the Theoretical Computation of *GM* (*m*,*n*) Involving the Continuous Whole Life Annuities

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Abstract: *Considering the risk connected with the expectation of life at retirement as a result of the unavailable actuarially modeled life annuity to price life insurance products, this study explores the gains in an annuity that would be advantageous to lives who choose life annuity option at retirement under defined condition of actuarially fair annuity value. When continuous parsimonious parametric mortality intensities are Makehamised, then the life table functions used in computing the actuarial present values of the fully continuous whole life insurance and the fully continuous life annuity could be expressed in terms of special functions such as Gamma, Incomplete lower Gamma and Incomplete upper Gamma functions for a homogeneous insured population. In this study, the objective is to*

(i) *Construct mathematical estimations through single life parameterization through algebraic technique*
(ii) *Apply the mean value theorem to construct modification theorems under the framework of policy alterations*
(iii) *Employ the properties of the aforementioned special functions to construct estimations which could permit us to compute closed-form expressions for continuous whole life insurance and continuous whole life expectancy applicable in classical life contingencies.*
(iv) *Apply the commutation function to develop a mathematical model for the employer liability.*

From our results, Gradshteyn and Ryzhik's analytic integral technique presents an advanced technique for computing life insurance biometric functions and ignores the need for any algorithmic numerical procedures.

Keywords: $GM(1,2)$, *Whole life insurance, whole life annuity, Gradshteyn and Ryzhik's integral, Gamma functions*.

1. Introduction to *GM* (*m*,*n*) Class

In human mortality, intensity is applied to define the trends of mortality where the management of life office assets and liability depend on the death rate of the insured Siswono, Azmi, and Syaifudin(2021). Following Lageras, 2009; Missov and Lennart (2013), continuous parametric functions such as assume that the mortality rate increases as age advances. However, the intensity law adds an ageindependent parameter that is not associated with senescence. In human populations, issues connected with overestimation in observed death rates at senescence in mortality trajectories aroused the study of continuous parsimonious parametric mortality models which are responsible for the unobserved heterogeneity and consequently, the cohort population is then partitioned into strata in accordance with an observed measure of insured's exposure to the risk of death. However, in Dragan(2022), we have observed that the methods of generating mortality tables were initially developed for cohorts whose members have varying characteristics in connection with longevity measures.

Numerical Computation of the $GM(1,2)$ Parameters

In Debon, Montes and Sala (2005); Debon, Montes and Sala (2005), the $GM(1,2)$ is defined as

$$\mu_x = \rho + GH^x \tag{1}$$

Let $\zeta = e^{\rho}$ and $G = -\log_e \delta \log_e H$, $\zeta > 0$ and $\delta > 0$

The right hand side must be multiplied by (-1) throughout by definition of the force of mortality

$$\mu_x = -\log_e \zeta + (-\log_e \delta \log_e H)H^x \tag{2}$$

$$\mu_x = -\frac{1}{l_x}\frac{dl_x}{d_x} = -\frac{d\log_e l_x}{dx} \tag{3}$$

$$\mu_x = -\frac{d\log_e l_x}{dx} = -\log_e \zeta + (-\log_e \delta \log_e H)H^x \tag{4}$$

$$-\int \frac{d\log_e l_x}{dx}dx = \int -\log_e \zeta + (-\log_e \delta \log_e H)H^x dx + K \tag{5}$$

$$\log_e l_x = x\log_e \zeta + (\log_e \delta \log_e H)\frac{H^x}{\log_e H} + \log_e \lambda \tag{6}$$

$K = \log_e \lambda$; taking K as the constant of integration

$$\log_e l_x = \log_e \zeta^x + (\log_e \delta \log_e H)\frac{H^x}{\log_e H} + \log_e \lambda \tag{7}$$

$$\log_e l_x = \log_e \zeta^x + (\log_e \delta)H^x + \log_e \lambda \tag{8}$$

where $\log_e \lambda$, is the constant of integration.

$$\log_e l_x = \log_e \zeta^x + \left(\log_e \delta^{H^X}\right) + \log_e \lambda = \log_e \lambda\zeta^x\delta^{H^X} \tag{9}$$

Now, equating both sides, we have

$$l_x = \lambda\zeta^x\delta^{H^x} \Rightarrow \int_0^\infty l_{x+s}\mu_{x+s}ds = \lambda\zeta^x\delta^{H^X} \tag{10}$$

Note that the age of the insured is chronological. We can take four of such age with equal intervals at the points $\{x+0, x+s, x+2s, x+3s\}$ to have four systems of simultaneous equations

$$l_{x+s} = \lambda \zeta^{x+s} \delta^{H^{x+s}} \tag{11}$$

$$l_{x+2s} = \lambda \zeta^{x+2s} \delta^{H^{x+2s}} \tag{12}$$

$$l_{x+3s} = \lambda \zeta^{x+3s} \delta^{H^{x+3s}} \tag{13}$$

$$_sP_x = \frac{l_{x+s}}{l_x} = \frac{\lambda \zeta^{x+s} \delta^{H^{x+s}}}{\lambda \zeta^{x} \delta^{H^{x}}} = \frac{\zeta^s \delta^{H^{x+s}}}{\delta^{H^x}} = \zeta^s \delta^{H^{x\left(H^S-1\right)}} \tag{14}$$

