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On a Generalization of Chatterjee's Fixed Point Theorem in *b*-metric Space

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Abstract: *Banach's Fixed Point Theorem (BFT) deals with the certain contraction mappings of a complete metric space into itself. It states sufficient conditions for the existence and uniqueness of a fixed point. In the study of fixed point theory, BCP has been extended and generalized in many different directions in usual metric spaces. One of those generalizations is a b-metric space. Such generalizations have resulted in generalizing some popular metric fixed point theorems in the context of a b-metric space. In 2013, Kir and Kiziltunc [8] attempted to generalize Chatterjee's Fixed Point Theorem (CFPT) in the context of b-metric spaces. The proof of that generalization, however, had a minor flaw and an unstated assumption. This paper attempts to fix these issues by introducing new conditions.*

Keywords: *Convergence, Compactness, Cauchy sequence, Metric space, b-Metric space.*

1. Introduction and Motivation:

The concept of fixed point theories is one of the most important results in Functional Analysis. The famous fixed point result called Banach Contraction Principle(BCP) is generalized and improved in many directions. One usual way of studying the Banach contraction principle is to replace the metric space with certain generalized metric spaces. Some problems, particularly the problem of the convergence of measurable functions with respect to measure led Czerwik[6] to a generalization of metric space and introduced the concept of b-metric space. The concept of b-metric space was generalized in different directions, for instance, we refer to a few: Alamari and Ahamad [1], Bakhtin[2], Iqbal, Batool, Ege and Sen[7], Ojha and Pahari [10] and, Shoaib, and et al [12]. Several authors proved fixed-point results of single-valued and multi-valued operators in b-metric spaces. Also, Kumar, Mishra, and Mishra [9] studied common fixed point theorems in b-metric space. In the present article, we shall study on a generalization of Chatterjee's Fixed Point Theorem studied in [4]in b-metric space.

Before proceeding with the main work, we shall define some important definitions, examples, and key results related to b-metric spaces, which are used in our further discussion.

Definition 1.1 (***b*-metric space**, Bakhtin **[2]**) Let X be any non-empty set and $b \geq 1$ be some given real number. Let $d : X \times X \to [0, \infty)$ be a function which satisfies the following properties:

(b1) For all $x, y \in X$, $d(x,y) \geq 0$ and $d(x,y) = 0 \Leftrightarrow x = y$.
(b2) For all $x, y \in X$, $d(x,y) = d(y,x)$.
(b3) For all $x, y, z \in X$, $d(x,y) \leq b[d(x,z) + d(y,z)]$.

Then, we say that d is a b-metric defined on X and that X along with d forms a b-metric space and is denoted by the ordered pair (X, d).In some cases, if we need a distinction between b-metrics defined on different spaces, we write the space in its suffix. For example, we may write d as d_X in the above discussion. We define b as a triangular constant and refer to (b3) as relaxed triangle inequality or b-triangle inequality (Cobzas,**[5]**), and (Czerwik, **[6]**).

The following are examples of b-metric spaces:

Example 1.2. Every metric space is an example of a b-metric space because we have b = 1 validating the condition, (b3).

Example 1.3.(Bakhtin **[2]**)

The set $L_p(\mathbb{R})$ where $L_p(\mathbb{R}) = \{\{x_n\} \subseteq \mathbb{R} : \sum |x_n|^p < \infty\}$ (with $0 < p < 1$) together with the function $d : L_p(\mathbb{R}) \times L_p(\mathbb{R}) \to [0, \infty)$ defined by

$$d(x, y) = \left(\sum_{i=0}^{n} |x - y|^p\right)^{1/p}$$

where $x = \{x_n\}, y = \{y_n\} \in L_p(\mathbb{R})$ forms a b-metric with $b = 2^{1/p}$.

Example 1.4.(Bakhtin **[2]**)

The space $L_p[0, 1]$ (where $0 < p < 1$) of all real functions $x(t)$, $t \in [0, 1]$ such that $\int_0^1 |x(t)|^p dt < \infty$ forms a b-metric by defining

$$d(x, y) = \left(\int_0^1 |x(t) - y(t)|^p\right)^{1/p} dt \text{ for each } x, y \in L_p[0, 1], \text{ with } b = 2^{1/p}.$$

It is clear that definition of b-metric is an extension of usual metric space. Obviously, each metric space is a b-metric space with b = 1. However, Czerwik [6] has shown that a b-metric on X need not be a metric on X. The following example illustrates this situation.

Example 1.5.

Let (X, d) be a metric space. Define $\rho(x, y) = [d(x, y)]^p$, where $p > 1$ is a real number. Then we can verify that ρ forms a b-metric with $b = 2^{p-1}$. However, if (X, d) is a metric space, then (X, ρ) is not necessarily a metric space.