Considering 4 consecutive values of function $\log_e l_x$

$$\log_e l_{x+0} = x \log_e \zeta + (\log_e \delta) H^x + \log_e \lambda \tag{15}$$

$$\log_e l_{x+s} = (x+s)\log_e \zeta + (\log_e \delta) H^{x+s} + \log_e \lambda \tag{16}$$

$$\log_e l_{x+2s} = (x+2s)\log_e \zeta + (\log_e \delta) H^{x+2s} + \log_e \lambda \tag{17}$$

$$\log_e l_{x+3s} = (x+3s)\log_e \zeta + (\log_e \delta) H^{x+3s} + \log_e \lambda \tag{18}$$

$$\begin{aligned}\log_e l_{x+s} - \log_e l_x &= (x+s)\log_e \zeta + (\log_e \delta) H^{x+s} + \log_e \lambda \\ &-x \log_e \zeta + (\log_e \delta) H^x + \log_e \lambda\end{aligned} \tag{19}$$

$$\log_e l_{x+s} - \log_e l_x = (s \log_e \zeta) + (\log_e \delta) H^s H^x - (\log_e \delta) H^x \tag{20}$$

$$\log_e l_{x+s} - \log_e l_x = s \log_e \zeta + H^x \left(H^s - 1\right) \log_e \delta \tag{21}$$

$$\log_e l_{x+2s} - \log_e l_{x+s} = s \log_e \zeta + H^{x+s} \left(H^s - 1\right) \log_e \delta \tag{22}$$

$$\log_e l_{x+3s} - \log_e l_{x+2s} = s \log_e \zeta + H^{x+2s} \left(H^s - 1\right) \log_e \delta \tag{23}$$

$$\begin{aligned}\log_e l_{x+2s} - 2\log_e l_{x+s} + \log_e l_x &= (x+2s)\log_e \zeta + (\log_e \delta) H^{x+2s} + \log_e \lambda - \\ 2\Big[(x+s)(\log_e \zeta) + (\log_e \delta) H^{x+s} + \log_e \lambda\Big] &+ x(\log_e \zeta) + (\log_e \delta) H^x + (\log_e \lambda)\end{aligned} \tag{24}$$

$$\begin{aligned}\log_e l_{x+2s} - 2\log_e l_{x+s} + \log_e l_x &= x \log_e \zeta + 2s \log_e \zeta + (\log_e \delta) H^{x+2s} + \log_e \lambda \\ -2x \log_e \zeta - 2s \log_e \zeta - 2(\log_e \delta) H^{x+s} &- 2\log_e \lambda + x \log_e \zeta + (\log_e \delta) H^x + \log_e \lambda\end{aligned} \tag{25}$$

$$\log_e l_{x+2s} - 2\log_e l_{x+s} + \log_e l_x = (\log_e \delta) H^{x+2s} - 2(\log_e \delta) H^{x+s} + (\log_e \delta) H^x \tag{26}$$

$$\log_e l_{x+2s} - 2\log_e l_{x+s} + \log_e l_x = (\log_e \delta) H^x \left[H^{2s} - 2H^s + 1\right] \tag{27}$$

Let $U = H^s$, then

$$\log_e l_{x+2s} - 2\log_e l_{x+s} + \log_e l_x = (\log_e \delta) U \left[U^2 - 2U^s + 1\right] \tag{28}$$

$$\log_e l_{x+2s} - 2\log_e l_{x+s} + \log_e l_x = (\log_e \delta) U (U-1)^2 \quad (29)$$

$$\log_e l_{x+2s} - 2\log_e l_{x+s} + \log_e l_x = (\log_e \delta) H^x (H^x-1)^2 \quad (30)$$

Similarly,

$$\begin{aligned}\log_e l_{x+3s} - 2\log_e l_{x+2s} + \log_e l_{x+s} &= (x+3s)\log_e \zeta + (\log_e \delta) H^{x+3s} + \log_e \lambda \\ &-2\left[(x+2s)\log_e \zeta + (\log_e \delta) H^{x+2s} + \log_e \lambda\right] + (x+s)\log_e \zeta + (\log_e \delta) H^{x+s} + \log_e \lambda\end{aligned} \quad (31)$$

$$\begin{aligned}\log_e l_{x+3s} - 2\log_e l_{x+2s} + \log_e l_{x+s} &= x\log_e \zeta + 3s\log_e \zeta + (\log_e \delta) H^{x+3s} + \log_e \lambda \\ &-2x\log_e \zeta - 4s\log_e \zeta - 2(\log_e \delta) H^{x+2s} - 2\log_e \lambda + x\log_e \zeta + s\log_e \zeta + (\log_e \delta) H^{x+s} \\ &+ \log_e \lambda\end{aligned} \quad (32)$$

$$\log_e l_{x+3s} - 2\log_e l_{x+2s} + \log_e l_{x+s} = (\log_e \delta) H^{x+3s} - 2(\log_e \delta) H^{x+2s} + (\log_e \delta) H^{x+s} \quad (33)$$

$$\log_e l_{x+3s} - 2\log_e l_{x+2s} + \log_e l_{x+s} = (\log_e \delta) H^{x+s}\left[H^{2s} - 2H^s + 1\right] \quad (34)$$

$$\log_e l_{x+3s} - 2\log_e l_{x+2s} + \log_e l_{x+s} = H^{x+s}(H^s-1)^2 \log_e \delta \quad (35)$$

$$\frac{\log_e l_{x+3s} - 2\log_e l_{x+2s} + \log_e l_{x+s}}{\log_e l_{x+2s} - 2\log_e l_{x+s} + \log_e l_x} = \frac{= H^{x+s}(H^s-1)^2 \log_e \delta}{H^x (H^s-1)^2 \log_e \delta} = H^x \quad (36)$$

Let $\log_e l_{x+2s} - 2\log_e l_{x+s} + \log_e l_x = \alpha$ (37)

$$\log_e l_{x+3s} - 2\log_e l_{x+2s} + \log_e l_{x+s} = \beta \quad (38)$$

$$H^x (H^s-1)^2 \log_e \delta = \alpha \quad (39)$$

$$H^{x+s}(H^s-1)^2 \log_e \delta = \beta \quad (40)$$

Taking logarithms of the two equations above, we have

$$x\log_e H + 2\log_e(H^s-1) + \log_e \log_e \delta = \log_e \alpha \quad (41)$$

$$(x+s)\log_e H + 2\log_e(H^s-1) + \log_e \log_e \delta = \log_e \beta \quad (42)$$

Subtracting equation (41) from(42), we obtain

$$\begin{aligned}&x\log_e H + s\log_e H + 2\log(H^s-1) + \log_e \log_e \delta - x\log_e H - 2\log_e(H^s-1) - \log_e \log_e \delta \\ &= \log_e \beta - \log_e \alpha\end{aligned} \quad (43)$$

$$s\log_e H = \log_e \beta - \log_e \alpha \quad (44)$$

$$\log_e H = \frac{\log_e \beta - \log_e \alpha}{s} = \frac{\log_e \frac{\beta}{\alpha}}{s} \tag{45}$$

$$x \log_e H + 2\log_e\left(H^s - 1\right) + \log_e \log_e \delta = \log_e \alpha \tag{46}$$

substitute (45) into(41)

$$\frac{x}{s}\log_e \frac{\beta}{\alpha} + 2\log_e\left(H^s - 1\right) + \log_e \log_e \delta = \log_e \alpha \tag{47}$$

$$\log_e \log_e \delta = \log_e \alpha - \frac{x}{s}\log_e \frac{\beta}{\alpha} - 2\log_e\left(H^s - 1\right) \tag{48}$$

$$\log_e\left[\log_e \delta\right] = \log_e \alpha + \log_e\left(\frac{\beta}{\alpha}\right)^{-\frac{x}{s}} + \log_e\left(H^s - 1\right)^{-2} = \log_e\left(\frac{\beta}{\alpha}\right)^{-\frac{x}{s}}\left(H^s - 1\right)^{-2} \tag{49}$$

Equation (49) then becomes $\log_e \delta = \alpha\left(\frac{\beta}{\alpha}\right)^{-\frac{x}{s}}\left(H^s - 1\right)^{-2}$ (50)

$$x \log_e H + 2\log_e\left(H^s - 1\right) + \log_e \log_e \delta = \log_e \alpha \tag{51}$$

Eqn (46) is re-expressed as

$$x \log_e H + \log_e\left(H^s - 1\right) + \log_e \log_e \delta = \log_e \alpha - \log_e\left(H^s - 1\right) \tag{52}$$

$$\log_e\left[H^x\left(H^s - 1\right)\log_e \delta\right] = \log_e \frac{\alpha}{\left(H^s - 1\right)} \tag{53}$$

$$H^x\left(H^s - 1\right)\log_e \delta = \frac{\alpha}{\left(H^s - 1\right)} \tag{54}$$

$$\log_e l_{x+s} - \log_e l_x = s\log_e \zeta + H^x\left(H^s - 1\right)\log_e \delta \tag{55}$$

Substituting equation (54) in (55), we have

$$\log_e l_{x+s} - \log_e l_x = s\log_e \zeta + \frac{\alpha}{\left(H^s - 1\right)} \tag{56}$$

$$s\log_e \zeta = \log_e l_{x+s} - \log_e l_x - \frac{\alpha}{\left(H^s - 1\right)} \tag{57}$$

$$\log_e \zeta = \left[\frac{\log_e l_{x+s} - \log_e l_x - \frac{\alpha}{\left(H^s - 1\right)}}{s}\right] \tag{58}$$

Recall from (52) that $x \log_e H + \log_e\left(H^s - 1\right) + \log_e \log_e \delta = \log_e \alpha - \log_e\left(H^s - 1\right)$ (59)

$$x\log_e H + \log_e \log_e \delta = \log_e \alpha + \log_e \left(H^s - 1\right)^{-2} \tag{60}$$

$$\log_e\left[H^x \log_e \delta\right] = \log_e \alpha\left(H^s - 1\right)^{-2} \tag{61}$$

$$H^x \log_e \delta = \alpha\left(H^s - 1\right)^{-2} \tag{62}$$

$$H^x = \frac{\alpha}{\log_e \delta}\left(H^s - 1\right)^{-2} \tag{63}$$

Recall that $x\log_e \zeta + (\log_e \delta)H^x + \log_e \lambda = \log_e l_x$ (64)

$$\log_e \lambda = \log_e l_x - x\log_e \zeta - (\log \delta)H^x \tag{65}$$

Inserting (58), (63) into (64)

$$\log_e \lambda = \log_e l_x - x\left[\frac{\log_e l_{x+s} - \log_e l_x - \dfrac{\alpha}{\left(H^s - 1\right)}}{s}\right] - \alpha\left(H^s - 1\right)^{-2} \tag{66}$$

$$\log_e \zeta = \rho$$

$$\rho = \left[\frac{\log_e l_{x+s} - \log_e l_x - \dfrac{\alpha}{\left(H^s - 1\right)}}{s}\right] \tag{67}$$

and by the initial definition $G = -\log_e \delta \log_e H$

recall $\log_e H = \dfrac{\log_e \dfrac{\beta}{\alpha}}{s} = \log_e\left(\dfrac{\beta}{\alpha}\right)^{\frac{1}{s}}$ (68)

$$H = \left(\frac{\beta}{\alpha}\right)^{\frac{1}{s}} \tag{69}$$

Note that $G = \dfrac{(-\log_e \delta)}{s}\log_e \dfrac{\beta}{\alpha}$ (70)