Example 1.6 (Bota , Molnar, and Varga,**[3]**)**.** Let X be a set with three elements. Let$X = X_1 \cup X_2$ such that X_1 has two elements and $X_1 \cap X_2 = \emptyset$. Define $d : X \times X \to \mathbb{R}$ by

$$d(x,y) = \begin{cases} 0, & for\ x = y \\ 4, & for\ x, y \in X_1 \ \ and\ \ x \neq y \\ 1, & for\ x \in X_1, y \in X_2 \ \ and\ \ x \neq y \end{cases}$$

Then (X, d) is a b-metric space but not a metric space.

It is noted that the class of b-metric spaces is larger than the class of metric spaces. The following are the concepts related to sequences which we shall use in the main result.

Definition 1.7(Bota ,Molnar, and Varga**, [3]).** Let (X, d) be a b-metric space. A sequence$(x_n)_{n=1}^{\infty}$ in X is said to converge to some $x \in X$ if for every $\varepsilon > 0$ there exists a positive integer N such that

$$n \geq N \Rightarrow d(x_n, x) < \varepsilon.$$

It is denoted by $\lim_{n \to \infty} x_n = x$.

Since$(d(x_n, x))_{n=1}^{\infty}$is a sequence of positive real numbers, this definition suggests the convergence of this sequence to zero is a characterization of convergent sequence in b-metric space. This is analogical to a similar characterization in a metric space.

Definition 1.8 (Bota , Molnar, and Varga**, [3]).** Let (X, d) be a b-metric space. A sequence$(x_n)_{n=1}^{\infty}$ in X is said to be a Cauchy sequence if for every $\varepsilon > 0$ there exists a positive integer N such that

$$m, n \geq N \Rightarrow d(x_m, x_n) < \varepsilon.$$

Thus, just like in the case of metric spaces, we can equivalently say that $(x_n)_{n=1}^{\infty}$ in X is a Cauchy sequence if $d(x_m, x_n) \to 0$ as $m, n \to \infty$.

Definition 1.9 (Bota , Molnar, and Varga**, [3]).** If a b-metric space(X, d) is such that every Cauchy sequence in space(X, d) is convergent, then it is complete b-metric space.

Definition 1.10 (Cobzas**, [5]).** Let (X, d) be a b-metric space. Then, d is said to be continuous if for any two convergent sequences $(x_n)_{n=1}^{\infty}$ and$(y_n)_{n=1}^{\infty}$ of points in X, we have

$$\lim_{n \to \infty} d(x_n, y_n) = d(x, y), \text{ where } \lim_{n \to \infty} x_n = x \text{ and } \lim_{n \to \infty} y_n = y.$$

Definition 1.11 (Panthi,**[11]).** A point u is a fixed point of the function $f(x)$ if $f(u) = u$. In other words, $f(x)$ has a root at u iff $g(x) = x - f(x)$ has a fixed point at u.

2. A Critical Study of Kir and Kiziltunc's Generalization of Chatterjee's Fixed Point Theorem (CFPT)

Kir and Kiziltunc**[8]** gave generalizations of Banach Fixed Point Theorem (BFPT), Kannan Fixed Point Theorem (KFPT) and Chatterjee's Fixed Point Theorem (CFPT). These theorems have been listed respectively as Theorem 2.1, Theorem 2.2 and Theorem 2.3 below, in the same order as they appear in Kir and Kiziltunc**[8].**The theorems have been restructured here in order to make them consistent with the notations that we have used in this paper.

Theorem 2.1 (Kir and Kiziltunc,**[8]).** Let (X, d) be a complete b-metric space with a triangular constant $b \geq 1$. Let $T: X \to X$ be a function then there exists $\lambda > 0$such that $\lambda \in (0, 1)$ and $b\lambda < 1$ which also satisfies

$$d(Tx, Ty) \leq \lambda\, d(x, y), \quad \forall\, x, y \in X.$$

Then, T has a unique fixed point.

Theorem 2.2 (Kir and Kiziltunc,**[8]).**Let (X, d) be a complete b-metric space with a triangular constant $b \geq 1$. Let $T: X \to X$ be a function for which there exists $\lambda > 0$such that $\lambda \in \left(0, \frac{1}{2}\right)$which also satisfies

$$d(Tx, Ty) \leq \lambda\, [d(x, Tx) + d(y, Ty)], \quad \forall\, x, y \in X.$$

Then, T has a unique fixed point.