And $H^x = \dfrac{\alpha}{\log_e \delta}\left(H^s - 1\right)^{-2}$ (71)

$\mu_x = \rho + GH^x$ becomes

$$\mu_x = \left[\frac{\log_e l_{x+s} - \log_e l_x - \dfrac{\alpha}{\left(H^s - 1\right)}}{s}\right] + \left[\frac{\left(-\log_e \delta\right)}{s}\log_e \frac{\beta}{\alpha}\right]\left(\frac{\beta}{\alpha}\right)^{\frac{x}{s}} \tag{72}$$

$$\mu_x = \left[\frac{\log_e \dfrac{l_{x+s}}{l_x} - \dfrac{\alpha}{\left(H^s - 1\right)}}{s}\right] + \left(\frac{\beta}{\alpha}\right)^{\frac{x}{s}} \frac{\left(-\log_e \delta\right)}{s}\log_e \frac{\beta}{\alpha} \tag{73}$$

$$\mu_x = \left[\frac{\log_e\left({}_sP_x\right) - \dfrac{\alpha}{\left(H^s - 1\right)}}{s}\right] + \left(\frac{\beta}{\alpha}\right)^{\frac{x}{s}} \frac{\left(-\log_e \delta\right)}{s}\log_e \frac{\beta}{\alpha} \tag{74}$$

$$\mu_x = \left[\frac{\log_e \zeta^2 \delta^{H^x\left(H^s-1\right)} - \dfrac{\alpha}{\left(H^s - 1\right)}}{s}\right] + \left(\frac{\beta}{\alpha}\right)^{\frac{x}{s}} \frac{\left(-\log_e \delta\right)}{s}\log_e \frac{\beta}{\alpha} \tag{75}$$

Recall $H = \left(\frac{\beta}{\alpha}\right)^{\frac{1}{s}} \Rightarrow H^s = \left(\frac{\beta}{\alpha}\right)$ (76)

So when $s = x$, we have

$$H^x = \left(\frac{\beta}{\alpha}\right) \tag{77}$$

$$\frac{\log_e l_{x+3s} - 2\log_e l_{x+2s} + \log_e l_{x+s}}{\log_e l_{x+2s} - 2\log_e l_{x+s} + \log_e l_x} = \frac{H^{x+s}\left(H^s - 1\right)^2 \log_e \delta}{H^x\left(H^s - 1\right)^2 \log_e \delta} = H^x = \left(\frac{\beta}{\alpha}\right) \tag{78}$$

Inserting Equation (37) and (38) into equation (78), we have

Hence, we obtain $\mu_x = \left\{ \begin{array}{l} \left[\dfrac{\log_e \zeta^2 \delta^{H^x\left(H^s-1\right)} - \dfrac{\alpha}{\left(H^s - 1\right)}}{s}\right] + \\ \left(\dfrac{\beta}{\alpha}\right)^{\frac{x}{s}} \dfrac{\left(-\log_e \delta\right)}{s}\log_e \dfrac{\left(\log_e l_{x+3s} - 2\log_e l_{x+2s}\log_e l_{x+s}\right)}{\log_e l_{x+2s} - 2\log_e l_{x+s} + \log_e l_x} \end{array} \right\}$ (79)

$$\mu_x=\left[\frac{\log_e \zeta^s \delta^{H^x\left(H^s-1\right)}-\dfrac{\alpha}{\left(H^s-1\right)}}{s}\right]+ \tag{80}$$

$$\left(\frac{\beta}{\alpha}\right)^{\frac{x}{s}}\frac{\left(-\log_e\delta\right)}{s}\log_e\frac{\left(\log_e\lambda\zeta^{x+3s}\delta^{H^{x+3s}}-2\log_e\lambda\zeta^{x+2s}\delta^{H^{x+2s}}+\log_e\lambda\zeta^{x+s}\delta^{H^{x+s}}\right)}{\log_e\lambda\zeta^{x+2s}\delta^{H^{x+3s}}-2\log_e\lambda\zeta^{x+s}\delta^{H^{x+s}}+\log_e\lambda\zeta^{x}\delta^{H^{x}}}$$

$$\mu_x=\left[\frac{\log_e \zeta^s \delta^{H^x\left(H^s-1\right)}-\dfrac{\alpha}{\left(H^s-1\right)}}{s}\right]+ \tag{81}$$

$$\left(\frac{\beta}{\alpha}\right)^{\frac{x}{s}}\frac{\left(-\log_e\delta\right)}{s}\log_e\frac{\left(\begin{array}{l}\log_e\lambda\zeta^{x+3s}\delta^{H^{x+3s}}-\log_e\lambda\zeta^{x+2s}\delta^{H^{x+2s}}+\log_e\lambda\zeta^{x+s}\delta^{H^{x+s}}\\-\log_e\lambda\zeta^{x+2s}\delta^{H^{x+2s}}\end{array}\right)}{\begin{array}{l}\log_e\lambda\zeta^{x+2s}\delta^{H^{x+3s}}-\log_e\lambda\zeta^{x+s}\delta^{H^{x+s}}+\log_e\lambda\zeta^{x}\delta^{H^{x}}\\-\log_e\lambda\zeta^{x+s}\delta^{H^{x+s}}\end{array}}$$

$$\mu_x=\left[\frac{\log_e \zeta^s \delta^{H^x\left(H^X-1\right)}-\dfrac{\alpha}{\left(H^s-1\right)}}{s}\right]+ \tag{81a}$$

$$\left(\frac{\beta}{\alpha}\right)^{\frac{x}{s}}\frac{\left(-\log_e\delta\right)}{s}\log_e\frac{\left(\log_e\dfrac{\lambda\zeta^{x+3s}\delta^{H^{x+3s}}}{\log_e\lambda\zeta^{x+2s}\delta^{H^{x+2s}}}+\log_e\dfrac{\lambda\zeta^{x+s}\delta^{H^{x+s}}}{\lambda\zeta^{x+2s}\delta^{H^{x+2s}}}\right)}{\log_e\dfrac{\lambda\zeta^{x+2s}\delta^{H^{x+3s}}}{\log_e\lambda\zeta^{x+s}\delta^{H^{x+s}}}+\log_e\dfrac{\log_e\lambda\zeta^{x}\delta^{H^{x}}}{\lambda\zeta^{x+s}\delta^{H^{x+s}}}}$$

$$\mu_x=\left[\frac{\log_e \zeta^s \delta^{H^x\left(H^X-1\right)}-\dfrac{\alpha}{\left(H^s-1\right)}}{s}\right]+ \tag{82}$$

$$\left(\frac{\beta}{\alpha}\right)^{\frac{x}{s}}\frac{\left(-\log_e\delta\right)}{s}\log_e\left[\frac{\left(\log_e\dfrac{\delta^{H^{x+3s}}}{\delta^{H^{x+2s}}}\dfrac{\delta^{H^{x+s}}}{\delta^{H^{x+2s}}}\right)}{\log_e\dfrac{\delta^{H^{x+3s}}}{\delta^{H^{x+s}}}\dfrac{\delta^{H^{x}}}{\delta^{H^{x+s}}}}\right]$$

Materials and Methods

Let $\sigma = \log_e(1+i)$ be the force of interest where i is the valuation interest rate

Following Neil (1979); Chowdhury (2012); Kara (2021); Patricio, Castellares and Queiroz (2023), the continuous whole life annuity. Suppose, Ω is the maximum age in a mortality table, then

$$\mathbf{E}\left(a_{\overline{T}|}\right) = \bar{a}_x = \frac{1}{l_x}\int_0^{\Omega-x} v^t l_{x+t}\,dt \tag{83}$$

$$\bar{a}_x = \int_0^{\Omega-x} e^{-\sigma t}\left({}_tP_x\right)dt \tag{84}$$

$$\bar{a}_x = \int_0^{\Omega-x} \zeta^t \delta^{H^x\left(H^t-1\right)} e^{-\sigma t}\,dt \tag{85}$$

$${}_tP_x e^{-\sigma t} = e^{-\sigma t}\zeta^t\delta^{H^x\left(H^t-1\right)} = e^{-\sigma t}\zeta^t\delta^{\left(H^{x+t}-H^x\right)} = \frac{e^{-\sigma t}\zeta^t\delta^{\left(H^{x+t}\right)}}{\delta^{H^x}} \tag{86}$$

$${}_tP_x e^{-\sigma t} = e^{-\sigma t}\zeta^t\delta^{H^x\left(H^t-1\right)} = \frac{e^{-\sigma t}\zeta^t\delta^{\left(H^{x+t}\right)}}{\delta^{H^x}} = \frac{\exp\left(H^tH^x\log_e\delta + t\log_e\left(e^{-\sigma}\zeta\right)\right)}{\delta^{H^x}} \tag{87}$$

Observe that $H^t = e^{\log_e H^t} = e^{t\log_e H}$ (88)

$${}_tP_x e^{-\sigma t} = e^{-\sigma t}\zeta^t\delta^{H^x\left(H^t-1\right)} = \frac{e^{-\sigma t}\zeta^t g^{\left(H^{x+t}\right)}}{\delta^{H^x}} = \frac{\exp\left(\left(e^{t\log_e H}\right)\left(H^x\times\log_e g\right)+t\log_e e^{-\sigma}\zeta\right)}{\delta^{H^x}} \tag{89}$$

$$\bar{a}_x = \int_0^{\Omega-x}\frac{\exp\left(\left(e^{t\log_e H}\right)\left(H^x\times\log_e\delta\right)+t\log_e e^{-\sigma}\zeta\right)}{\delta^{H^x}}dt \tag{90}$$

Let $\eta = t\log_e H \Rightarrow \dfrac{\eta}{\log_e H} = t$ (91)

When $t=0$, $\eta=0$

When $t=\Omega-x$, $\eta=\left(\Omega-x\right)\log_e H = \log_e H^{(\Omega-x)}$ (92)

$$\frac{d\eta}{dt} = \log_e H \Rightarrow d\eta = \log_e H dt \Rightarrow \frac{d\eta}{\log_e H} = dt \tag{93}$$

$$\bar{a}_x = \int_0^{\log_e H^{(\Omega-x)}}\frac{\exp\left(e^{\eta}\left(H^x\times\log_e\delta\right)+\dfrac{\eta}{\log_e H}\times\log_e e^{-\sigma}\zeta\right)}{\delta^{H^x}}\frac{d\eta}{\log_e H} \tag{94}$$

$$\bar{a}_x = \frac{1}{(\log_e H)\left(\delta^{H^x}\right)} \int_0^{\log_e H^{(\Omega-x)}} \exp\left(\left(e^{\eta} \times H^x \times \log_e \delta\right) + \frac{\eta}{\log_e H} \times \log_e e^{-\sigma}\zeta\right) d\eta \tag{95}$$

$$\bar{a}_x = \frac{1}{(\log_e H)\left(\delta^{H^x}\right)} \int_0^{\log_e H^{(\Omega-x)}} \exp\left(\left(H^x \times \log_e \delta\right) e^{\eta} + \left(\frac{\log_e e^{-\sigma}\zeta}{\log_e H}\right)\eta\right) d\eta \tag{96}$$