Theorem 2.3 (Kir and Kiziltunc,**[8]).**Let (X, d) be a complete b-metric space with a triangular constant $b \geq 1$. Let $T: X \to X$ be a function for which there exists $\lambda > 0$ be such that $b\lambda \in \left(0, \frac{1}{2}\right)$which also satisfies

$$d(Tx, Ty) \leq \lambda\, [d(x, Ty) + d(y, Tx)] \quad \forall\, x, y \in X.$$

Then, T has a unique fixed point.

These theorems had one more condition, which was actually a hint to construct a Cauchy sequence for the proof, rather than a condition that was needed to construct a proof. It was to choose any $x_0 \in X$ and construct a sequence $(x_n)_{n=0}^{\infty}$ by $x_n = T^n x_0$. This sequence is then shown to be a Cauchy sequence using the conditions in the theorems. This construction has not been overlooked in this paper.

The proof of the third theorem had a flaw and the proof of $(x_n)_{n=0}^{\infty}$being a Cauchy sequence has some unstated assumptions as below.

a) The flaw is that the step marked in their proof has been obtained by assuming the continuity of the *b*-metric *d.* The theorem doesn't state that condition and it has been illustrated by Cobzas**[5]** that a *b*-metric is not necessarily continuous.

b) The proof of $(x_n)_{n=0}^{\infty}$ being a Cauchy sequence is said to be followed by using a similar method as used in the proof of Theorem 2.1 and Theorem 2.2 .Theorem 2.2 suggests the method similar to that of Theorem 2.1. So, basically the authors want us to use the procedure as used in Theorem 2.1. But while doing so, we obtain

$$d(x_m\,,x_n)\ \le bk^m[1+(bk)+(bk)^2+\cdots+(bk)^{n-m-1}]\,d(x_0\,,x_1)$$

The authors have assumed that $bk\ <\ 1$, which leads to the conclusion that the geometric series on the right was convergent and therefore the sequence was Cauchy.

Here, $k=\frac{b\lambda}{1-b\lambda}$. But ,if we have $b\ =\ 20$ and $\lambda=\frac{1}{80}$? In such a case, we have

$$bk=\frac{b^2\lambda}{1-b\lambda}=\frac{(400\times\frac{1}{80})}{(1-\frac{1}{4})}=\frac{20}{3}>1\,.$$

In this case, the convergence of the said geometric sequence will not follow at all. The authors have not considered or mentioned such possibilities, which makes the proof incomplete.

Here, we wish to alter the conditions prescribed by Theorem 2.3 so that the new conditions would generalize Chatterjee's Fixed Point Theorem studied in **[4]** to a *b*-metric space and has no such questionable assumptions and flaws.

3. Main Result

After critically analyzing the proof of Theorem 2.3, it was found that to fix the flaw of continuity of *d*, we need the assumption of continuity of *d*. And, to obtain a Cauchy sequence as we wished, it sufficed to take $bk\ <\ 1$. If $bk\ <\ 1$, then it was found that we can drop the original condition that $b\lambda\in\left(0,\ \frac{1}{2}\right)$. The necessary “corrections" were found to be trivial. This is stated and proved formally in Theorem 3.1.

Theorem 3.1.Let (X,d) be a complete *b*-metric space with a continuous *b*-metric *d* and a triangular constant $b\geq 1$. Let $T\colon X\to X$ be a function for which there exists $\lambda>0$ such that $0<\frac{b^2\lambda}{1-b\lambda}<1$ which also satisfies

$$d(Tx\,,Ty)\le\ \lambda\,[d(x,Ty)+d(y,Tx)]\quad\forall\,x\,,y\ \in X\,.$$

Then, *T* has a unique fixed point.

Proof. Let the given condition hold.
Since$0<\frac{b^2\lambda}{1-b\lambda}<1$ and $b^2\lambda>0$, it follows that $1-b\lambda>0$.
Consequently, we get

$$0<\frac{b\lambda}{1-b\lambda}<\frac{b^2\lambda}{1-b\lambda}<1$$

To construct a Cauchy sequence, let $s\ \in X$ be arbitrary. Define a sequence $(x_n)_{n=0}^{\infty}$ by $x_n\ =\ T^n\,s$ so that, in general we get $x_{n+1}\ =\ Tx_n$. This sequence will be shown to be a Cauchy sequence. Let $n\ \in\mathbb{N}$. Then