Following Gradshteyn and Ryzhik (n.d, pp. 356, formula, $ETI147(37)$) (97)

$$\int_0^{\infty} \exp\left(-\alpha e^{Y} - \bar{\delta} Y\right) dY = \alpha^{\delta}\Gamma\left(-\bar{\delta}, \alpha\right) \tag{98}$$

Where $\Gamma(.)$ is the gamma function

$$\alpha = a + ib,\ i = \sqrt{-1} \text{ and } a > 0 \text{ and } a \le \operatorname{Re}|\alpha| \tag{99}$$

$$\bar{a}_x = \frac{1}{(\log_e H)\left(\delta^{H^x}\right)} \int_0^{\log_e H^{(\Omega-x)}} \exp\left(-\left(-H^x \times \log_e \delta\right) e^{\eta} - \left(-\frac{\log_e e^{-\sigma}\zeta}{\log_e H}\right)\eta\right) d\eta \tag{100}$$

$$\int_0^{\infty} \exp\left(-\left(-H^x \times \log_e \delta\right) e^{\eta} - \left(-\frac{\log_e e^{-\sigma}\zeta}{\log_e H}\right)\eta\right) d\eta =$$
$$\left\{\begin{array}{l} \displaystyle\int_0^{\log_e H^{(\Omega-x)}} \exp\left(-\left(-H^x \times \log_e \delta\right) e^{\eta} - \left(-\frac{\log_e e^{-\sigma}\zeta}{\log_e H}\right)\eta\right) d\eta \\ \displaystyle + \int_{\log_e H^{(\Omega-x)}}^{\infty} \exp\left(-\left(-H^x \times \log_e \delta\right) e^{\eta} - \left(-\frac{\log_e e^{-\sigma}\zeta}{\log_e H}\right)\eta\right) d\eta \end{array}\right\} \tag{101}$$

Consequently,

$$\bar{a}_x = \frac{\left\{\begin{array}{l} \displaystyle\int_0^{\infty} \exp\left(-\left(-H^x \times \log_e \delta\right) e^{\eta} - \left(-\frac{\log_e e^{-\sigma}\zeta}{\log_e H}\right)\eta\right) d\eta - \\ \displaystyle \int_{\log_e H^{(\Omega-x)}}^{\infty} \exp\left(-\left(-H^x \times \log_e \delta\right) e^{\eta} - \left(-\frac{\log_e e^{-\sigma}\zeta}{\log_e H}\right)\eta\right) d\eta \end{array}\right\}}{(\log_e H)\left(\delta^{H^x}\right)} \tag{101}$$

$$\bar{a}_x = \frac{1}{(\log_e H)\left(\delta^{H^x}\right)} \left\{\begin{array}{l} \displaystyle\int_0^{\infty} \exp\left(-\left(-H^x \times \log_e \delta\right) e^{\eta} - \left(-\frac{\log_e e^{-\sigma}\zeta}{\log_e H}\right)\eta\right) d\eta - \\ \displaystyle \int_{\log_e H^{(\Omega-x)}}^{\infty} \exp\left(-\left(-H^x \times \log_e \delta\right) e^{\eta} - \left(-\frac{\log_e e^{-\sigma}\zeta}{\log_e H}\right)\eta\right) d\eta \end{array}\right\} \tag{102}$$

$$J_1 = \int_0^\infty \exp\left(-\left(-H^x \times \log_e \delta\right)e^\eta - \left(-\frac{\log_e e^{-\sigma}\zeta}{\log_e H}\right)\eta\right)d\eta =$$
$$\left[\left(-H^x \times \log_e \delta\right)\right]^{\left(-\frac{\log_e \zeta}{\log_e H}\right)}\Gamma\left(-\left(-\frac{\log_e e^{-\sigma}\zeta}{\log_e H}\right), \left(-H^x \times \log_e \delta\right)\right) \quad (103)$$

$$J_2 = \int_{\log_e H^{(\Omega-x)}}^\infty \exp\left(-\left(-H^x \times \log_e \delta\right)e^\eta - \left(-\frac{\log_e e^{-\sigma}\zeta}{\log_e H}\right)\eta\right)d\eta \quad (104)$$

Let $\xi = \eta - \log_e H^{\Omega-x}$ (105)

Let $\xi + \log_e H^{\Omega-x} = \eta$ (106)

$d\xi = d\eta$ (107)

When $\eta = \infty$, $\xi = \infty$ and when $\eta = \log_e H^{\Omega-x}$, $\xi = 0$

Therefore, $J_2 = \int_0^\infty \exp\left(-\left(-H^x \times \log_e \delta\right)e^{\xi+\log_e H^{\Omega-x}} - \left(-\frac{\log_e e^{-\sigma}\zeta}{\log_e H}\right)\left(\xi + \log_e H^{\Omega-x}\right)\right)d\xi$ (108)

$$J_2 = \int_0^\infty \exp\left\{-\left(-H^x \times \log_e \delta\right)e^{\log_e H^{\Omega-x}}e^\xi - \left(\left(-\frac{\log_e e^{-\sigma}\zeta}{\log_e H}\right)\xi + \left(\left(-\frac{\log_e e^{-\sigma}\zeta}{\log_e H}\right)\right)\log_e H^{\Omega-x}\right)\right\}d\xi \quad (109)$$

$$J_2 = \int_0^\infty \exp\left\{-\left(-H^x \times \log_e \delta\right)H^{\Omega-x}e^\xi + -\left(-\frac{\log_e e^{-\sigma}\zeta}{\log_e H}\right)\xi - \left(-\frac{\log_e e^{-\sigma}\zeta}{\log_e H}\right)(\Omega - x)\left(\log_e H\right)\right\}d\xi \quad (110)$$

$$J_2 = \int_0^\infty \exp\left\{-\left(-\log_e \delta\right)H^\Omega e^\xi + -\left(-\frac{\log_e e^{-\sigma}\zeta}{\log_e H}\right)\xi - \left(-\log_e e^{-\sigma}\zeta\right)(\Omega - x)\right\}d\xi \quad (111)$$

$$J_2 = \int_0^\infty \exp\left\{-\left(-\log_e \delta\right)H^\Omega e^\xi + -\left(-\frac{\log_e e^{-\sigma}\zeta}{\log_e H}\right)\xi + \log_e \zeta^{\Omega-x}\right\}d\xi \quad (112)$$

$$J_2 = \int_0^\infty \exp\left\{-\left(-\log_e \delta\right)H^\Omega e^\xi + -\left(-\frac{\log_e e^{-\sigma}\zeta}{\log_e H}\right)\xi + \log_e e^{-\sigma(\Omega-x)}\right\}d\xi \quad (113)$$

$$J_2 = \int_0^\infty \exp\left\{-\left(-\log_e \delta\right)H^\Omega e^\xi + -\left(-\frac{\log_e e^{-\sigma}\zeta}{\log_e H}\right)\xi - \sigma(\Omega - x)\right\}d\xi \quad (114)$$

$$J_2 = e^{-\sigma(\Omega-x)}\int_0^\infty \exp\left\{-\left(-\log_e \delta\right)H^\Omega \times e^\xi - \left(-\frac{\log_e e^{-\sigma}\zeta}{\log_e H}\right)\xi\right\}d\xi \quad (115)$$

$$J_2 = e^{-\sigma(\Omega - x)}\int_0^\infty \exp\left\{-\left(-\log_e \delta\right)H^\Omega \times e^\xi - \left(-\frac{\log_e e^{-\sigma}\zeta}{\log_e H}\right)\xi\right\}d\xi = e^{-\sigma(\Omega-x)}\left[\left(-\log_e \delta\right)H^\Omega\right]^{\left(-\frac{\log_e e^{-\sigma}\zeta}{\log_e H}\right)}\Gamma\left(-\left(-\frac{\log_e e^{-\sigma}\zeta}{\log_e H}\right), \left(-\log_e \delta\right)H^\Omega\right) \tag{116}$$

$$\bar{a}_x = \frac{1}{\left(\log_e H\right)\left(\delta^{H^x}\right)}\left(J_1 - J_2\right) \tag{117}$$

$$\bar{a}_x = \frac{1}{\left(\log_e H\right)\left(\delta^{H^x}\right)}\left\{\begin{array}{l}\left[\left(-H^x \times \log_e \delta\right)\right]^{\left(-\frac{\log_e e^{-\sigma}\zeta}{\log_e H}\right)}\Gamma\left(-\left(-\frac{\log_e e^{-\sigma}\zeta}{\log_e H}\right), \left(-H^x \times \log_e \delta\right)\right) - \\ e^{-\sigma(\Omega-x)}\left[\left(-\log_e \delta\right)H^\Omega\right]^{\left(-\frac{\log_e e^{-\sigma}\zeta}{\log_e H}\right)}\Gamma\left(-\left(-\frac{\log_e e^{-\sigma}\zeta}{\log_e H}\right), \left(-\log_e \delta\right)H^\Omega\right)\end{array}\right\} \tag{118}$$

$$\bar{a}_x = \frac{1}{\left(\log_e H\right)\left(\delta^{H^x}\right)}\left\{\begin{array}{l}\left[\left(-H^x \times \log_e \delta\right)\right]^{\left(-\frac{\log_e e^{-\sigma}\zeta}{\log_e H}\right)}\Gamma\left(\left(\frac{\log_e e^{-\sigma}\zeta}{\log_e H}\right), \left(-H^x \times \log_e \delta\right)\right) - \\ e^{-\sigma(\Omega-x)}\left[\left(-\log_e \delta\right)H^\Omega\right]^{\left(-\frac{\log_e e^{-\sigma}\zeta}{\log_e H}\right)}\Gamma\left(\left(\frac{\log_e e^{-\sigma}\zeta}{\log_e H}\right), \left(-\log_e \delta\right)H^\Omega\right)\end{array}\right\} \tag{119}$$