$$\begin{aligned} d(x_n , x_{n+1}) &= d(Tx_{n-1} , Tx_n) \\ &\le \lambda\, [d(x_{n-1} , Tx_n) + d(x_n , Tx_{n-1})] \\ &= \lambda\, d(x_{n-1} , Tx_n) \qquad [\because\ x_n = Tx_{n-1}] \\ &= \lambda\, d(x_{n-1} , x_{n+1}) \qquad [\because\ x_{n+1} = Tx_n] \\ &\le b\lambda\, [d(x_{n-1} , x_n) + d(x_n , x_{n+1})] \end{aligned}$$

which implies that

$$(1 - b\lambda)\, d(x_n , x_{n+1}) \le b\, \lambda\, d(x_{n-1} , x_n)$$

and therefore,

$$d(x_n , x_{n+1}) \le k d(x_{n-1} , x_n)$$

where, $k = \frac{b\lambda}{1-b\lambda}$, because $1 - b\lambda > 0$. Using this relation recursively, we get

$$d(x_n , x_{n+1}) \le k^n d(s , x_1)$$

Now, let $m, n \in \mathbb{N}, n > m$ and for $0 < bk < 1$, it follows that

$$\begin{aligned} d(x_m , x_n) &\le b[d(x_m , x_{m+1}) + d(x_{m+1} , x_n)] \\ &\le b[k^m d(s , x_1) + d(x_{m+1} , x_n)] \\ &= bk^m d(s , x_1) + b\, d(x_{m+1} , x_n) \\ &\le bk^m d(s , x_1) + b^2 k^{m+1} d(s , x_1) + b^2 d(x_{m+2} , x_n) \\ &\ \ \vdots \\ &\le bk^m [1 + (bk) + (bk)^2 + \cdots + (bk)^{n-m-1}]\, d(s , x_1) \\ &= bk^m \left[\frac{1-(bk)^{n-m}}{1-(bk)}\right] d(s , x_1) \\ &\le bk^m \left[\frac{1}{1-(bk)}\right] d(s , x_1) \end{aligned}$$

So, $(x_n)_{n=0}^{\infty}$ is a Cauchy sequence since $d(x_m , x_n) \to 0$ as $m , n \to \infty$.
Thus, by completeness of *X*, there exists $x \in X$ such that $\lim_{n \to \infty} x_n = x$.Now, we show that *x* is a fixed point of *T*. For $n \in \mathbb{N}$, we have

$$\begin{aligned} d(x, Tx) &\le b\, [d(x, x_{n+1}) + d(x_{n+1} , Tx)] \\ &= b\, d(x, x_{n+1}) + b\, d(Tx_n , Tx)] \\ &\le b\, d(x, x_{n+1}) + b\, \lambda\, d(x, Tx_n) + b\lambda\, d(x_n , Tx) \\ &= b\, d(x, x_{n+1}) + b\, \lambda\, d(x, x_{n+1}) + b\lambda\, d(x_n , Tx) \end{aligned}$$

Due to the continuity of *d,* we get

$$d(x_n , Tx) \to d(x , Tx) \text{ as } n \to \infty.$$

So taking limits as $n \to \infty$ in above inequality, we get

$$d(x, Tx) \le b\lambda\, d(x, Tx).$$

Now, as $1 - b\,\lambda > 0$, we have

$$\begin{aligned} & d(x, Tx) \le b\lambda\, d(x, Tx) \\ & \Rightarrow (1 - b\lambda) d(x, Tx) \le 0 \\ & \Rightarrow d(x, Tx) \le 0 \\ & \Rightarrow d(x, Tx) = 0. \end{aligned}$$

Therefore, $Tx = x$, which makes *x* a fixed point of *T*.

To establish the uniqueness, let *y* be a different fixed point than *x* so that we have $y = Ty$. As $x \neq y$, we have $d(x, y) > 0$. Since *x* and *y* are fixed points of *T*, we have

$$d(x, Ty) = d(y, Tx) = d(x, y).$$

So, we obtain

$$\begin{aligned} d(x,y) &= d(Tx,Ty) \\ &\le \lambda\,[\,d(x,Ty)+d(y,Tx)] \\ &\le 2\lambda\, d(x,y) \end{aligned}$$

Now, as $0 < \frac{b\lambda}{1-b\lambda} < 1$.

It follows that $b\lambda < 1 - b\lambda$ and so, $2b\lambda < 1$.

Since $b \geq 1$, it follows that $2\lambda < 2b\lambda < 1$.

Therefore, the last inequality reduces to $d(x,y) < d(x,y)$.This is absurd. Hence, x is a unique fixed point of T.

Conclusion

In this paper, we have introduced some existing properties of b-metric space as the usual notion of a metric space. Besides this, we have studied a generalization of Chatterjee's Fixed Point Theorem in *b*-metric space. In fact, this result can be used for further research work in fixed point theory in Metric space and extends many other authors' existing works.

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