Where $H^x = \frac{\alpha}{\log_e \delta}\left(H^s - 1\right)^{-2}$ (120)

$$\log_e \zeta = \left[\frac{\log_e l_{x+s} - \log_e l_x - \frac{\alpha}{\left(H^s - 1\right)}}{s}\right] \Rightarrow \zeta = e^{\left(\frac{\log_e l_{x+s} - \log_e l_x - \frac{\alpha}{\left(H^s-1\right)}}{s}\right)} \tag{121}$$

$$\log_e \delta = \frac{\alpha\left(H^s - 1\right)^{-2}}{H^x} \Rightarrow H^x \log_e \delta = \alpha\left(H^s - 1\right)^{-2} \Rightarrow \log_e \delta^{H^x} = \alpha\left(H^s - 1\right)^{-2} \Rightarrow \delta^{H^x} = e^{\alpha\left(H^s - 1\right)^{-2}} \tag{122}$$

$$H^x = \frac{\alpha}{\log_e \delta}\left(H^s - 1\right)^{-2} \tag{123}$$

$$\bar{a}_x = \frac{1}{\left(\log_e H\right)\left(\delta^{H^x}\right)}\left\{\begin{array}{l}\left[\left(-H^x \times \log_e \delta\right)\right]^{\left(-\frac{\log_e e^{-\sigma}\zeta}{\log_e H}\right)}\Gamma\left(\left(\frac{\log_e e^{-\sigma}\zeta}{\log_e H}\right), \left(-H^x \times \log_e \delta\right)\right) - \\ e^{-\sigma(\Omega-x)}\left[\left(-\log_e \delta\right)H^\Omega\right]^{\left(-\frac{\log_e e^{-\sigma}\zeta}{\log_e H}\right)}\Gamma\left(\left(\frac{\log_e e^{-\sigma}\zeta}{\log_e H}\right), \left(-\log_e \delta\right)H^\Omega\right)\end{array}\right\} \tag{124}$$

If $l_x = \int_0^\infty l_{x+s}\mu_{x+s}ds$ and $l_{x+\xi} = \int_0^\infty l_{x+s+\xi}\mu_{x+s}\left(\xi\right)d\xi$, then following de Souza (2018), the continuous annuity in equation (124) is bounded as follows

$$\left(\frac{l_{x+n} - l_{x+n}e^{-\delta n} + l_{x+n}\delta e^{-\delta n}\overline{a}_{x+n}}{\delta l_x}\right) \le \overline{a}_x \le \left(\frac{l_x - l_x e^{-\delta n} + \delta l_{x+n}e^{-\delta n}\overline{a}_{x+n}}{l_x \delta}\right) \tag{125}$$

Discussion of results

$$\lim_{\sigma\to 0}\overline{a}_x = \int_0^\infty {}_\xi p_x d\xi = \overline{e}_x \tag{126}$$

$$\lim_{\sigma\to 0}\overline{a}_x = \frac{1}{\left(\log_e H\right)\left(\delta^{H^x}\right)}\lim_{\sigma\to 0}\left\{\begin{array}{l}\left[\left(-H^x \times \log_e \delta\right)\right]^{\left(-\frac{\log_e e^{-\sigma}\zeta}{\log_e H}\right)}\Gamma\left(\left(\dfrac{\log_e e^{-\sigma}\zeta}{\log_e H}\right),\left(-H^x \times \log_e \delta\right)\right) - \\ e^{-\sigma(\Omega - x)}\left[\left(-\log_e \delta\right)H^\Omega\right]^{\left(-\frac{\log_e e^{-\sigma}\zeta}{\log_e H}\right)}\Gamma\left(\left(\dfrac{\log_e e^{-\sigma}\zeta}{\log_e H}\right),\left(-\log_e \delta\right)H^\Omega\right)\end{array}\right\} \tag{127}$$

$$\int_0^\infty {}_\xi p_x d\xi = \frac{1}{\left(\log_e H\right)\left(\delta^{H^x}\right)}\left\{\begin{array}{l}\left[\left(-H^x \times \log_e \delta\right)\right]^{\left(-\frac{\log_e \zeta}{\log_e H}\right)}\Gamma\left(\left(\dfrac{\log_e \zeta}{\log_e H}\right),\left(-H^x \times \log_e \delta\right)\right) - \\ \left[\left(-\log_e \delta\right)H^\Omega\right]^{\left(-\frac{\log_e \zeta}{\log_e H}\right)}\Gamma\left(\left(\dfrac{\log_e \zeta}{\log_e H}\right),\left(-\log_e \delta\right)H^\Omega\right)\end{array}\right\} \tag{128}$$

Conclusion

Life annuity plays an important role in defined benefits schemes under defined contribution pension plans and hence it represents a modified version of a defined benefit structure. Consequently, it lends itself as a good alternative measure to defined benefits schemes to assist retirees in earning income streams provided the annuitant survives. This paper contributes to this field by providing an analytical technique for computing the fully continuous life annuities and continuous life insurance under the framework of mortality rate intensity defining the trend of human mortality. The development of actuarially robust analytical computation of fully continuous life annuities and fully continuous life insurance has continually posed core challenges for actuaries and life offices. In insured populations having reasonably good track records of death statistics, there seems to be disturbances in the function of a low number of events representing limitations in the information on the survival data at different ages. The applicable pricing assumptions available in life insurance especially in annuity-linked securities take into account changes in demographic statistics and mortality changes. The mathematical technique through the Gamma function is used to evaluate attempts to model and generate mortality rate intensities further employed in computing pension and death benefits. The continuous life annuities in a probabilistic mortality model aptly defines the actuarial present value of the underlying death density function such that the analytically closed form solution for the annuity integral contains special function in the form Gamma, upper Incomplete Gamma, and Lower Incomplete Gamma function. In particular, the lower Incomplete Gamma function was constructed with series representation to allow approximations of first-order and second-order basis when the initial level of mortality is infinitesimally small.

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